$$\lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-2ip\pi/n}\,d\mu(t) = c_\mu\left(\frac{2\pi p}{n}\right) = 1.$$

Cela est facile: en désignant par δ_m la masse unité au point m, on prend

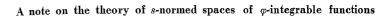$$d\mu = \sum_{j=1}^{\infty} (n_{j+1}-n_j)\,\delta_{n_j}$$

la suite $\{n_j\}$ étant choisie de telle façon que 1° $n_{j+1}-n_j = \sigma(n_j)$, 2° n_j est multiple de $n!$ quand j est assez grand ($j > j_n$). Ensuite on ,,régularise'' $d\mu$ en remplaçant chaque δ_{n_j} par une fonction ≥ 0, d'intégrale égale à 1, et de support $[n_j-\varepsilon_j,\, n_j+\varepsilon_j]$ avec $\lim_{j\to\infty}\varepsilon_j = 0$.

Comme me l'a signalé S. Hartman, le théorème 1 admet le corollaire suivant:

THÉORÈME 2. *Soit f une fonction localement sommable sur $[0,\infty)$. Alors $c(\lambda)$, définie par (4), ne peut exister et être différent de zéro sur un ensemble non dénombrable.*

Démonstration. Le premier membre de (4) ne change pas si l'on astreint T à ne prendre que des valeurs entières. Ainsi l'ensemble E des valeurs des λ pour lesquelles $c(\lambda)$ existe est l'ensemble de convergence d'une suite de fonctions continues; c'est donc un ensemble borélien. Il en est encore de même de l'ensemble $E_0 \subset E$ où $c(\lambda) \neq 0$. Si E_0 n'était pas dénombrable, il existerait un parfait $F \subset E_0$, et l'application du théorème 1 à F aboutirait à une contradiction.

A note on the theory of s-normed spaces of φ-integrable functions

by

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1.1. In this paper, a function nondecreasing and continuous for $u \geq 0$, vanishing only at $u = 0$ and tending to ∞ as $u \to \infty$ will be called a *φ-function*. Two φ-functions φ, ψ are called *equivalent for large u*, in symbols $\varphi \overset{l}{\sim} \psi$, if

$(+)$ $$a\varphi(k_1 u) \leq \psi(u) \leq b\varphi(k_2 u)$$

for $u \geq u_0 \geq 0$ and for some constants $a, b, k_1, k_2 > 0$.

If $(+)$ holds for $u \geq u_0 = 0$, then φ, ψ are called *equivalent for all u*, in symbols $\varphi \overset{a}{\sim} \psi$; if $(+)$ holds for $0 \leq u \leq u_0$, then φ, ψ are called *equivalent for small u*, in symbols $\varphi \overset{s}{\sim} \psi$ (cf. [2], [3]).

1.2. The following conditions are of importance in our considerations:

(o_s) $\lim_{u\to 0+}\varphi(u)u^{-s} = 0$, (∞_s) $\lim_{u\to\infty}\varphi(u)u^{-s} = \infty$,

(o_s^b) $\overline{\lim}_{u\to 0+}\varphi(u)u^{-s} < \infty$, (∞_s^b) $\overline{\lim}_{u\to\infty}\varphi(u)u^{-s} < \infty$,

where $0 < s \leq 1$.

Easy calculation shows, (o_s) is an invariant of the relation $\overset{s}{\sim}$, and (∞_s) of the relation $\overset{l}{\sim}$ and both these properties are invariants of the relation $\overset{a}{\sim}$. The same holds for (o_s^b) and (∞_s^b).

1.3. Let φ be a φ-function satisfying (o_1) and (∞_1). The function

$$\varphi^*(v) = \sup_{u\geq 0}\big(uv - \varphi(u)\big) \quad \text{for} \quad v \geq 0$$

is called the *function complementary to φ*. It is proved easily that φ^* is a convex φ-function satisfying (o_1), (∞_1) and $(\varphi^*)^* = \varphi$ if and only if φ is convex. For arbitrary φ-functions satisfying (o_1), (∞_1) there holds $\big(\varphi^*(u)\big)^* \leq \varphi(u)$ for $u \geq 0$. We shall call $\bar\varphi(u) = \big(\varphi^*(u)\big)^*$ the *function associated with φ*.

