References

[1] E. Čech, *On bicompact spaces*, Ann. of Math. 33 (1937), p. 823-844.

[2] E. Hewitt, *Rings and real-valued continuous functions*, Trans. Amer. Math. Soc. 56 (1948), p. 45-99.

[3] S. Kakutani, *Concrete representation of abstract (M)-spaces*, Ann. of Math. 42 (1941), p. 944-1024.

[4] S. Mazur, *Sur la structure des fonctionnelles linéaires dans certains espaces (L)*, C. R. des séances de la Société Polonaise de Mathématique, Ann. Soc. Pol. Math. 19 (1946), p. 241.

[5] S. Mrówka, *On the form of certain functionals*, Bull. Acad. Pol. Sci., Cl. III, 5, (1957), p. 1061-1067.

[6] — *A generalization of a theorem of S. Mazur concerning linear multiplicative functionals*, ibidem 6 (1958).

[7] — *Functionals on uniformly closed rings of continuous functions*, Fund. Math. 46 (1958), p. 81-87.

Reçu par la Rédaction le 31. 10. 1958

STUDIA MATHEMATICA, T. XXI. (1961)

On quasi-modular spaces

by

SHÔZÔ KOSHI and TETSUYA SHIMOGAKI (Sapporo)

§ 1. Introduction. Let R be a *universally continuous semi-ordered linear space* (i. e. a lattice ordered linear space in which there exists $\bigcap_{\lambda\in\Lambda} a_\lambda$ for every system of positive elements $\{a_\lambda;\ \lambda\in\Lambda\}$ of R).

H. Nakano has considered a kind of functional on R which is called a *modular* (¹), and constructed the most important parts of the theory of modular spaces (i. e. spaces on which modulars are defined).

In this paper we shall consider a functional ϱ on R which satisfies the following conditions, weaker than those of modulars:

(ρ.1) $0 \leqslant \varrho(x) = \varrho(-x) \leqslant +\infty$ for all $x\in R$;

(ρ.2) $\varrho(x+y) = \varrho(x)+\varrho(y)$ for every $x, y\in R$ with $|x|\frown|y| = 0$;

(ρ.3) for any system $\{x_\lambda;\ \lambda\in\Lambda\}$ such that $|x_\lambda|\frown|x_\gamma| = 0$ for $\lambda\neq\gamma$, $\lambda, \gamma\in\Lambda$ and $\sum_{\lambda\in\Lambda}\varrho(x_\lambda) < +\infty$, there exists $x_0\in R$ with $\sum_{\lambda\in\Lambda} x_\lambda = x_0$ and $\sum_{\lambda\in\Lambda}\varrho(x_\lambda) = \varrho(x_0)$;

(ρ.4) $\overline{\lim_{\alpha\to 0}}\,\varrho(\alpha x) < +\infty$ for all $x\in R$.

R is called a *quasi-modular space* if the above ϱ is defined on R and ϱ is called a *quasi-modular*. This quasi-modular is considered as a generalization of a Nakano's monotone complete modular or of a concave modular [4 and 6].

Recently, J. Musielak and W. Orlicz considered the *pseudo-modular* on a linear space in [8]. If we add the order structure to linear spaces and additive conditions: (ρ.2) and (ρ.3) to those of a pseudo-modular, then a quasi-modular can be considered as a pseudo-modular in the case of semi-ordered linear spaces.

Some of the examples of a pseudo-modular established in [8] are regarded as those of a quasi-modular.

(¹) For the definition of a modular see § 2.

The main object of this paper is to prove the following fact:

If R is a quasi-modular space which is non-atomic and semi-regular, then R can be considered as a modular space, that is, there is defined a modular on R (Theorem 3.1).

By virtue of this theorem, *the theory of quasi-modular spaces which are semi-regular and non-atomic is to a great extent contained in that of modular spaces*. In general, a universally continuous semi-ordered linear space R is decomposed into a non-atomic part and a discrete part. In the case of discrete spaces, the above theorem does not remain true. But if we assume a property concerning the basis of a discrete space R, we shall also prove that *R is a modular space with a modular defined by the quasi-modular* (Theorem 3.3).

When ϱ satisfies

$$(\rho.5) \qquad \varrho(x) < +\infty \text{ for all } x \in R$$

in addition to (ρ.1)-(ρ.4), ϱ is called *finite*. In this case, the assumption that R is semi-regular becomes unnecessary, since semi-regularity is implied by the finiteness of ϱ (Theorem 4.1).

As a concrete example of quasi-modular spaces, we consider a function space $L_{M(s,t)}$ as follows. Let $M(s,t)$ ($s \in [0,+\infty)$, $t \in [0,1]$) be a real-valued function with

(1.1) $0 \leqslant M(s,t) \leqslant +\infty$ for all $(s,t) \in [0,+\infty) \times [0,1]$;

(1.2) $M(s,t)$ is a non-decreasing and left-hand continuous function of s for a fixed $t \in [0,1]$;

(1.3) $M(s,t)$ is a measurable function of t for a fixed $s \in [0,+\infty)$ with $M(0,t) = 0$ a. e.

And let $L_{M(s,t)}$ be *the totality of all measurable functions $x(t)$ ($t \in [0,1]$) such that*

$$\varrho(ax) = \int_0^1 M\big(a|x(t)|, t\big)\,dt < +\infty \quad \textit{for some} \quad a > 0\ (^2).$$

Then $L_{M(s,t)}$ is considered as a universally continuous semi-ordered linear space (we define $x \geqslant y$, $x, y \in L_{M(s,t)}$ if and only if $x(t) \geqslant y(t)$ a. e. $t \in [0,1]$). And putting

$$(1.4) \qquad \varrho(x) = \int_0^1 M\big(|x(t)|, t\big)\,dt \quad \text{for} \quad x(t) \in L_{M(s,t)},$$

we obtain a quasi-modular space $L_{M(s,t)}$ with the quasi-modular ϱ. $L_{M(s,t)}$ is investigated by S. Mazur and W. Orlicz in [7] under the more restricted

(2) $M(a|x(t)|, t)$ is a measurable function of t for every $a > 0$ and measurable $x(t)$, if $M(s,t)$ satisfies (1.1), (1.2) and (1.3).

condition: $M(s,t) = M(s)$ for all s, t. Therefore, Theorem 3.2 can be regarded as a generalization of Theorem 3 in [7] (i. e. *$L_{M(s)}$ is a Banach space if and only if $L_{M(s)} = L_\Phi$, where Φ is a convex function with $\Phi(0) = 0$ and $\Phi(s) \to +\infty$ as $s \to \infty$*.

In § 2 we shall state the definitions and results already known which are necessary for the following sections. In § 3 we prove the main theorems and in § 4 we consider the supplementaries of § 3, i. e. the finite cases, and the function spaces $L_{M(s,t)}$ are considered as an application in § 5. The conditions under which $L_{M(s,t)}$ is semi-regular or is a Banach space with respect to a certain norm are obtained there.

