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Reçu par la Rédaction le 6. 2. 1961

STUDIA MATHEMATICA, T. XXI. (1961)

Fourier analysis in Marcinkiewicz spaces

by

K. URBANIK (Wrocław)

The *Marcinkiewicz space* $\mathcal{M}^p$ $(p \geqslant 1)$ consists of all complex-valued Lebesgue measurable and locally integrable functions f on the real line such that

$$\|f\|_p = \overline{\lim_{T\to\infty}}\left(\frac{1}{2T}\int_{-T}^{T}|f(t)|^p\,dt\right)^{1/p} < \infty.$$

J. Marcinkiewicz [10] proved that the quotient space $\mathcal{M}^p/Q^p$, where Q^p denotes the set of all elements f belonging to $\mathcal{M}^p$, with $\|f\|_p = 0$, is a Banach space having $\| \; \|_p$ as its norm. It is easy to verify that for any $p \geqslant 1$ the inclusion $\mathcal{M}^p \subset \mathcal{M}^1$ holds. The closure in the norm $\| \; \|_p$ of the set of all trigonometric polynomials $\sum_{k=1}^{n} a_k e^{i\lambda_k t}$ with arbitrary real exponents $\lambda_1, \lambda_2, \ldots, \lambda_n$ and complex coefficients $a_1, a_2, \ldots, a_n$ is the well known Besicovitch space $\mathcal{B}^p$, whose elements are so-called *$\mathcal{B}^p$-almost periodic functions* ([4], Chapter II, § 7).

Every Besicovitch almost periodic function g has the mean value $m(g)$, which is defined by the limit

$$m(g) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(t)\,dt.$$

The Fourier coefficients $\{a_g(\lambda)\}$ are defined by the formula

$$a_g(\lambda) = m(ge^{-i\lambda t}) \qquad (-\infty < \lambda < \infty).$$

The Fundamental Uniqueness Theorem says that two almost periodic functions from $\mathcal{B}^p$ having the same Fourier coefficients are identical in the sense of the norm $\| \; \|_p$. Moreover, there exists at most an enumerably infinite set of values λ for which $a_g(\lambda)$ differs from nought.

The mean value m on $\mathcal{B}^1$ is a continuous linear functional and $|m(g)| \leqslant \|g\|_1$ for $g \in \mathcal{B}^1$. Every extension m of the functional m from $\mathcal{B}^1$ to $\mathcal{M}^1$ satisfying the inequality $|m(f)| < \|f\|_1$ $(f \in \mathcal{M}^1)$ will be called a *generalized mean value* on $\mathcal{M}^1$. The well-known Hahn-Banach extension theorem (for complex spaces proved in [6] and [13]) implies the existence of many generalized mean values on $\mathscr{M}^1$.

Now let f be a function from the Marcinkiewicz space $\mathscr{M}^1$. For any real λ the product $f(t)e^{-i\lambda t}$ also belongs to $\mathscr{M}^1$. We introduce the notation $a_f^{\mathfrak{m}}(\lambda) = \mathfrak{m}(fe^{-i\lambda t})$. The family $\{a_f^{\mathfrak{m}}(\lambda)\}$ can be regarded as a family of generalized Fourier coefficients with respect to the generalized mean value $\mathfrak{m}$. The $\mathfrak{m}$-spectrum of a function f from $\mathscr{M}^1$ consists of all exponents λ for which $a_f^{\mathfrak{m}}(\lambda) \neq 0$. The aim of this paper is to study some properties of generalized Fourier coefficients.

In the sequel we shall use the well-known method of the Bohr compact. Let us consider the Cartesian product $\mathscr{P}_{\lambda} I_\lambda$ $(-\infty < \lambda < \infty)$ with Tychonoff topology, where I_λ denotes the multiplicative group of complex numbers z, with $|z| = 1$. Let R_0 be the closure of the subset $\{\langle e^{i\lambda u}\rangle : -\infty < u < \infty\}$ of $\mathscr{P}_{\lambda} I_\lambda$. Of course, R_0 is a compact subgroup of $\mathscr{P}_{\lambda} I_\lambda$. The theory of Besicovitch almost periodic functions from $\mathscr{B}^p$ and the theory of L^p-spaces over the Bohr compact R_0 with respect to the Haar measure on R_0 are equivalent. The natural mapping of the function $g_0(t) = e^{i\lambda_0 t}$ onto the coordinate $\hat{g}_0(x) = x_{\lambda_0}$ $(x = \langle x_\lambda\rangle)$ can be extended to an isomorphism of the whole space $\mathscr{B}^p$ onto the L^p-space over R_0. Moreover, this mapping can be extended to an isomorphism between the space of all uniformly almost periodic functions (cf. [4], Chapter I) and the space $C(R_0)$ of all continuous functions on R_0.