1.41. *Let* φ, ψ *be* φ-functions satisfying (o_1), (∞_1), where ψ is convex *and satisfies the inequality*

$$(+) \qquad\qquad \psi(u) \leqslant \varphi(u)$$

for $u \geqslant u_0$. *Then*

$$(++) \qquad\qquad \psi(u) \leqslant \bar{\varphi}(u)$$

for $u \geqslant u_*$, *where* u_* *is sufficiently large (or* $u_* = 0$, *if* $u_0 = 0$). *If* $(+)$ *is satisfied for* $0 < u \leqslant u_0$, *then* $(++)$ *holds for* $0 < u \leqslant u_*$ *for sufficiently small* u_*.

In order to prove this theorem, note that to every $v > 0$ there exists the least number $u_v > 0$ for which

$$u_v v - \varphi(u_v) = \varphi^*(v).$$

It is easily seen that $u_v \to 0$ as $v \to 0$ and $u_v \to \infty$ as $v \to \infty$. Choosing v_0 so that $u_v \geqslant u_0$ for $v \geqslant v_0$, we obtain

$$\psi^*(v) \geqslant u_v v - \psi(u_v) \geqslant u_v v - \varphi(u_v) = \varphi^*(v) \qquad \text{for} \quad v \geqslant v_0.$$

The same arguments applied to the pair of functions φ^*, ψ^* lead to the relations

$$\psi(u) = \big(\psi^*(u)\big)^* \leqslant \big(\varphi^*(u)\big)^* = \bar{\varphi}(u)$$

for sufficiently large u. In the remaining cases, the proof is performed similarly.

The previous statement implies

1.42. *Let* φ *satisfy the conditions* (o_1), (∞_1). *There holds* $\varphi \overset{l}{\sim} \psi$ *with a convex* φ-function ψ satisfying (o_1), (∞_1) *if and only if* $\varphi \overset{l}{\sim} \bar{\varphi}$.

Analogous theorems are valid if we replace $\overset{l}{\sim}$ by $\overset{a}{\sim}$ or $\overset{s}{\sim}$.

1.43. *Assume* φ *satisfies the conditions* (o_s), (∞_s); *then the function* $\varphi_s(u) = \varphi(u^{1/s})$ *satisfies* (o_1), (∞_1) *and the associated function* $\bar{\varphi}_s(u)$ *is defined. Let* $\chi(u) = \psi(u^s)$, *where* ψ *is a* φ-function satisfying (o_1), (∞_1). *If* $\varphi \overset{l}{\sim} \chi$, *then* $\varphi_s \overset{l}{\sim} \psi$, *and conversely.*

Hence 1.42 implies the following statement:

Assume φ *satisfies the conditions* (o_s), (∞_s); *then there holds* $\varphi \overset{l}{\sim} \chi$, *where* $\chi(u) = \psi(u^s)$ *and* ψ *is a convex function satisfying* (o_1), (∞_1) *if and only if* $\varphi \overset{l}{\sim} \bar{\chi}_s$, *where* $\bar{\chi}_s = \bar{\varphi}_s(u^s)$.

This theorem remains true if we replace $\overset{l}{\sim}$ by $\overset{a}{\sim}$ or by $\overset{s}{\sim}$.

1.5. *If* φ *satisfies* (o_1) *(or* (o_1^b)*) and* $\varphi \overset{l}{\sim} \psi$, *where* ψ *is a convex function, then there exists a convex function* $\bar{\bar{\varphi}}$ *satisfying* (o_1) *(or* (o_1^b)*) such that* $\bar{\bar{\varphi}} \overset{l}{\sim} \psi$ *and* $\varphi(u) \leqslant \bar{\bar{\varphi}}(u)$ *for* $u \geqslant 0$.