Here, we give thanks to Dr. J. Musielak for his kind advice.

§ 2. Preliminaries. In this section, we shall explain the definitions, notations and results which are necessary for the following sections. The results are mainly due to H. Nakano and their proofs are found in his book [4].

Throughout this paper we write R to denote a *universally continuous semi-ordered linear space*. For any $x \in R$, x^+, x^- and $|x|$ denote $x \cup 0$, $-x \cup 0$ and $x \cup -x$ respectively. $|x|$ is called the *absolute* of x and $|x| = x^+ + x^-$. $x \perp y$ ($x, y \in R$) denotes $|x| \cap |y| = 0$ and we call x and y *mutually orthogonal*. For any subset $A \subset R$, $A^\perp = \{x : x \perp y \text{ for all } y \in A\}$ is called the *orthogonal complement* of A. We always have

$$(2.1) \qquad A^\perp = A^{\perp\perp\perp} \quad \text{and} \quad R = A^\perp \oplus A^{\perp\perp},$$

which means that for any $x \in R$ there exist $x_1 \in A^\perp$ and $x_2 \in A^{\perp\perp}$ such that $x = x_1 + x_2$.

Hence we can define an operator $[A]$ with $[A]x = x_2$. $[A]$ is called a *projection operator* and $A^{\perp\perp}$ is called a *normal manifold generated by A*. A linear subset $N \subset R$ is said to be a *normal manifold* if it satisfies

$$(2.2) \qquad x \in N \text{ and } |x| \geqslant |y| \quad \text{imply} \quad y \in N,$$

and

$$(2.3) \qquad x = \bigcup_{\lambda \in \Lambda} x_\lambda \text{ and } x_\lambda \in N\ (\lambda \in \Lambda) \quad \text{imply} \quad x \in N \ [4;\ \text{Theorem } 4.9].$$

For a normal manifold N, we have $N = N^{\perp\perp}$. A linear subset M is called *semi-normal* if it satisfies only the above condition (2.2). When $M^\perp = \{0\}$, M is called a *complete semi-normal manifold of R*. If M is a complete semi-normal manifold, then every $0 \leqslant x \in R$ can be written as $x = \bigcup_{\lambda \in \Lambda} x_\lambda$ with $0 \leqslant x_\lambda \in M$ ($\lambda \in \Lambda$). For a normal manifold N, it follows from the definitions that $x \in N$ if and only if $[N]x = x$.

If A consists of only one element, then $[A] = [\{a\}] = [a]$ is called a *projector* by a. Then we have $[a]x = \bigcup_{n \geqslant 1} (n|a| \cap x)$ for every $x \geqslant 0$.

A linear functional $\tilde{a}$ on R is called *bounded* if

(2.4) $\sup\limits_{|x|\leqslant|a|} |\tilde{a}(x)| < +\infty$ for any $a \in R$.

And the totality of all bounded linear functionals on R is denoted by $\tilde{R}$. A linear functional $\bar{a}$ on R is called *universally continuous* if

(2.5) $\inf\limits_{\lambda\in\Lambda} |\bar{a}(x_\lambda)| = 0$ for any $x_\lambda \downarrow_{\lambda\in\Lambda} 0$.

We denote by $\bar{R}$ the totality of all universally continuous linear functionals on R and call it the *conjugate space* of R. It is known that $\tilde{R}$ and $\bar{R}$ are both universally continuous semi-ordered linear spaces (in these cases $\tilde{a} \geqslant \tilde{b}$ (or $\bar{a} \geqslant \bar{b}$) means $\tilde{a}(x) \geqslant \tilde{b}(x)$ (resp. $\bar{a}(x) \geqslant \bar{b}(x)$) for all $0 \leqslant x \in R$), and $\bar{R} \subset \tilde{R}$. R is said to be *semi-regular* if $\bar{a}(x) = 0$ for all $\bar{a} \in \bar{R}$ implies $x = 0$. When R is semi-regular, putting for any $a \in R$

(2.6) $f_a(\bar{x}) = \bar{x}(a) \quad (\bar{x} \in \bar{R})$,

we obtain a universally continuous linear functional on $\bar{R}$. Therefore there exists an isomorphism from R into $\bar{\bar{R}} = (\overline{\bar{R}})$ by the correspondence: $R \ni a \to f_a \in \bar{\bar{R}}$. Hence we can find that R is embedded in $\bar{\bar{R}}$. In this sense we write in the sequel $R \subseteq \bar{\bar{R}}$ when R is semi-regular. R is a complete semi-normal manifold of $\bar{\bar{R}}$. R is said to be *reflexive* if R coincides with $\bar{\bar{R}}$. $\bar{R}$ *is always reflexive* [4; Theorem 24.5].

For any projection operator $[\bar{A}]$ of $\bar{R}$ we put $C_{\bar{A}} = \{x : x \in R,\ |\bar{a}(x)| = 0$ for all $\bar{a} \in \bar{A}\}$ and $[\bar{A}]^R = [C_{\bar{A}}^\perp]$. $[\bar{A}]^R$ is a projection operator on R such that

(2.7) $([\bar{A}]\bar{a})(x) = \bar{a}([\bar{A}]^R x)$ for all $x \in R$,

and

(2.8) $[\bar{A}][\bar{B}] = 0, \quad \bar{A}, \bar{B} \subset \bar{R}$ if and only if $[\bar{A}]^R[\bar{B}]^R = 0$.

An element $0 \neq a \in R$ is called *atomic* if $[a] \geqslant [b]$ implies $[a] = [b]$ or $[b] = 0$. When R has no atomic elements, R is said to be *non-atomic*, and when there exist no elements in R except 0 which is orthogonal to every atomic element of R, R is said to be *discrete*. If R is discrete, then there exists a system of positive atomic elements of R: $\{e_\lambda;\ \lambda \in \Lambda\}$ such that any $0 \leqslant a \in R$ is written uniquely as

(2.9) $$a = \bigcup_{\lambda\in\Lambda} \xi_\lambda e_\lambda$$

where ξ_λ is a positive real number for each $\lambda \in \Lambda$. In general, R is decomposed into $R = R_c \oplus R_d$, where R_c is non-atomic and R_d is discrete.

Throughout this paper, let a norm $\|\cdot\|$ on R satisfy the condition:

(N.1) $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$.

A norm $\|\cdot\|$ on R is called *semi-continuous* if

(N.2) $\|x\| = \sup\limits_{\lambda\in\Lambda} \|x_\lambda\|$ for $0 \leqslant x_\lambda \uparrow_{\lambda\in\Lambda} x$,

and called *monotone complete* if $0 \leqslant x_\lambda \uparrow_{\lambda\in\Lambda}$ and $\sup\limits_{\lambda\in\Lambda} \|x_\lambda\| < +\infty$ imply $\bigcup\limits_{\lambda\in\Lambda} x_\lambda \in R$. If $\|\cdot\|$ is monotone complete, $\|\cdot\|$ is complete (i. e. R is a Banach space with respect to $\|\cdot\|$). $\bar{R}^*$ denotes the totality of all $\bar{a} \in \bar{R}$ with $\sup\limits_{\|x\|\leqslant 1} |\bar{a}(x)| < +\infty$. We have $\bar{R}^* \subset \bar{R}$ and $\bar{R}^* = \bar{R}$ if $\|\cdot\|$ is complete.