THEOREM 1. *There exists a generalized mean value $\mathfrak{m}_1$ such that every function from $\mathscr{M}^1$ has at most denumerable $\mathfrak{m}_1$-spectrum. Moreover, for any $f \in \mathscr{M}^1$, $\{a_f^{\mathfrak{m}_1}(\lambda)\}$ is the set of Fourier coefficients of a Besicovitch almost periodic function from $\mathscr{B}^1$.*

Proof. Let $\mathfrak{m}$ be an arbitrary generalized mean value on $\mathscr{M}^1$ and let f be a function from $\mathscr{M}^1$. Putting $l(\hat{g}) = \mathfrak{m}(fg)$ for any uniformly almost periodic function g, we define the linear functional l on $C(R_0)$. From the inequality

$$|l(\hat{g})| \leqslant \|fg\|_1 \leqslant \sup_{-\infty<t<\infty} |g(t)|\,\|f\|_1 = \|f\|_1 \max_{x \in R_0} |\hat{g}(x)|$$

it follows that l is a continuous functional on $C(R_0)$. Consequently, there exists a finite complex measure μ_f on R_0 such that

$$l(\hat{g}) = \int_{R_0} \hat{g}(x)\,\mu_f(dx).$$

The correspondence between f and μ_f is linear, the absolute variation of μ_f is not greater than $\|f\|_1$; and for any uniformly almost periodic function g_0, and for any Borel subset E of R_0,

$$\mu_{fg_0}(E) = \int_E \hat{g}_0(x)\,\mu_f(dx).$$

Let $\tilde{h}_f$ denote the density function with respect to the Haar measure on R_0 of the absolutely continuous component of the measure μ_f. Since $\tilde{h}_f$ is integrable with respect to the Haar measure on R_0, there exists a Besicovitch almost periodic function $h_f \in \mathscr{B}^1$ such that $\hat{h}_f = \tilde{h}_f$. We define the generalized mean value $\mathfrak{m}_1$ by the formula $\mathfrak{m}_1(f) = m(h_f)$ $(f \in \mathscr{M}^1)$. Since $m(h_{fg}) = m(h_f g)$ for every uniformly almost periodic function g, we have the equality

$$a_f^{\mathfrak{m}_1}(\lambda) = \mathfrak{m}_1(fe^{-i\lambda t}) = m(h_f e^{-i\lambda t}) = a_{h_f}(\lambda) \qquad (f \in \mathscr{M}^1).$$

Thus $\{a_f^{\mathfrak{m}_1}(\lambda)\}$ is the set of Fourier coefficients of the Besicovitch almost periodic function h_f. The first assertion of our Theorem is a direct consequence of the second one.

THEOREM 2. *There exists a generalized mean value $\mathfrak{m}_2$ on $\mathscr{M}^1$ such that the $\mathfrak{m}_2$-spectrum of a function belonging to $\mathscr{M}^1$ is non-denumerable.*

Proof. For any function f belonging to $\mathscr{M}^1$ the sequence of complex numbers

$$\left\{\frac{1}{2n!}\int_{-n!}^{n!} f(t)\,dt\right\}$$

is bounded. Moreover, we have the inequality

$$\overline{\lim_{n\to\infty}}\, \frac{1}{2n!}\int_{-n!}^{n!} |f(t)|\,dt \leqslant \|f\|_1.$$

We define the generalized mean value $\mathfrak{m}_2$ by the formula

$$\mathfrak{m}_2(f) = \operatorname*{Lim}_{n\to\infty} \frac{1}{2n!}\int_{-n!}^{n!} f(t)\,dt \qquad (f \in \mathscr{M}^1),$$

where $\operatorname*{Lim}_{n\to\infty} z_n$ denotes the generalized Banach-Mazur limit of bounded sequences of complex numbers ([3], p. 34; [11]). We may assume that the generalized limit $\operatorname*{Lim}_{n\to\infty} z_n$ is equal to the usual limit of a subsequence of the sequence $\{z_n\}$. Of course, $\mathfrak{m}_2$ is an extension of the mean value m.