The analogous theorem remains valid if we replace $\overset{l}{\sim}$ by $\overset{a}{\sim}$ or $\overset{s}{\sim}$. Define

$$r(u) = \sup_{0 < t \leqslant u} \varphi(t) t^{-1} \qquad \text{for} \quad u > 0, \qquad r(0) = 0.$$

Denote by u_0 a number such that $\varphi(u) \leqslant b\psi(k_2 u)$, b, $k_2 > 0$, for $u \geqslant u_0$ (resp. for $0 \leqslant u \leqslant u_0$ in the case $\varphi \overset{s}{\sim} \psi$). Choose $A > 1$ so that $r(u_0) \leqslant Ab\psi(k_2 u_0) u_0^{-1}$. The function

$$\bar{\bar{\varphi}}(u) = 2 \int_0^u r(2t)\, dt$$

has the required properties. Indeed, $\bar{\bar{\varphi}}$ is convex and satisfies (o_1), for $r(t) \to 0$ as $t \to 0+$ (if φ satisfies (o_1^b), then $\bar{\bar{\varphi}}$ satisfies (o_1^b), too). Moreover, the inequality

$$\bar{\bar{\varphi}}(u) \geqslant \frac{u}{2} 2r(u) \geqslant \varphi(u)$$

holds for $u \geqslant 0$, for $ur(u) \geqslant \varphi(u)$, and the inequality $\bar{\bar{\varphi}}(u) \leqslant 2ur(2u)$ holds also for $u \geqslant 0$. Finally, $r(u) \leqslant Ab\psi(k_2 u) u^{-1}$ for $u \geqslant u_0$ (resp. for $0 < u \leqslant u_0$). Hence $\bar{\bar{\varphi}}(u) \leqslant Ab\psi(2k_2 u)$ for $u \geqslant u_0$ (resp. for $0 \leqslant u \leqslant \frac{1}{2}u_0$).

1.6. *If* φ *satisfies the condition* (o_s) *(resp.* (o_s^b)*) and* $\varphi \overset{l}{\sim} \chi$, $\chi(u) = \psi(u^s)$, *where* ψ *is a convex function, then there exists a convex function* $\bar{\bar{\varphi}}_s$ *satisfying* (o_1) *(resp.* (o_1^b)*) such that* $\varphi \overset{l}{\sim} \chi_s$, $\chi_s(u) = \bar{\bar{\varphi}}_s(u^s)$, $\varphi(u) \leqslant \bar{\bar{\varphi}}_s(u^s)$ *for* $u \geqslant 0$.

An analogous theorem is true replacing $\overset{l}{\sim}$ by $\overset{a}{\sim}$ or $\overset{s}{\sim}$.

It is sufficient to apply 1.5 to $\varphi_s(u) = \varphi(u^{1/s})$.

2.1. Let E denote an abstract set, $\mathscr{E}$ — a σ-algebra of subsets of E and let a σ-additive and σ-finite measure μ be defined on $\mathscr{E}$. Given a function x with complex values, defined in E and μ-measurable, we shall write

$$\mathscr{I}_\varphi(x) = \int_E \varphi\big(|x(t)|\big)\, d\mu.$$

The set of μ-measurable functions such that $\mathscr{I}_\varphi(x) < \infty$ we denote by $L^\varphi(E, \mu)$. Moreover, we denote by $L^{*\varphi}(E, \mu)$ the class of functions x such that $\lambda x \in L^\varphi(E, \mu)$, λ being a positive constant (in general dependent on x). Functions $x \in L^{*\varphi}(E, \mu)$ are called φ-integrable. In the space of μ-measurable functions, $L^\varphi(E, \mu)$ is a convex set and $L^{*\varphi}(E, \mu)$ a linear set. In $L^{*\varphi}(E, \mu)$, an F-norm can be defined as follows

$$\|x\|_\varphi = \inf\left\{\varepsilon > 0 : \mathscr{I}_\varphi\left(\frac{x}{\varepsilon}\right) \leqslant \varepsilon\right\}.$$

We shall call this norm the *norm generated by* φ. If $\varphi(u) = \psi(u^s)$, where $0 < s \leqslant 1$ and ψ is a convex function, then an *s*-homogeneous norm can be defined in $L^{*\varphi}(E, \mu)$ by means of the formula

$$\|x\|_{s\varphi} = \inf \left\{ \varepsilon > 0 : \mathscr{I}_\varphi \left(\frac{x}{\varepsilon^{1/s}} \right) \leqslant 1 \right\}.$$

It is easily shown that $\| \ \|_\varphi$-is equivalent to the norm $\| \ \|_{s\varphi}$. The above definitions imply

$(+)$ $\qquad\qquad\qquad \mathscr{I}_\varphi(x) \leqslant \|x\| \quad$ if $\quad \|x\| \leqslant 1,$

where $\| \ \|$ denotes the norm $\| \ \|_\varphi$ or $\| \ \|_{s\varphi}$ in the case when φ is of the form $\varphi(u) = \psi(u^s)$. The above definitions were introduced in [5], [2], [4] by the assumption that E is an interval and μ — the Lebesgue measure.