A *modular* m on R is a functional which satisfies the following:

(M.1) $0 \leqslant m(x) \leqslant +\infty$ for all $x \in R$;

(M.2) $m(\xi x) = 0$ for every ξ implies $x = 0$;

(M.3) $m(\alpha x) < +\infty$ for some $\alpha > 0$;

(M.4) $|x| \leqslant |y|$ implies $m(x) \leqslant m(y)$;

(M.5) $\alpha + \beta = 1,\ \alpha, \beta \geqslant 0$ implies $m(\alpha x + \beta y) \leqslant \alpha m(x) + \beta m(y)$, for any $x, y \in R$;

(M.6) $x \perp y$ implies $m(x+y) = m(x) + m(y)$;

(M.7) $0 \leqslant x_\lambda \uparrow_{\lambda\in\Lambda} x$ implies $\sup\limits_{\lambda\in\Lambda} m(x_\lambda) = m(x)$.

If a modular m is defined on R, R is called a *modular space*. A modular m on R is said to be *monotone complete* if

(M.8) $0 \leqslant x_\lambda \uparrow_{\lambda\in\Lambda}$ and $\sup\limits_{\lambda\in\Lambda} m(x_\lambda) < +\infty$ imply $\bigcup\limits_{\lambda\in\Lambda} x_\lambda \in R$.

The condition (M.8) is equivalent to the apparently weaker one:

(M.8′) for any $\{x_\lambda;\ \lambda \in \Lambda\}$ with $x_\lambda \perp x_{\lambda'}$, $\lambda \neq \lambda'$, $\lambda, \lambda' \in \Lambda$ and $\sum\limits_{\lambda\in\Lambda} m(x_\lambda) < +\infty$, there exists $\bigcup\limits_{\lambda\in\Lambda} x_\lambda \in R$.

$\bar{a} \in \bar{R}$ is called *modular bounded* if there exist $\alpha, \beta \geqslant 0$ such that

(2.10) $$|\bar{a}(x)| \leqslant \alpha + \beta m(x) \quad \text{for all} \quad x \in R.$$

$\bar{R}^m$ denotes the totality of all modular bounded $\bar{a} \in \bar{R}$ and is called the *modular conjugate space of* R.

If we define on $\bar{R}^m$ a functional $\bar{m}$ by

(2.11) $$\bar{m}(\bar{a}) = \sup_{x\in R} \{|\bar{a}(x)| - m(x)\} \quad (\bar{a} \in \bar{R}^m),$$

$\bar{m}$ becomes a modular on $\bar{R}^m$ and is called *the conjugate modular of* m.

If R is semi-regular, we have

(2.12) $\bar{\bar{m}}(a) = \sup\limits_{\bar{x}\in\bar{R}^m} \{|\bar{x}(a)| - \bar{m}(\bar{x})\} = m(a)$ for all $a \in R$,

that is, m *is reflexive as a functional.*

Let R be a modular space, then R is a normed space with the norm $|||\cdot|||$:

$$(2.13)\qquad |||x||| = \inf_{m(\xi x)\leqslant 1} \frac{1}{|\xi|} \qquad (x \in R).$$

This norm $|||x|||$ is called the *modular norm by* m [3]. It is always semi-continuous, whence it is reflexive as a norm [3].

A sequence of elements $\{x_\nu;\ (\nu \geqslant 1)\}$ is said to be *modular convergent to* x_0 and denoted by $m\text{-}\lim\limits_{\nu\to\infty} x_\nu = x_0$ if

$$(2.14)\qquad \lim_{\nu\to\infty} m\big(a(x_\nu - x_0)\big) = 0 \quad \text{for every} \quad a \geqslant 0.$$

(2.14) is equivalent to $\lim\limits_{\nu\to\infty} |||x_\nu - x_0||| = 0$.

A modular m is called *complete* if $\lim\limits_{\nu,\mu\to\infty} m\big(a(x_\nu - x_\mu)\big) = 0$ $(a \geqslant 0)$ implies the existence of $x \in R$ with $m\text{-}\lim\limits_{\nu\to\infty} x_\nu = x$. It is easily seen that a modular m is complete (or monotone complete) if and only if the modular norm $|||\cdot|||$ is complete (resp. monotone complete) and hence, monotone completeness of m implies completeness of m. Monotone completeness of modulars (or norms) plays an important rôle in the theory of modular (resp. normed) spaces because in that case R is reflexive (i. e. $R = \bar{\bar{R}}$).

§ 3. Fundamental theorems. Let ϱ be a quasi-modular on R, that is, let the functional ϱ satisfy (ρ.1)-(ρ.4) in § 1.

From the definition of a quasi-modular, we have the following

LEMMA 1. *If ϱ is a quasi-modular on R, we have*

$$(3.1)\qquad \varrho(0) = 0;$$

$$(3.2)\qquad \varrho([p]x) \leqslant \varrho(x) \quad \textit{for all} \quad [p] \ \textit{and}\ x \in R;$$

$$(3.3)\qquad \varrho(|x|) = \varrho(x) \quad \textit{for all} \quad x \in R;$$

$$(3.4)\qquad \sup_{\lambda\in\Lambda} \varrho([p_\lambda]x) = \varrho([p]x) \quad \textit{for any} \quad [p_\lambda]\uparrow_{\lambda\in\Lambda}[p] \ \textit{and}\ x \in R.$$

The proofs are direct consequences of (ρ.1)-(ρ.4). Thus we omit them here.

When we consider a modular on R, it is convenient that *the modular be sufficiently compatible with the structure of R as a semi-ordered linear space*. For instance, it is very convenient if the modular is monotone complete or complete; because R can be considered as a Banach space with respect to the modular norms. A semi-regular quasi-modular space, however, cannot be a complete modular space by defining a modular in general, as shown below. Therefore we must consider a slightly weaker property than completeness.

(3) In [4 or 5] this norm is called the *second norm by* m, since another norm, called the *first one*,

$$\|x\| = \inf_{\xi>0} \frac{1+m(\xi x)}{\xi} \qquad (x \in R),$$

is defined there.

We shall say in the sequel that a modular (or norm) on R is *perfect* if R is semi-regular and $\bar{R} = \bar{R}^m$ (resp. $\bar{R} = \bar{R}^*$). Namely it means that *every universally continuous linear functional is modular* (resp. *norm*-) *bounded*.

It is clear that a modular m is perfect if and only if the modular norm is perfect. If R is semi-regular and a modular (or a norm) on R is complete, then it is perfect [4; §§ 31, 38]. But the converse of this is not true in general.