We define the function f_0 by assuming $f_0(t) = 2n!(n-1)!(n-1)$ if $n!-(n!)^{-1} \leqslant t \leqslant n!$ $(n = 2, 3, \ldots)$ and $f_0(t) = 0$ in other cases. We have

$$\|f_0\|_1 = \lim_{n\to\infty} \frac{1}{2n!}\int_{-n!}^{n!} f_0(t)\,dt = \lim_{n\to\infty} \frac{1}{n!}\sum_{k=2}^{n} (k-1)!(k-1) = 1.$$

Thus $f_0 \in \mathscr{M}^1$. Further, for any real number λ we have the equality

$$(1) \qquad \frac{1}{2n!}\int_{-n!}^{n!} f_0(t)e^{-i\lambda t}\,dt = \frac{1}{n!}\sum_{k=2}^{\infty} (k-1)!(k-1)e^{-ik!\lambda} + o(1).$$

Let Λ be the set of all numbers λ of the form

$$\lambda = 2\pi \sum_{r=0}^{\infty} \frac{a_r}{r!},$$

where $a_r = 0$ or 1 $(r = 0, 1, \ldots)$. Obviously, the set Λ is non-denumerable. For any positive integer k, $k!\sum_{r=0}^{k} a_r/r!$ is also an integer and

$$0 \leqslant k! \sum_{r=k+1}^{\infty} \frac{a_r}{r!} \leqslant k! \sum_{r=k+1}^{\infty} \frac{1}{r!} \leqslant \frac{e}{k+1}.$$

Consequently, for any $\lambda \epsilon \Lambda$ we have the inequality

$$|e^{-ik!\lambda} - 1| \leqslant \frac{c}{k+1} \qquad (k = 1, 2, \ldots),$$

where c is a constant. We conclude further, on account of (1), that

$$\frac{1}{2n!} \int_{-n!}^{n!} f_0(t)\cdot e^{-i\lambda t} dt = 1 + o(1) \qquad (\lambda \epsilon \Lambda).$$

Thus $a_{f_0}^{m_2}(\lambda) = 1$ for every $\lambda \epsilon \Lambda$, which completes the proof.

A real function Φ is said to be an *N-function* if it is of the form

$$\Phi(u) = \int_0^{|u|} \varphi(t)\,dt,$$

where $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$, $\varphi(t)$ is non-decreasing, right continuous and $\lim_{t\to\infty} \varphi(t) = \infty$. Setting $\psi(t) = \sup_{\varphi(u)\leqslant t} u$, we define

$$\Psi(u) = \int_0^{|u|} \psi(t)\,dt.$$

Then Ψ is also an N-function and Φ, Ψ are called *complementary* functions ([5]). They satisfy Young's inequality

$$uv \leqslant \Phi(u) + \Psi(v),$$

for arbitrary u and v. We say that the function Φ satisfies the Δ_2-*condition* if

$$\overline{\lim_{u\to\infty}} \frac{\Phi(2u)}{\Phi(u)} < \infty.$$

Putting $\Phi(u) = |u|^p/p$, $\Psi(u) = |u|^q/q$, where $p > 1$, $q > 1$ and $1/p + 1/q = 1$, we obtain an important example of complementary N-functions satisfying the Δ_2-condition.

We assume that (X, μ) is a measure space and $\mu(X) = 1$. If Φ is an N-function and f is an arbitrary complex-valued μ-measurable function on X, the number $\varrho_\Phi(f)$ is now defined by

$$\varrho_\Phi(f) = \int_X \Phi(|f(x)|)\,\mu(dx).$$

By $L_\Phi(X, \mu)$ we shall denote the class of all functions f for which $\varrho_\Phi(f)$ is finite. The class $L_\Phi(X, \mu)$ is called the *Orlicz class*. The Orlicz class is a convex set, but not in general a linear set. A necessary and sufficient condition for the Orlicz class to be linear is that the function Φ fulfills the Δ_2-condition. Furthermore, if Φ and Ψ are complementary functions, then, by Young's inequality,