2.2. The set E may be expressed as $E = E_0 \cup \bigcup_{\nu=1}^{\infty} e_\nu^a$, where $E_0 \in \mathscr{E}$ is the non-atomic part of E and e_ν^a are different atoms (we do not consider the case when the number of atoms is finite). It is clear that μ-measurable functions in E are exactly the functions measurable in E_0 and constant in each of the sets e_n^a, $x(t) = u_n$ for $t \in e_n^a$. Write $\mu(e_n^a) = p_n$, then

$$\mathscr{I}_\varphi(x) = \int\limits_{E_0} \varphi(|x(t)|) \, d\mu + \sum_{\nu=1}^{\infty} \mu(e_\nu^a) \varphi(|u_\nu|).$$

The case $\mu(E_0) = 0$ means that $L^{*\varphi}(E, \mu)$ is a space of sequences for which $\sum_\nu p_\nu \varphi(\lambda|u_\nu|) < \infty$ for a $\lambda > 0$. In particular, if $0 < c_1 < p_n < c_2$ for $n = 1, 2, \ldots$, we obtain the space $l^{*\varphi}$ of sequences for which $\sum_\nu \varphi(\lambda|u_\nu|) < \infty$ for a $\lambda > 0$.

2.21. The following lemma is important in further considerations:

If $\varphi \overset{l}{\sim} \psi$, $\mu(E) < \infty$, *then* $L^{*\varphi}(E, \mu) = L^{*\psi}(E, \mu)$ *and the norms generated by these* φ-*functions are equivalent.*

The theorem remains true in the case $\mu(E) = \infty$ if we replace $\overset{l}{\sim}$ by $\overset{a}{\sim}$. Moreover, the theorem is also true if $E = \bigcup_\nu e_\nu^a$, where e_n^a are atoms and we take $\overset{s}{\sim}$ in place of $\overset{l}{\sim}$ when $\inf_n \mu(e_n^a) > 0$ and $\overset{a}{\sim}$ in place of $\overset{l}{\sim}$ when $\inf_n \mu(e_n^a) = 0$.

The proof is similar to that in the case of the space of functions φ-integrable with respect to the Lebesgue measure and will be omitted here (cf. [4]).

2.3. *An s-homogeneous norm equivalent to the norm generated by* φ *exists in* $L^{*\varphi}(E, \mu)$ *if and only if*

(a) $\varphi \overset{l}{\sim} \chi$, $\chi(u) = \psi(u^s)$, ψ *is a convex* φ-*function, if* $E = E_0$, $\mu(E) < \infty$;

(b) $\varphi \overset{a}{\sim} \chi$, *where* χ *has the same meaning as in* (a), *if* $E = E_0$, $\mu(E) = \infty$;

(c) $\varphi \overset{s}{\sim} \chi$, *where* χ *has the same meaning as in* (a), *if* μ *is purely atomic,*

i. e. $E = \bigcup_{\nu=1}^{\infty} e_\nu^a$, e_ν^a *being different atoms, and* $0 < c_1 < \mu(e_n^a) < c_2$ *for* $n = 1, 2, \ldots$

Cases (a) and (b) of the theorem may be found in [2], [4], assuming that E is an interval and μ — a Lebesgue measure. Special cases of the theorem were already proved in [5] and [6]. In all three cases, the proof of sufficiency is based on the remarks 2.1 and on the fact (cf. 2.21) that the norms generated by two l-equivalent (resp. a-equivalent, s-equivalent) φ-functions are equivalent in $L^{*\varphi}(E, \mu)$ if E satisfies the assumptions as in (a) (resp. (b), (c)). The proof given below of necessity simultaneously for all cases (a)-(c) is based on a slightly different idea than proofs in the above mentioned papers.