Though a slight improvement is made, the following Lemma 2 is essentially the same as proved in [11] and plays important role in the proofs of the theorems stated in this section:

LEMMA 2. *Let n_ν $(\nu = 0, 1, 2, \ldots)$ be a sequence of functionals on R such that*

$$(3.5)\qquad 0 \leqslant n_\nu(x) \leqslant +\infty \quad (\nu \geqslant 0) \quad \textit{for all}\ x \in R;$$

$$(3.6)\qquad n_\nu(x+y) \leqslant n_\nu(x) + n_\nu(y) \quad (\nu \geqslant 0) \quad \textit{if}\ x \perp y;$$

$$(3.7)\qquad \sup_{\lambda\in\Lambda} n_\nu([p_\lambda]x) = n_\nu([p]x) \quad (\nu \geqslant 0)\ \textit{for}\ [p_\lambda]\uparrow_{\lambda\in\Lambda}[p]\ \textit{and}\ x \in R;$$

(3.8) *for any* $\{x_\nu\}$, $x_\nu \in R$ $(\nu = 1, 2, \ldots)$ *with* $x_\nu \perp x_\mu$ *for* $\nu \neq \mu$ *and* $\sum\limits_{\nu=1}^{\infty} n_0(x_\nu) < +\infty$, *there exists* $\sum\limits_{\nu=1}^{\infty} x_\nu \in R$;

$$(3.9)\qquad \lim_{\nu\to\infty} n_\nu(x) < +\infty \quad \textit{for all} \quad x \in R.$$

Then there exist positive numbers ε, γ a natural number ν_0 and a finite dimensional normal manifold N_0 such that $n_0(x) \leqslant \varepsilon$ and $x \in [N_0^\perp]R$ imply $n_\nu(x) \leqslant \gamma$ for all $\nu \geqslant \nu_0$.

We must deal with the subject in slightly different manners according to whether R is non-atomic or discrete. First we suppose that R is non-atomic.

THEOREM 3.1. *Let R be a quasi-modular space which is semi-regular and non-atomic. Then R is a modular space with a perfect modular m_ϱ* (4) *which is constructed by ϱ.*

Proof. We define a functional $\bar{\varrho}$ on $\bar{R}$ by

$$(3.10)\qquad \bar{\varrho}(\bar{x}) = \sup_{x\in R}\{|\bar{x}(x)| - \varrho(x)\} \qquad (\bar{x} \in \bar{R}),$$

(4) For the relation between ϱ and m_ϱ see (3.10) and (3.12) in the proof of this theorem and Remark 3.3 below.

and we shall show that $\bar{\varrho}$ is a *monotone complete modular on* $\bar{R}$. Since $0 \leqslant \bar{\varrho}(\bar{x}) \leqslant +\infty$ for all $\bar{x} \in \bar{R}$ by definition, the modular condition (M.1) follows. It $\bar{\varrho}(\xi\bar{x}) = 0$ for all $\xi \geqslant 0$, then we have

$$|\xi\bar{x}(x)| - \varrho(x) \leqslant 0 \quad \text{for all} \quad x \in R \text{ and } \xi \geqslant 0.$$

For any $x_0 \in R$, (ρ.4) ensures the existence of $\alpha > 0$ with $\varrho(\alpha x_0) < +\infty$, which implies

$$|\xi\bar{x}(\alpha x_0)| \leqslant \varrho(\alpha x_0) \quad (\xi \geqslant 0).$$

Thus we obtain $\bar{x}(x_0) = 0$, and since $x_0 \in R$ is arbitrary, we have $\bar{x} = 0$. This shows that (M.2) holds. For any $\bar{x}, \bar{y} \in \bar{R}$ and real numbers $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$, we have

$$\begin{aligned}\bar{\varrho}(\alpha\bar{x}+\beta\bar{y}) &= \sup_{x \in R}\{|(\alpha\bar{x}+\beta\bar{y})(x)| - \varrho(x)\}\\ &\leqslant \sup_{x \in R}\alpha\{|\bar{x}(x)| - \varrho(x)\} + \sup_{x \in R}\beta\{|\bar{y}(x)| - \varrho(x)\}\\ &\leqslant \alpha\bar{\varrho}(\bar{x}) + \beta\bar{\varrho}(\bar{y}).\end{aligned}$$

Hence (M.4) follows.

If $\bar{x} \perp \bar{y}$, $\bar{x}, \bar{y} \in \bar{R}$, then it follows from (ρ.2) and (2.7) that

$$\begin{aligned}&|(\bar{x}+\bar{y})(x)| - \varrho(x)\\ &\qquad \leqslant |\bar{x}([\bar{x}]^R x)| - \varrho([\bar{x}]^R x) + |\bar{y}([\bar{y}]^R x| - \varrho([\bar{y}]^R x)\end{aligned}$$

for all $x \in R$, which implies (M.6). (M.6) and (3.3) yield directly (M.5). (M.7) is an immediate consequence of the form (3.10).

Therefore, in order to prove that $\bar{\varrho}$ is a modular on $\bar{R}$, we need to verify that $\bar{\varrho}$ satisfies (M.3). Let $\bar{x}_0 \in \bar{R}$ be fixed. We put

$$n_0(x) = \varrho(x) \quad (x \in R),$$
$$n_\nu(x) = |\bar{x}_0|(|x|) \quad \text{for all} \quad \nu \geqslant 1.$$

Since $\bar{x}_0 \in \bar{R}$, and ϱ is a quasi-modular, a sequence of functionals $\{n_\nu\}_{(\nu \geqslant 0)}$ satisfies the conditions (3.5)-(3.9).

Hence by virtue of Lemma 2 and of the fact that R is non-atomic, we can find the positive numbers ε, γ such that

$$\varrho(x) = n_0(x) \leqslant \varepsilon \text{ implies } n_\nu(x) = |\bar{x}_0|(|x|) \leqslant \gamma.$$

For any $x \in R$ with $\varepsilon < \varrho(x) < +\infty$, there exists a natural number n such that

$$(3.11) \qquad n\varepsilon < \varrho(x) \leqslant (n+1)\varepsilon.$$

Since R is non-atomic and ϱ satisfies (ρ.3) and (3.4), x can be decomposed orthogonally with $x = \sum_{i=1}^{k} x_i$, $\varepsilon \geqslant \varrho(x_i) > \frac{1}{2}\varepsilon$ $(i = 1, 2, \ldots, k)$. It follows from (3.11) that $k < 2(n+1)$ and

$$|\bar{x}_0|(|x|) = \sum_{i=1}^{k} |\bar{x}_0|(|x_i|) \leqslant k\cdot\gamma < 2(n+1)\gamma \leqslant \frac{4n}{\varepsilon}\varepsilon\gamma \leqslant \frac{4\gamma}{\varepsilon}\varrho(x).$$

Putting $4\gamma/\varepsilon = \delta = \delta(\bar{x}_0)$, we obtain

$$|\bar{x}_0|(|x|) \leqslant \delta\varrho(x) \quad \text{for all} \quad x \in R \text{ with } \varrho(x) > \varepsilon,$$

that is

$$\frac{1}{\delta}|\bar{x}_0|(|x|) - \varrho(x) \leqslant 0.$$

Therefore we have

$$\bar{\varrho}\left(\frac{1}{\delta}\bar{x}_0\right) \leqslant \sup_{x \in R}\left\{\frac{1}{\delta}|\bar{x}_0|(|x|) - \varrho(x)\right\} \leqslant \sup_{\varrho(x) \leqslant \varepsilon}\left\{\frac{1}{\delta}|\bar{x}_0|(|x|) - \varrho(x)\right\} \leqslant \frac{\gamma}{\delta}.$$

Thus (M.3) is ascertained.