$$\int_X |f(x)g(x)|\,\mu(dx) \leqslant \varrho_\Phi(f) + \varrho_\Psi(g) \tag{2}$$

for every pair $f \epsilon L_\Phi(X, \mu)$ and $g \epsilon L_\Psi(X, \mu)$. Now we denote by $L_\Phi^*(X, \mu)$ the class of all μ-measurable functions f satisfying the condition

$$\int_X |f(x)g(x)|\,\mu(dx) < \infty \quad \text{for every} \quad g \epsilon L_\Psi(X, \mu).$$

In the class $L_\Phi^*(X, \mu)$ we define the norm $\| \ \|_\Phi$ by the formula

$$\|f\|_\Phi = \sup \int_X |f(x)g(x)|\,\mu(dx), \tag{3}$$

where the supremum is taken for all $g \epsilon L_\Psi(X, \mu)$ satisfying the inequality $\varrho_\Psi(g) \leqslant 1$. The class $L_\Phi^*(X, \mu)$ under the norm $\| \ \|_\Phi$ is a Banach space [12] and will be called the *Orlicz space*. The inclusion $L_\Phi(X, \mu) \subset L_\Phi^*(X, \mu)$ si evident and the equality $L_\Phi(X, \mu) = L_\Phi^*(X, \mu)$ is equivalent to the Δ_2-condition. It is still possible to define in the Orlicz space $L_\Phi^*(X, \mu)$ a second norm $\| \ \|_{(\Phi)}$. We define it by the formula

$$\|f\|_{(\Phi)} = \inf\left\{a\colon \varrho_\Phi\left(\frac{f}{a}\right) \leqslant 1,\ a > 0\right\}. \tag{4}$$

The norms $\| \ \|_\Phi$ and $\| \ \|_{(\Phi)}$ are equivalent: $\| \ \|_{(\Phi)} \leqslant \| \ \|_\Phi \leqslant 2\| \ \|_{(\Phi)}$ (see [9], p. 97). It may be proved ([9], p. 98) that for any pair $f \epsilon L_\Phi^*(X, \mu)$ and $g \epsilon L_\Phi^*(X, \mu)$, where Φ and Ψ are complementary functions, the generalized Hölder inequality holds

$$\int_X |f(x)g(x)|\,\mu(dx) \leqslant \|f\|_\Phi \|g\|_{(\Psi)}. \tag{5}$$

By $\mathscr{M}_\Phi$ we shall denote the *Marcinkiewicz-Orlicz class* induced by the N-function Φ, i. e. the class of all functions f from $\mathscr{M}^1$ for which $\|\Phi(|f|)\|_1 < \infty$. This class was introduced by J. Albrycht [1]. Further, the class $\mathscr{B}_\Phi$ is defined to be the closure in $\mathscr{M}_\Phi$ of the set of all trigonometric polynomials (see [2]). Its elements are called *$\mathscr{B}_\Phi$-almost periodic functions*.

THEOREM 3. *Let Φ be an N-function satisfying the Δ_2-condition and let $\mathfrak{m}$ be a generalized mean value on $\mathscr{M}^1$. Then the $\mathfrak{m}$-spectrum of every function from the Marcinkiewicz-Orlicz class $\mathscr{M}_\Phi$ is at most denumerable. Moreover, $\{a_f^{\mathfrak{m}}(\lambda)\}$ $(f\in\mathscr{M}_\Phi)$ is the set of Fourier coefficients of a Besicovitch almost periodic function belonging to $\mathscr{B}_\Phi$.*

Proof. Let us consider the Orlicz class $L_\Psi(R_0,\mu_0)$, where μ_0 is the Haar measure on the Bohr compact R_0 normalized by assuming $\mu_0(R_0)=1$ and Ψ is the complementary function of Φ. It may be proved that the natural mapping $g\to\hat g$ transforms $\mathscr{B}_\Psi$ onto $L_\Psi(R_0,\mu_0)$. Furthermore, for any positive a we have the equality

$$m\left(\psi\left(\frac{|g|}{a}\right)\right)=\int_{R_0}\Psi\left(\frac{|\hat g(x)|}{a}\right)\mu_0(dx)\qquad (g\in\mathscr{B}_\Psi).$$