Let $\delta > 0$ denote a constant $< \inf(1, \mu(E))$ such that $\|x\|_0 \leqslant 2\delta$ implies $\|x\|_\varphi \leqslant 1$ and $\|x\|_\varphi \leqslant 2\delta$ implies $\|x\|_0 \leqslant 1$, where $\| \ \|_0$ is an *s*-homogeneous norm equivalent to $\| \ \|_\varphi$. We shall prove the inequality

$(+)$ $\qquad\qquad\qquad\qquad \varphi(au) \leqslant \dfrac{2}{\delta^2} a^s \varphi(u\delta^{-1})$

to be satisfied, namely: (α) for $0 < a^s \leqslant \delta$, $a^s \varphi(u\delta^{-1}) \geqslant 1$ in the case (a), (β) for $0 < a^s \leqslant \delta$, $u \geqslant 0$ in the case (b), (γ) for $0 < a^s \leqslant \delta$, $\delta^{-1}\varphi(u\delta^{-1}) \leqslant (2c_2)^{-1}$ in the case (c).

Indirect proof. Suppose

$$\varphi(a^* u^*) > \frac{2}{\delta^2} (a^*)^s \varphi(u^* \delta^{-1})$$

holds for a pair a^*, u^* satisfying conditions as in (α), (β), (γ), resp. Define a positive integer l as follows. If $(a^*)^s \geqslant \frac{1}{2}\delta$ we define $l = 1$, if $0 < (a^*)^s < \frac{1}{2}\delta$ we choose $l \geqslant 1$ so that $\frac{1}{2}\delta \leqslant l(a^*)^s < \delta$. In all cases, l disjoint μ-measurable sets e_n may be defined so that $\mu(e_n) = \delta\varphi(u^*\delta^{-1})^{-1}k_n$, where $1 \leqslant k_n \leqslant 2$. This is possible in the case (α), for the measure is non-atomic and $\mu(E) \geqslant \delta \geqslant l(a^*)^s \geqslant l\varphi(u^*\delta^{-1})^{-1}k_n$, $k_n = 1$. In the case (β) it follows from the assumption that the measure is non-atomic and $\mu(E) = \infty$; here $k_n = 1$. Finally, in the case (γ) we choose a set $e_n = e_n^a \cup e_{n+1}^a \cup \ldots \cup e_{n+r}^a$ choosing $r(n)$ so that $\mu(e_n) \geqslant \delta\varphi(u^*\delta^{-1})^{-1} > \mu(e_n) - \mu(e_{n+r}^a)$. Since $\mu(e_{n+r}^a) \leqslant c_2 \leqslant \delta\varphi(u^*\delta^{-1})^{-1} \cdot \frac{1}{2} \leqslant \frac{1}{2}\mu(e_n)$, we have $\mu(e_n) = \delta\varphi(u^*\delta^{-1})^{-1}k_n$, $1 \leqslant k_n \leqslant 2$. Since n may be chosen arbitrarily large, l disjoint sets e_n may be obtained. We have $\mathscr{I}_\varphi(u^*\chi_{e_n}\delta^{-1}) = \mu(e_n)\varphi(u^*\delta^{-1}) = \delta k_n \leqslant 2\delta$; thus $\|u^*\chi_{e_n}\|_\varphi \leqslant 2\delta$; whence $\|u^*\chi_{e_n}\|_0 \leqslant 1$. Hence the inequality

$$\left\| \sum_1^l a_\nu u^* \chi_{e_\nu} \right\|_0 \leqslant \sum_1^l a_\nu^s \|u^* \chi_{e_\nu}\|_0 \leqslant \sum_1^l a_\nu^s$$

is satisfied. Assuming $a_\nu = a^*$ we have $\sum_1^l a_\nu^s = l(a^*)^s < \delta$; i. e., there holds $\|\sum_1^l a^* u^* \chi_{e_\nu}\|_\varphi \leqslant 1$, whence

$$1 \geqslant \mathscr{I}_\varphi\left(\sum_1^l a^* u^* \chi_{e_\nu}\right) = \sum_1^l \varphi(a^* u^*) \mu(e_\nu)$$

$$\geqslant \delta l \varphi(a^* u^*) \varphi(u^* \delta^{-1})^{-1} > \delta l \frac{2}{\delta^2}(a^*)^s \geqslant \frac{2}{\delta} \cdot \frac{\delta}{2} = 1$$

and we have a contradiction.