If $0 \leqslant \bar{x}_\lambda \uparrow_{\lambda \in \Lambda}$ and $\sup_{\lambda \in \Lambda}\bar{\varrho}(\bar{x}_\lambda) < +\infty$, then for any $0 \leqslant x \in R$ with $\varrho(x) < +\infty$ we have

$$\sup_{\lambda \in \Lambda}\bar{x}_\lambda(x) \leqslant \bar{\varrho}(\bar{x}_\lambda) + \varrho(x) \leqslant \gamma' + \varrho(x) \quad \text{for some} \quad \gamma' > 0.$$

This implies $\sup_{\lambda \in \Lambda}\bar{x}_\lambda(x) < +\infty$ for all $0 \leqslant x \in R$ by (ρ.4). Hence we have $\bigcup_{\lambda \in \Lambda}\bar{x}_\lambda \in \bar{R}$, which shows that (M.8) holds, i. e. the modular $\bar{\varrho}$ on $\bar{R}$ is monotone complete.

Since $\bar{R}$ is a monotone complete modular space with the modular $\bar{\varrho}$, we can define the conjugate modular $\bar{\bar{\varrho}}$ of $\bar{\varrho}$ on $\bar{\bar{R}} = \bar{\bar{R}}^{\bar{\varrho}}$. As stated in § 2, R can be considered as a complete semi-normal manifold of $\bar{\bar{R}}$, because R is semi-regular. Now putting

$$(3.12) \qquad m_\varrho(x) = \bar{\bar{\varrho}}(x) \quad \text{for all} \quad x \in R \subset \bar{\bar{R}},$$

we obtain a modular m_ϱ on R. And for any $\bar{x} \in \bar{R}$

$$(3.13) \qquad \bar{\varrho}(\bar{x}) = \sup_{\bar{\bar{a}} \in \bar{\bar{R}}}\{|\bar{\bar{a}}(x)| - \bar{\bar{\varrho}}(x)\} = \sup_{x \in R}\{|\bar{x}(x)| - m_\varrho(x)\}$$

holds, because R is a complete semi-normal manifold of $\bar{\bar{R}}$ and the modular $\bar{\varrho}$ on $\bar{R}$ is reflexive (cf. § 2). Namely $\bar{\varrho}$ is the conjugate modular of m_ϱ. Therefore m_ϱ is a perfect modular on R, q. e. d.

Remark 3.1. From the above proof, it follows that Theorem 3.1 remains true if we replace the condition (ρ.4) by a weaker one, such as

(ρ.4′) *for any $x \in R$ there exists $\alpha > 0$ such that $\varrho(\alpha x) < +\infty$.*

Remark 3.2. From the definition of m_ϱ we have

(3.14) $m_\varrho(x) \leqslant \varrho(x)$ for all $x \in R$.

Thus, if R is semi-regular and non-atomic we have

$$\varliminf_{\xi\to+\infty} \frac{\varrho(\xi x)}{\xi} \geqslant \varliminf_{\xi\to+\infty} \frac{m_\varrho(\xi x)}{\xi} > 0 \quad \text{for all} \quad x \in R,$$

because $m(\xi x)$ is a convex function of $\xi \geqslant 0$ for each $x \in R$.

In the sequel, m_ϱ denotes the modular defined by (3.10) and (3.12).

We say that a sequence $\{x_\nu;\ \nu = 1, 2, \ldots\}$ of R is *ϱ-convergent* to x_0 and write $\varrho\text{-}\lim_{\nu\to\infty} x_\nu = x_0$ if there exists a constant number $k > 0$ such that

(3.15) $$\varlimsup_{\nu\to\infty} \varrho\big(\xi(x_\nu - x_0)\big) \leqslant k \quad \text{for all} \quad \xi \geqslant 0\ ({}^5).$$

Remark 3.3. Let R be the same as in Theorem 3.1. For the modular m_ϱ, we find that ϱ-convergence implies m_ϱ-convergence. Namely, if $\varrho\text{-}\lim_{\nu\to\infty} x_\nu = x_0$, there exists a constant number $k > 0$ with $\varlimsup_{\nu\to\infty} \varrho\big(\xi^2(x_\nu - x_0)\big) \leqslant$ $\leqslant k$ for all $\xi \geqslant 0$. It follows by (3.14) that

$$\varlimsup_{\nu\to\infty} m_\varrho\big(\xi^2(x_\nu - x_0)\big) \leqslant k \quad \text{for all} \quad \xi \geqslant 0.$$

Since $m_\varrho(\xi x)$ is a convex function of $\xi \geqslant 0$, for each x, we have

$$\varlimsup_{\nu\to\infty} m_\varrho\big(\xi(x_\nu - x_0)\big) \leqslant \frac{k}{\xi} \quad \text{for all} \quad \xi \geqslant 1.$$

It follows that $m_\varrho\text{-}\lim_{\nu\to\infty} x_\nu = x_0$.

The converse of Remark 3.3 is not true in general. However, if m_ϱ is complete, m_ϱ-convergence implies ϱ-convergence, that is, we obtain the following theorem showing the conditions under which m_ϱ is complete; a fortiori R becomes a Banach space with the norm defined by m_ϱ.

THEOREM 3.2. *Let R be a quasi-modular space which is semi-regular and non-atomic. Then the following four conditions are equivalent to each other:*

(i) *m_ϱ is complete;*

(ii) *m_ϱ is monotone complete;*

(iii) *R is reflexive (i. e. $R = \overline{\overline{R}}$);*

(iv) *m_ϱ-convergence implies ϱ-convergence.*

Proof. We denote by $|||\cdot|||_\varrho$ the modular norm of m_ϱ. First we assume (i). We put

$$n_0(x) = |||x|||_\varrho \quad \text{and} \quad n_\nu(x) = \varrho\left(\frac{1}{\nu}\,x\right)$$

for all $x \in R$ and $\nu \geqslant 1$. A sequence of functionals $\{n_\nu\}_{(\nu\geqslant 0)}$ satisfies conditions (3.5)-(3.9). Here we shall show only that (3.8) is true, as the others are obtained without difficulty. Let $\{x_\nu;\ x_\nu \in R\ (\nu \geqslant 1)\}$ be $\sum_{\nu=1}^{\infty} n_0(x_\nu) < +\infty$ and $x_\nu \perp x_\mu$ ($\nu \neq \mu$). Putting $y_\mu = \sum_{\nu=1}^{\mu} |x_\nu|$ ($\mu \geqslant 1$), we have

$$0 \leqslant y_\mu\uparrow_{\mu=1,2,\ldots} \quad \text{and} \quad |||y_\mu - y'_\mu|||_\varrho \leqslant \sum_{\nu=\mu+1}^{\mu'} |||x_\nu|||_\varrho = \sum_{\nu=\mu+1}^{\mu'} n_0(x_\nu)$$
$$\text{for} \quad \mu' > \mu \geqslant 1.$$