Denoting by $\|\ \|_{(\Psi)}$ the norm on $L_\Psi(R_0,\mu_0)$ defined by formula (4), we get the inequality

$$m\left(\Psi\left(\frac{|g|}{\|\hat g\|_{(\Psi)}}\right)\right)\leqslant 1\qquad (0\neq g\in\mathscr{B}_\Psi).$$

Consequently, for any $g\in\mathscr{B}_\Psi$, $g\neq 0$, and every number ε satisfying the inequality $0<\varepsilon\leqslant 1$ there exists a number T_0 such that for $T\geqslant T_0$ the inequality

$$\frac{1}{2T}\int_{-T}^{T}\Psi\left(\frac{|g(t)|}{\|\hat g\|_{(\Psi)}}\right)dt\leqslant 1+\varepsilon \tag{6}$$

holds. Let

$$\Psi_\varepsilon(u)=\frac{1}{1+\varepsilon}\Psi(u).$$

Then the complementary function Φ_ε of Ψ_ε is given by the formula

$$\Phi_\varepsilon(u)=\frac{1}{1+\varepsilon}\Phi\big((1+\varepsilon)u\big).$$

Consider the Orlicz classes $L_{\Phi_\varepsilon}(I_T,\mu_T)$, $L_{\Psi_\varepsilon}(I_T,\mu_T)$, where I_T is the interval $-T\leqslant t\leqslant T$ and μ_T is the Lebesgue measure on I_T, with $\mu_T(I_T)=1$. We denote by $\|\ \|_{\Phi_\varepsilon}^T$ and $\|\ \|_{(\Phi_\varepsilon)}^T$ the norms on $L_{\Phi_\varepsilon}(I_T,\mu_T)$ and $L_{\Psi_\varepsilon}(I_T,\mu_T)$ defined by (3) and (4) respectively. Let $g\in\mathscr{B}_\Psi$ and $f\in\mathscr{M}_\Phi$. Then the restrictions g_T and f_T of g and f to I_T belong to $L_{\Psi_\varepsilon}(I_T,\mu_T)$ and $L_{\Phi_\varepsilon}(I_T,\mu_T)$ respectively. In fact, the first relation is evident and the second one follows from the Δ_2-condition

$$\int_{I_T}\Phi_\varepsilon\big(|f_T(t)|\big)\mu_T(dt)\leqslant\frac{1}{2T}\int_{-T}^{T}\Phi\big((1+\varepsilon)|f(t)|\big)dt\leqslant\frac{1}{2T}\int_{-T}^{T}\Phi\big(2|f(t)|\big)dt<\infty.$$

Furthermore, inequality (6) implies the following one:

$$\|g_T\|_{(\Psi_\varepsilon)}^T\leqslant\|\hat g\|_{(\Psi)}. \tag{7}$$

Taking into account Young's inequality and definition (3) we have

$$\|f_T\|_{\Phi_\varepsilon}^T\leqslant 1+\int_{I_T}\Phi_\varepsilon\big(|f_T(t)|\big)\mu_T(dt)\leqslant 1+\frac{1}{2T}\int_{-T}^{T}\Phi\big(2|f(t)|\big)dt.$$

Hence and from (7), using the generalized Hölder inequality (5), we get the inequality

$$\frac{1}{2T}\int_{-T}^{T}|f(t)g(t)|\,dt=\int_{I_T}|f_T(t)g_T(t)|\,\mu_T(dt)\leqslant\|f_T\|_{\Phi_\varepsilon}^T\|g_T\|_{(\Psi_\varepsilon)}^T$$

$$\leqslant\left(1+\frac{1}{2T}\int_{-T}^{T}\Phi\big(2|f(t)|\big)dt\right)\|\hat g\|_{(\Psi)}.$$

Thus

$$\|fg\|_1\leqslant C\|\hat g\|_\Psi\qquad (f\in\mathscr{M}_\Phi,\ g\in\mathscr{B}_\Psi), \tag{8}$$

where C is a constant depending on f.