In all three cases the inequality (+) holds for $\delta \leqslant a^s \leqslant 1$ and $u \geqslant 0$, for

$$\varphi(au) \leqslant \frac{\delta}{\delta} \varphi(u\delta^{-1}) \leqslant \frac{1}{\delta^2} a^s \varphi(u\delta^{-1}).$$

Substituting $u = u_2$, $a = u_1/u_2$, where $u_1 \leqslant u_2$, (+) **yields the inequality**

$$(++) \qquad \varphi(u_1) u_1^{-s} \leqslant \frac{2}{\delta^2} \varphi(u_2) u_2^{-s},$$

which is satisfied for $u_2 \geqslant u_1 > u_0 = 0$ in the case (β) and for $0 < u_1 \leqslant u_2 \leqslant u_0$, where u_0 is sufficiently small, in the case (γ). In the case (α) we verify first the inequality

$$\lim_{u \to \infty} \varphi(u\delta^{-1}) u^{s} > 0,$$

substituting $a^s = \varphi(u\delta^{-1})^{-1}$ in (+). Hence it follows that a constant u_0 exists having the following property: the inequality (+) is satisfied for all a, u such that $a^s u^s \geqslant u_0$. Substituting again $u = u_2$, $a = u_1/u_2$, where $u_2 \geqslant u_1 \geqslant u_0$, we verify that (++) is satisfied in the case (α) for $u_2 \geqslant u_1 \geqslant u_0$.

Define a function

$$r(t) = \sup_{u \in \Delta(t)} \frac{\varphi(u^{1/s})}{u},$$

where $t \geqslant u_0^s$, $\Delta(t) = \langle u_0^s, t\rangle$ in the case (α), $t > 0$, $\Delta(t) = (0, t\rangle$ in the case (β) and $0 < t < u_0^s$, $\Delta(t) = (0, t\rangle$ in the case (γ). In the case (α) we complete the definition of $r(t)$ in $\langle 0, u_0^s\rangle$ linearly and in the case (γ) we complete $r(t)$ linearly in (u_0^s, ∞), in order to get a monotone function. (++) implies $r(t)$ to be finite in the considered cases and the convex function

$$\psi(t) = \int_0^t r(\tau) d\tau$$

satisfies the required conditions. We shall consider e. g. the case (α). The following inequalities are satisfied:

$$\psi(u^s) \leqslant u^s r(u^s) \leqslant u^s \frac{2}{\delta^2} \varphi(u) u^{-s} = \frac{2}{\delta^2} \varphi(u) \quad \text{for} \quad u \geqslant u_0,$$

$$\psi(u^s) \geqslant \tfrac{1}{2} u^s r(\tfrac{1}{2} u^s) \geqslant \varphi(u \cdot 2^{-1/s}) \quad \text{for} \quad u \geqslant 2^{1/s} u_0.$$

2.4. Let $\mu(E) < \infty$ and let φ satisfy the condition (o_s^b). Assume $\varphi \overset{l}{\sim} \chi$, where $\chi(u) = \psi(u^s)$ and ψ is a convex function. By these assumptions, an s-homogeneous norm may be defined in $L^{*\varphi}(E, \mu)$ equivalent to the norm generated by φ and such that

$$\mathscr{I}_\varphi(x) \leqslant \|x\|_s^0 \quad \text{for} \quad \|x\|_s^0 \leqslant 1.$$

The theorem remains true for $\mu(E) = \infty$, replacing $\overset{l}{\sim}$ by $\overset{a}{\sim}$, and for $E = \bigcup_1^\infty e_\nu^a$, where e_n^a are atoms with $\inf_n \mu(e_n^a) > 0$, replacing $\overset{l}{\sim}$ by $\overset{s}{\sim}$.

Consider the function $\overline{\overline{\varphi}}_s$ defined in 1.6. Since

$$\mathscr{I}_\varphi(x) \leqslant \mathscr{I}_{\chi_s}(x) \quad \text{for} \quad x \in L^{*\varphi}(E, \mu),$$

it is sufficient to take $\|x\|_s^0 = \|x\|_{s\chi_s}$ and to apply 2.21.