Therefore $\{y_\mu\}$ ($\mu \geqslant 1$) is a Cauchy sequence. Since $|||\cdot|||_\varrho$ is complete, there exists $y_0 \in R$ with $\lim_{\mu\to\infty} |||y_\mu - y_0||| = 0$. It follows from the above that

$$y_0 = \bigcup_{\nu=1}^{\infty} y_\nu = \sum_{\nu=1}^{\infty} |x_\nu| \quad \text{and} \quad \sum_{\nu=1}^{\infty} x_\nu \in R,$$

which shows that $n_0(x) = |||x|||_\varrho$ ($x \in R$) satisfies (3.8).

Therefore, by virtue of Lemma 2, we can find positive numbers ε', γ' and a natural number ν_0 such that $|||x|||_\varrho \leqslant \varepsilon'$ implies

$$n_{\nu_0}(x) = \varrho\left(\frac{1}{\nu_0}\,x\right) \leqslant \gamma'.$$

Let $\{x_\lambda\}_{(\lambda\in\Lambda)}$ be a system of mutually orthogonal elements with $\sum_{\lambda\in\Lambda} m_\varrho(x_\lambda) < +\infty$. Then we can find $\alpha > 0$ with $\sum_{\lambda\in\Lambda} m_\varrho(\alpha x_\lambda) \leqslant 1$, and also $\alpha' > 0$ with $\big|\big|\big|\sum_{\lambda\in\Lambda'} \alpha' x_\lambda\big|\big|\big|_\varrho \leqslant \varepsilon'$, where Λ' is an arbitrary finite subset of Λ. It follows that

$$\varrho\left(\frac{\alpha'}{\nu_0} \sum_{\lambda\in\Lambda'} x_\lambda\right) = \sum_{\lambda\in\Lambda'} \varrho\left(\frac{\alpha'}{\nu_0}\,x_\lambda\right) \leqslant \gamma',$$

and hence

$$\sum_{\lambda\in\Lambda} \frac{\alpha'}{\nu_0}\,x_\lambda \in R \quad \text{and} \quad \sum_{\lambda\in\Lambda} x_\lambda \in R.$$

---

[5] When ϱ is a modular m, (3.15) is equivalent to $m\text{-}\lim x_\nu = x_0$ in the sense of (2.4), because $\varrho(\xi x)$ is a convex function of $\xi \geqslant 0$ for every $x \in R$ in this case. This fact is verified in the same way as in Remark 3.3 below. On the other hand, for an arbitrary quasi-modular ϱ, (3.15) is not equivalent to $\lim_{\nu\to\infty} \varrho(\alpha(x_\nu - x_0)) = 0$ for each $\alpha > 0$.

Thus m_ϱ is monotone complete by virtue of (M.8′), and so (i) → (ii) is proved.

(ii) → (iii) is already known [4; § 38].

Now we shall prove (iii) → (iv). Since $\overline{\overline{R}} = R$, m_ϱ is monotone complete, and a fortiori complete. Then as shown above, we can find positive numbers ε', γ' and a natural number ν_0 such that $|||x|||_\varrho \leqslant \varepsilon'$ implies

$$\varrho\left(\frac{1}{\nu_0}x\right) \leqslant \gamma'.$$

Let $m_\varrho\text{-}\lim_{\nu\to\infty} x_\nu = 0$. Since $m_\varrho\text{-}\lim_{\nu\to\infty} x_\nu = 0$ is equivalent to $\lim_{\nu\to\infty}|||x_\nu|||_\varrho = 0$, for any $\xi > 0$ we can find $\mu_0 = \mu_0(\xi)$ such that $|||\xi x_\mu|||_\varrho \leqslant \varepsilon'$ for all $\mu \geqslant \mu_0$. From this it follows that

$$\varrho\left(\frac{\xi}{\nu_0}x_\mu\right) \leqslant \gamma' \quad (\mu \geqslant \mu_0) \qquad \text{and} \qquad \overline{\lim_{\nu\to\infty}}\,\varrho\left(\frac{\xi}{\nu_0}x_\nu\right) \leqslant \gamma'.$$

Here ξ is arbitrary and we have (iii) → (iv).

If we assume (iv), we can find positive numbers ε' and γ' such that $|||x|||_\varrho \leqslant \varepsilon'$ implies $\varrho(x) \leqslant \gamma'$. This implies completeness of $|||\cdot|||_\varrho$, i. e. that of m_ϱ, similarly to the foregoing proofs, q. e. d.

Next we consider the case of R being discrete. Since a discrete space is always semi-regular, the assumption of semi-regularity now becomes unnecessary. But instead of this, we must assume the following two conditions on the discrete basis $\{e_\lambda; \lambda \epsilon \Lambda\}$ of R (see (2.9)):

(d.1) for any $x \epsilon R$, there exists $M_x > 0$ for which $|\xi_\lambda| \leqslant M_x$ $(\lambda \epsilon \Lambda)$, where $x = \sum_{\lambda\epsilon\Lambda}\xi_\lambda e_\lambda$.

(d.2) $\{e_\lambda\}_{\lambda\epsilon\Lambda}$ is weakly bounded, namely, $\sup_{\lambda\epsilon\Lambda}|\overline{a}(e_\lambda)| < +\infty$ for each $\overline{a}\,\epsilon\,\overline{R}$.

We say that a basis $\{e_\lambda\}_{\lambda\epsilon\Lambda}$ of R is *normal* if it satisfies (d.1) and (d.2). A basis which satisfies (d.1) is normal if and only if the basis of $\overline{R}$, $\{\overline{e}_\lambda\}_{\lambda\epsilon\Lambda}$, defined by the formula

$$(\overline{e}_\lambda, e_{\lambda'}) = \begin{cases} 1 & \text{if} \quad \lambda = \lambda', \\ 0 & \text{if} \quad \lambda \neq \lambda', \end{cases}$$

also fulfils (d.1) in $\overline{R}$. When R is a normed discrete space with a perfect norm $\|\cdot\|$, R always has a normal basis. Indeed we may select a basis system $\{e_\lambda\}_{\lambda\epsilon\Lambda}$ out of all atomic elements of R with $\|e\|_\lambda = 1$.