Consider now a generalized mean value $\mathfrak{m}$ on $\mathscr{M}^1$. Given a function $f\in\mathscr{M}_\Phi$, we denote by $l(\hat g)$ the mean $\mathfrak{m}(fg)$ $(g\in\mathscr{B}_\Psi)$. From (8) it follows that l is a continuous functional on $L_\Phi(R_0,\mu_0)$. Let E_Ψ denote the closure of the set of all bounded functions in $L_\Psi(R_0,\mu_0)$. The class E_Ψ is a Banach space under the norm $\|\ \|_\Psi$. The conjugate space of E_Ψ is the Orlicz space $L_\Phi^*(R_0,\mu_0)$ [8], which, according to the Δ_2-condition coincides with $L_\Phi(R_0,\mu_0)$. Since l is a continuous linear functional on E_Ψ, there exists a function $h_f\in\mathscr{B}_\Phi$ such that

$$\mathfrak{m}(fg)=l(\hat g)=\int_{R_0}\hat g(x)\hat h_f(x)\mu_0(dx)\qquad (\hat g\in E_\Psi).$$

Hence, it follows that

$$a_f^{\mathfrak{m}}(\lambda)=\mathfrak{m}(fe^{-i\lambda t})=m(h_fe^{-i\lambda t})=a_{h_f}(\lambda).$$

which completes the proof of the Theorem.

Setting $\Phi(u)=|u|^p$ $(p>1)$, we get the following

COROLLARY 1. *Let* $\mathfrak{m}$ *be a generalized mean value on* $\mathscr{M}^1$. *Then, for every function* f *belonging to* $\mathscr{M}^p$ $(p > 1)$ $\{a_f^{\mathfrak{m}}(\lambda)\}$ *is the set of Fourier coefficients of a Besicovitch almost periodic function belonging to* $\mathscr{B}^p$.

COROLLARY 2. *Let* Φ *be an* N*-function satisfying the* Δ_2*-condition. If* f *is a function from* $\mathscr{M}_\Phi$ *and for every* λ *the limit*

$$a_f(\lambda) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\lambda t} dt$$

exists, then $\{a_f(\lambda)\}$ *is the set of Fourier coefficients of a Besicovitch almost periodic function belonging to* $\mathscr{B}_\Phi$.

In order to prove this assertion it is sufficient to take an extension $\mathfrak{m}_0$ of the linear functional m from the subspace of those functions f for which $m(fe^{-i\lambda t})$ exists for all λ, and apply Theorem 3. This assertion is connected with the following problem raised by S. Hartman and affirmatively solved by J. P. Kahane [7]: Let $f \in \mathscr{M}^1$ and for any λ the limit

$$a_f(\lambda) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\lambda t} dt$$

exists.

Is the set of those λ's for which $a_f(\lambda) \neq 0$ at most denumerable?

A function $f \in \mathscr{M}^1$ is said to be $\mathfrak{m}$-*periodic of period* ω $(\omega > 0)$ if the $\mathfrak{m}$-spectrum of f consists only of multiples of $2\pi/\omega$. A complex-valued Borel measure ν on the real line, finite on compact sets, is said to be *periodic of period* ω if, for every Borel set E, $\nu(E) = \nu(E+\omega)$, where $E+\omega = \{t+\omega : t \in E\}$. The family of coefficients

$$b_\nu(\lambda) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} \nu(dt) \qquad (-\infty < \lambda < \infty)$$

will be called the family of *Fourier-Stieltjes coefficients* of ν.

For any periodic measure ν *of period* ω *there exists a function* $f_0 \in \mathscr{M}^1$ *such that*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_0(t) e^{-i\lambda t} dt = b_\nu(\lambda)$$

for every real λ.

In fact, we can choose a sequence $\{h_n\}$ of integrable functions defined on the interval $0 \leqslant t \leqslant \omega$ such that for every continuous function g

$$\lim_{n\to\infty} \int_0^\omega g(t) h_n(t) dt = \int_0^\omega g(t) \nu(dt).$$

We define the function $f_0 \in \mathscr{M}^1$ by means of the formula

$$f_0(t) = h_n(t - n\omega) \quad \text{if} \quad n\omega \leqslant |t| < (n+1)\omega \quad (n = 0, 1, \ldots).$$

It is very easy to verify that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_0(t) e^{-i\lambda t} dt = b_\nu(\lambda).$$

If, for instance, $\nu(E)$ is the number of integers contained in E, then $a_{f_0}(\lambda) = 1$ or 0 according as λ is or not a multiple of 2π.