2.5. *Let* $E = E_0$, $\mu(E) < \infty$; *the relation*

$$(+) \qquad \lim_{t \to 0+} \mathscr{I}_\varphi(tx) t^{-s} = 0 \quad \text{for} \quad x \in L^{*\varphi}(E, \mu)$$

holds if and only if φ *satisfies* (o_s) *and* $\varphi \overset{l}{\sim} \chi$, $\chi(u) = \psi(u^s)$, *where* ψ *is a convex function.*

This theorem remains true when $E = E_0$, $\mu(E) = \infty$ and $\overset{l}{\sim}$ is replaced by $\overset{a}{\sim}$, or when $E = \bigcup_1^\infty e_\nu^a$, e_n^a are atoms satisfying the inequalities $0 < c_1 < \mu(e_n^a) < c_2$ for $n = 1, 2, \ldots$ and $\overset{l}{\sim}$ is replaced by $\overset{s}{\sim}$.

Let us consider the case $E = E_0$, $\mu(E) < \infty$.

Sufficiency. Write $e_n' = \{t : |x(t)| \geqslant n, t \in E\}$, $e_n'' = E - e_n'$. By 1.6, there exists a convex function $\overline{\overline{\psi}}_s$ satisfying the inequality $\varphi(u) \leqslant \overline{\overline{\psi}}_s(u^s)$ for $u \geqslant 0$ and $\varphi \overset{l}{\sim} \overline{\overline{\psi}}_s$. Hence it follows

$$(*) \qquad \mathscr{I}_\varphi(tx\chi_{e_n'}) \leqslant \mathscr{I}_{\overline{\overline{\psi}}_s}(tx\chi_{e_n'}) \leqslant \frac{t^s}{\lambda_0^s} \mathscr{I}_{\overline{\overline{\psi}}_s}(\lambda_0 x\chi_{e_n'}), \quad 0 \leqslant t \leqslant \lambda_0.$$

Let $\mathscr{I}_\varphi(x) < \infty$, then $\mathscr{I}_{\overline{\overline{\psi}}_s}(\lambda_0 x) < \infty$ for a certain $\lambda_0 > 0$. Choose n_0 so that $\mathscr{I}_{\overline{\overline{\psi}}_s}(\lambda_0 \chi_{e_{n_0}'}) < \varepsilon \lambda_0^s$. The inequality $\mathscr{I}_\varphi(tx\chi_{e_{n_0}''}) \leqslant \varphi(tn_0)\mu(e_{n_0}'')$ holds; hence by (o_s), we have $\mathscr{I}_\varphi(tx\chi_{e_{n_0}''}) t^{-s} < \varepsilon$ for sufficiently small t. Thus

$$\mathscr{I}_\varphi(tx)t^{-s} \leqslant \mathscr{I}_\varphi(tx\chi_{e'_{n_0}})t^{-s} + \mathscr{I}_\varphi(tx\chi_{e''_{n_0}})t^{-s} < 2\varepsilon$$

for sufficiently small t.

Necessity. If $x(t) = 1$ for $t\epsilon E$, there is $\varphi(t)\mu(E)t^{-s} = \mathscr{I}_\varphi(tx)t^{-s} \to 0$, whence (o_s) is satisfied. If $(+)$ holds, then the inequality

$$(**) \qquad\qquad \varphi(au) \leqslant ca^s\varphi(u)$$

is satisfied for $0 \leqslant a \leqslant a_0,\ u \geqslant u_0$, where $a_0, u_0, c > 0$ are certain constants. For in the contrary case, a sequence $a_n \to 0$ and numbers u_n would exist such that

$$\sum_\nu 2^{-\nu}\big(\varphi(u_\nu)\big)^{-1} \leqslant \mu(E), \qquad \varphi(a_nu_n) \geqslant 2^{2n}a_n^s\varphi(u_n) \qquad \text{for} \qquad n = 1, 2, \ldots$$

We choose disjoint intervals e_n, $\mu(e_n) = 2^{-n}\big(\varphi(u_n)\big)^{-1}$ and we define $x(t) = u_n$ for $t\epsilon e_n$, $x(t) = 0$ for $t\epsilon E - \bigcup_\nu e_\nu$. There is $\mathscr{I}_\varphi(x) = \sum_\nu \varphi(u_\nu)\mu(e_\nu) = 1$ and

$$\mathscr{I}_\varphi(a_kx)a_k^{-s} = a_k^{-s}\sum_{\nu=1}^\infty \varphi(a_ku_\nu)\big(\varphi(u_\nu)\big)^{-1}2^{-\nu} \geqslant a_k^{-s}\varphi(a_ku_k)\big(\varphi(u_k)\big)^{-1}2^{-k} \geqslant 2^k,$$

whence we get a contradiction.