On the other hand, we can prove the sufficiency of this condition for the existence of a perfect norm, in the case of R being a quasi-modular space. That is, we have

Theorem 3.3. *Let R be a discrete quasi-modular space and have a normal basis $\{e_\lambda\}_{\lambda\epsilon\Lambda}$. Then R is a modular space with a perfect modular m_ϱ.*

Proof. We denote by S the totality of all $x \epsilon R$ for which $|\xi_\lambda| \leqslant 1$ $(\lambda\epsilon\Lambda)$, where $x = \sum_{\lambda\epsilon\Lambda}\xi_\lambda e_\lambda$.

For any $x\epsilon R$, there exists $a > 0$ such that $ax \epsilon S$ and $\varrho(ax) < +\infty$, because $\{e_\lambda\}_{\lambda\epsilon\Lambda}$ is a normal basis and ϱ satisfies (ρ.4). We define a functional $\overline{m}$ on $\overline{R}$ as

$$\overline{m}(\overline{x}) = \sup_{x\epsilon S}\{|\overline{x}|(|x|) - \varrho(x)\}. \tag{3.16}$$

Then, similarly to Theorem 3.1, we can see without difficulty that $\overline{m}$ satisfies the modular conditions: (M.1)-(M.7) except (M.3). For any $\overline{x}_0 \epsilon \overline{R}$ we put

$$n_0(x) = \varrho(x) \quad \text{and} \quad n_\nu(x) = |\overline{x}_0|(|x|) \qquad (x\epsilon R \text{ and } \nu \geqslant 1).$$

Lemma 2 also ensures the existence of positive numbers ε, γ and a finite dimensional normal manifold $N_0 = N_0(\overline{x}_0)$ such that

$$n_0(x) = \varrho(x) \leqslant \varepsilon \qquad (x\epsilon N_0^\perp)$$

implies

$$n_{\nu_0}(x) = |\overline{x}_0|(|x|) \leqslant \gamma.$$

If $x\epsilon[N_0^\perp]S$ and $(n+1)\varepsilon > \varrho(x) > n\varepsilon$ for some n, there exists an orthogonal decomposition such as

$$x = x_1 + x_2 + \ldots + x_k + y_1 + \ldots + y_{k'} + z$$

where $\varepsilon/2 < \varrho(x_i) \leqslant \varepsilon$ $(i = 1, 2, \ldots, k)$, y_i is an atomic element with $+\infty > \varrho(y_i) > \varepsilon$ for each $1 \leqslant i \leqslant k'$ and $\varrho(z) \leqslant \varepsilon/2$. We easily get $k \leqslant 2(n+1)$ and $k' < (n+1)$. Since $y_i = \xi_i e_i$ and $|\xi_i| \leqslant 1$ for every $1 \leqslant i \leqslant k'$ (because $x\epsilon S$), we can find a positive number $\delta = \delta(\overline{x}_0) > 0$ for which

$$\sup_{\lambda\epsilon\Lambda}|\overline{x}_0(e_\lambda)| \leqslant \delta$$

holds. Then we obtain

$$\begin{aligned} |\overline{x}_0|(|x|) &= \sum_{i=1}^{k}|\overline{x}_0|(|x_i|) + \sum_{i=1}^{k'}|\overline{x}_0|(|y_i|) + |\overline{x}_0|(|z|) \\ &\leqslant 2(n+1)\gamma + (n+1)\delta + \gamma \\ &\leqslant \frac{4}{\varepsilon}n\varepsilon\gamma + \frac{n+1}{\varepsilon}\varepsilon\delta \leqslant \frac{4\gamma + 2\delta}{\varepsilon}\varrho(x). \end{aligned}$$

Putting $\varepsilon/(4\gamma+2\delta) = \sigma$, we have

$$|\bar{x}_0|(|x|) \leqslant \frac{1}{\sigma}\,\varrho(x) \qquad \text{for all } x$$

such that $\varepsilon < \varrho(x) < +\infty$ and $x \in N_0^{\perp}$. Hence

$$\begin{aligned}\overline{m}(\sigma\bar{x}_0) &= \sup_{x\in N_0^{\perp}\cap S}\{\sigma|\bar{x}_0|(|x|) - \varrho(x)\} + \sup_{x\in N_0\cap S}\{\sigma|\bar{x}_0|(|x|) - \varrho(x)\}\\ &\leqslant \sup_{x\in N_0^{\perp}\cap S,\ \varrho(x)\leqslant\varepsilon}\{\sigma|\bar{x}_0|(|x|) - \varrho(x)\} + \sup_{x\in N_0\cap S}\{\sigma|\bar{x}_0|(|x|) - \varrho(x)\}\\ &\leqslant \sigma\gamma + \sup_{x\in N_0\cap S}\sigma|\bar{x}_0|(|x|) < +\infty,\end{aligned}$$

because N_0 is finite dimensional. Therefore (M.3) holds. Applying the same method as in Theorem 3.1, we can show that

$$m_\varrho(x) = \sup_{\bar{x}\in\bar{R}}\{|\bar{x}(x)| - \overline{m}(\bar{x})\} \qquad (x\in R)$$

is a perfect modular on R, q. e. d.

Finally, we deal with the general case, that is, no assumptions of the existence of atomic elements are made. The following theorem is nothing but a generalization of those of [7; Theorems 3.6]:

THEOREM 3.4. *Let R be a quasi-modular space with a quasi-modular ϱ and semi-regular. In order that R be a Banach space with a semi-continuous norm $\|\ \|_0$, it is necessary and sufficient that we be able to define a monotone complete modular m_ϱ on R. In this case the following three concepts of convergence:*

(i) $\|\cdot\|_0$-*convergence,*

(ii) ϱ-*convergence,*

(iii) m_ϱ-*convergence,*

are mutually equivalent.

The assertion of this theorem follows immediately from Theorems 3.1, 3.2, 3.3 and Lemma 2 (here we can prove the monotone completeness of $\|\cdot\|_0$ by putting $n_0(x) = \|\cdot\|_0$ and $n_\nu(x) = \varrho\left(\frac{1}{\nu}x\right)$ for each $\nu \geqslant 1$ and $x\in R$).

§ 4. Finite quasi-modulars. In this section, we shall explain the supplementary results regarding quasi-modulars.

A quasi-modular ϱ is said to be *finite* if the following condition is satisfied:

(ϱ.5) $\varrho(x) < +\infty$ for all $x\in R$.