We shall now prove the following

THEOREM 4. *Let* $\mathfrak{m}$ *be an arbitrary mean value on* $\mathscr{M}^1$. *For any* $\mathfrak{m}$*-periodic function* $f \in \mathscr{M}^1$ *of period* ω, $\{a_f^{\mathfrak{m}}(\lambda)\}$ *is the set of Fourier-Stieltjes coefficients of a complex periodic measure of the same period.*

Proof. Let f be an $\mathfrak{m}$-periodic function of period ω. Then $\mathfrak{m}(fg)$ is a linear functional on the space of all continuous periodic functions g of period ω. Moreover, from the inequality

$$|\mathfrak{m}(fg)| \leqslant \|fg\|_1 \leqslant \max_{0\leqslant t\leqslant\omega} |g(t)| \, \|f\|_1$$

we get the continuity of the functional $\mathfrak{m}(fg)$. There exists then a complex measure ν_0 on the interval $0 \leqslant t \leqslant \omega$ such that

$$\mathfrak{m}(fg) = \frac{1}{\omega} \int_0^\omega g(t) \nu_0(dt).$$

Let ν be the periodic extension of ν_0 of period ω to the whole line. Then

$$b_\nu(\lambda) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} \nu(dt) = \lim_{T\to\infty} \frac{1}{2n\omega} \sum_{k=-n}^{n} \int_{k\omega}^{(k+1)\omega} e^{-i\lambda t} \nu(dt)$$

$$= \frac{1}{\omega} \int_0^\omega e^{-i\lambda t} \nu_0(dt) \Big(\lim_{n\to\infty} \frac{1}{2n} \sum_{k=-n}^{n} e^{-ik\lambda\omega} \Big).$$

Hence we get the equality

$$b_\nu\Big(\frac{2\pi n}{\omega}\Big) = \frac{1}{\omega} \int_0^\omega \exp(-2\pi i n t/\omega) \nu_0(dt) = \mathfrak{m}\big(f \exp(-2\pi i n t/\omega)\big)$$

and $b_\nu(\lambda) = 0$ if $\lambda \neq 2\pi n/\omega$ $(n = 0, \pm 1, \ldots)$. Consequently, $a_f^{\mathfrak{m}}(\lambda) = b_\nu(\lambda)$ for all λ. The Theorem is thus proved.

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MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

Reçu par la Rédaction le 10. 2. 1961

STUDIA MATHEMATICA, T. XXI. (1961)

Sur les coefficients de Fourier-Bohr

par

J.-P. KAHANE (Montpellier)

Cette note répond à la question suivante, posée par S. Hartman. Soit f une function localement sommable sur la droite, telle que

$$\int_{-T}^{T} |f(t)|\,dt = O(T) \qquad (T\to\infty). \tag{1}$$

On suppose que

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)e^{-i\lambda t}\,dt = a(\lambda) \tag{2}$$

existe pour tout λ réel. Peut-on affirmer que $a(\lambda)=0$ sauf sur un ensemble au plus dénombrable?

Si l'on remplace (1) par

$$\int_{-T}^{T} |f(t)|^p\,dt = O(T) \qquad (T\to\infty) \tag{3}$$

avec $p>1$, la réponse positive résulte d'une récente étude d'Urbanik (1). Nous allons montrer que la réponse est positive sans même astreindre f à la condition (1).

THÉORÈME 1. *Soit f une fonction localement sommable sur $[0,\infty)$. On suppose que*

$$\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T} f(t)e^{-i\lambda t}\,dt = c(\lambda) \tag{4}$$

existe pour un ensemble fermé F de valeurs de λ. Alors, sur F, $c(\lambda)=0$ sauf sur un ensemble au plus dénombrable, et l'ensemble E_ε des $\lambda\in F$ tel que $|c(\lambda)|>\varepsilon$ ($\varepsilon>0$ donné) est clairsemé (2).

(1) K. Urbanik, *Fourier analysis in Marcinkiewicz spaces*, Studia Math. 21 (1961), p. 93-102.

(2) Rappelons qu'un ensemble est dit *clairsemé* s'il ne contient aucun ensemble $\neq\emptyset$ dense en lui-même, c'est-à-dire sans point isolé.