Similarly as in the proof of 2.3, $(**)$ implies $\varphi \overset{l}{\sim} \chi$, where χ is a φ-function, as in our proposition. Note that if $E = E_0$, $\mu(E) = \infty$, then $(**)$ must hold for $u \geqslant 0$; in the case of purely atomic E, $(**)$ is proved for sufficiently small u based on a remark applied implicitly in the proof of 2.3: if numbers γ_n are sufficiently large, then there are disjoint sets e_n, $n = 1, 2, \ldots$, satisfying the condition $\mu(e_n) = \gamma_n k_n$ where $1 \leqslant k_n \leqslant 2$ for $n = 1, 2, \ldots$

It follows from the above proof of necessity that if

$$(++) \qquad\qquad \overline{\lim_{t\to 0+}}\, \mathscr{I}_\varphi(tx)t^{-s} < \infty \qquad \text{for} \qquad x\epsilon L^{*\varphi}(E, \mu),$$

then $\varphi \overset{l}{\sim} \chi$, assuming $E = E_0$, $\mu(E) < \infty$, and analogously in the remaining cases. So we have

2.51. By the same assumptions on E, μ as in 2.5, the relation $(++)$ holds if and only if φ satisfies (o_s^b) and $\varphi \overset{l}{\sim} \chi$ (resp. $\varphi \overset{a}{\sim} \chi$, $\varphi \overset{s}{\sim} \chi$), where χ has the same meaning as in 2.5.

The sufficiency is proved analogously as in 2.5, and the necessity follows from the remarks in concluding the proof of 2.5.

3. THEOREM. *Assuming $E = E_0$ or $E = \bigcup_{\nu=1}^\infty e_\nu^a$, where e_n^a are atoms such that $0 < c_1 < \mu(e_n^a) < c_2$ for $n = 1, 2, \ldots$, the following properties are equivalent:*

(a) *φ possesses the property (o_s^b), there exists an s-homogeneous norm in $L^{*\varphi}(E, \mu)$ equivalent to the norm generated by φ;*

(b) *there exists an s-homogeneous norm $\| \ \|_s$ in $L^{*\varphi}(E, \mu)$ such that $\mathscr{I}_\varphi(x) \leqslant \|x\|_s$, if $\|x\|_s \leqslant 1$;*

(c) *$\lim_{t\to 0+} \mathscr{I}_\varphi(tx)t^{-s} < \infty$.*

If φ satisfies the condition (o_s), then (c) may be replaced by $(+)$ from 2.5.

2.3 and 2.4 imply (a) $\Rightarrow$ (b) and substituting tx in place of x in (b) we state (b) $\Rightarrow$ (c); 2.52, 2.21 and the definition of s-homogeneous norm given in 2.1 imply (c) $\Rightarrow$ (a).

Let us yet note that if an s-homogeneous norm satisfying (b) exists, then it may always be chosen so that it is equivalent to the norm generated by φ. This follows from 2.4.

References

[1] Z. Birnbaum und W. Orlicz, *Über die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen*, Studia Math. 3 (1931), p. 1-67.

[2] W. Matuszewska, *On generalized Orlicz spaces*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. 8 (1960), p. 349-353.

[3] — *Przestrzenie funkcji φ-całkowalnych, I, Własności ogólne φ-funkcji i klas funkcji φ-całkowalnych*, to appear in Prace Matematyczne 5 (1961).

[4] — *Przestrzenie funkcji φ-całkowalnych, II, Uogólnione przestrzenie Orlicza*, to appear in Prace Matematyczne 5 (1961).

[5] S. Mazur and W. Orlicz, *On some classes of linear spaces*, Studia Math. 17 (1958), p. 97-119.

[6] S. Rolewicz, *Some remarks on the space N(L) and N(l)*, ibidem 18 (1959), p. 1-9.