THEOREM 4.1. *Let ϱ be a finite quasi-modular on R, and let R be a Banach space with a semi-continuous norm. Then the norm bounded linear functional is universally continuous. And hence R is semi-regular.*

Proof. Let f be a norm-bounded linear functional on R, and $\|\cdot\|$ be a complete and semi-continuous norm on R. Since $\|\cdot\|$ is semi-continuous, if we put

$$n_0(x) = \varrho(x) \tag{4.1}$$

and

$$n_\nu(x) = \|x\| \qquad (\nu = 1, 2, \ldots), \tag{4.2}$$

then (4.1) and (4.2) satisfy the conditions of Lemma 2. Hence there exist a finite dimensional space $N_0 \subset R$ and $\varepsilon,\ \gamma > 0$ such that

$$\varrho(x) \leqslant \varepsilon \text{ and } x\in(1-[N_0])R = R_1 \text{ imply } \|x\| \leqslant \gamma. \tag{4.3}$$

Let $[p_\lambda]\downarrow_{\lambda\in\Lambda} 0$. For every $x\in R_1$ and $\alpha > 0$, we have

$$\inf_{\lambda\in\Lambda}\varrho(\alpha[p_\lambda]x) = 0, \tag{4.4}$$

because of (3.4) of Lemma 1 and (ϱ.5). (4.4) implies by (4.3)

$$\inf_{\lambda\in\Lambda}|f(\alpha[p_\lambda]x)| \leqslant \gamma\|f\|. \tag{4.5}$$

Since α is arbitrary, (4.5) implies also

$$\inf_{\lambda\in\Lambda}|f([p_\lambda]x)| = 0 \qquad \text{for} \quad x\in R_1. \tag{4.6}$$

(4.6) means that f is universally continuous on R_1 (cf. 4, § 19). In a finite dimensional space N_0, any linear functional is universally continuous, therefore, f is universally continuous in the whole space R.

From this theorem and Theorem 3.4 we have

COROLLARY. *Let ϱ be a finite quasi-modular on R. Then R is a Banach space with a semi-continuous norm if and only if R has a complete modular.*

In this case, we do not assume that R is semi-regular.

THEOREM 4.2. *Let R be non-atomic. A quasi-modular ϱ on R is finite if and only if there exist ε and γ such that*

$$\varrho(2x) \leqslant \gamma\varrho(x) \quad \text{for all} \quad x\in R \text{ with } \varrho(x) > \varepsilon. \tag{4.7}$$

Theorem 4.2 is a generalization of Mazur-Orlicz's Theorem 4 in [7]. The proof of this theorem is similar to that of [1] or [12] and thus omitted.

We also have

THEOREM 4.3. *Let R be non-atomic and let ϱ and ϱ' be finite quasi-modulars on R. Then there exist ε, γ such that*

$$\varrho'(x) \leqslant \gamma\varrho(x) \quad \text{for all} \quad x\in R \text{ with } \varrho(x) > \varepsilon. \tag{4.8}$$

By the same method applied to [12], we have

THEOREM 4.4. *Let R be discrete. Then a quasi-modular ϱ is finite if and only if*

(4.9) $\varrho(\xi a) < +\infty$ *for all atomic elements* $a \epsilon R$,

(4.10) *there exist* $\varepsilon > \varepsilon' > 0$ *and* $\gamma > 0$ *such that* $\varepsilon > \varrho(x) > \varepsilon'$ *implies* $\varrho(2x) \leqslant \gamma\varrho(x)$.

In [6] H. Nakano has defined concave modulars. Theorem 4.2 of [6] is included in

THEOREM 4.5. *Let R be non-atomic with a finite quasi-modular ϱ such that $\varrho(a) \geqslant \varrho(b)$ if $|a| \geqslant |b|$ and $\lim\limits_{\xi\to\infty} \frac{\varrho(\xi a)}{\xi} < +\infty$ for every $a \epsilon R$. Then R is semi-regular if and only if*

$$\lim_{\xi\to\infty} \frac{\varrho(\xi a)}{\xi} > 0 \quad \textit{for every} \quad a > 0.$$

Proof. If R is semi-regular, there exists a modular m such that $m(x) \leqslant \varrho(x)$ for each $x \epsilon R$ by Theorem 3.1. Since

$$\lim_{\xi\to\infty} \frac{m(\xi x)}{\xi} > 0 \quad \text{for every} \quad x > 0,$$

it follows that

$$\lim_{\xi\to\infty} \frac{\varrho(\xi x)}{\xi} > 0 \quad \text{for every} \quad x > 0.$$

Conversely, if $\lim\limits_{\xi\to\infty} \frac{\varrho(\xi x)}{\xi} > 0$ for every $x > 0$, then, putting

$$p_1(x) = \lim_{\xi\to\infty} \frac{\varrho(\xi x)}{\xi} \quad \text{and} \quad p_2(x) = \inf_{0 \leqslant x_\lambda \uparrow_A |x|} \sup_{\lambda \epsilon A} p_1(x_\lambda),$$

we have

(4.11) $p_2(ax) = ap_2(x)$ for $a \geqslant 0$;

(4.12) $p_2(x+y) = p_2(x)+p_2(y)$ if $x \perp y$;

(4.13) $0 \leqslant x_\lambda \uparrow_A x$ implies $\sup\limits_{\lambda\epsilon A} p_2(x_\lambda) = p_2(x)$;

(4.14) $p_2(x) > 0$ if $x > 0$ (6).

---

(6) (4.14) follows from the facts that $p_1(x) > 0$ for any $0 < x \epsilon R$ and that for any $0 < a_{\mu,\nu} \uparrow_{\nu=1}^{\infty} x$ $(\mu = 1, 2, \ldots)$ there exist $0 < b \epsilon R$ and a subsequence $\{\nu_\mu\}_{\mu=1,2,\ldots}$ such that $b < a_{\mu,\nu_\mu}$ $(\mu = 1, 2, \ldots)$. Note that in this case R is totally continuous (cf. [4; § 14]).

From these facts we can see that $f(x) = p_2(x^+) - p_2(x^-)$ is a universally continuous linear functional on R such that $x > 0$ implies $f(x) > 0$. Hence R is semi-regular, q. e. d.

§ 5. Function spaces $L_{M(s,t)}$ and $l(M_n)$. In this section, we shall consider the function space $L_{M(s,t)}$ which is defined in § 1. Let $M(s,t)$ and $N(s,t)$ be real-valued functions satisfying (1.1), (1.2) and (1.3) in § 1.

LEMMA 3. *In order that $L_{N(s,t)} \subseteq L_{M(s,t)}$ it is necessary and sufficient that there exist positive numbers α, δ and a function $a(t)$ belonging to $L_1[0,1]$ such that*

(5.1) $N(\alpha s, t) \leqslant \delta M(s,t) + a(t)$

for all $s \geqslant 0$ and a. e. $t \epsilon [0,1]$.

The proof of Lemma 3 is essentially the same as in Theorem 1 in [2] and so it is omitted here.

Applying this lemma, we obtain a necessary and sufficient condition for the semi-regularity of $L_{M(s,t)}$, that is (cf. Theorem 4 in [10]).

THEOREM 5.1. *$L_{M(s,t)}$ is semi-regular if and only if*

(5.2) $$\varliminf_{s\to\infty} \frac{M(s,t)}{s} > 0 \quad \textit{for a. e.} \quad t \epsilon [0,1].$$

Proof. Let $L_{M(s,t)}$ be semi-regular; then for any measurable set e with $m_{es}(e) > 0$ we have a positive measurable function $\bar{a}(t) \epsilon \overline{L_{M(s,t)}}$ (i. e. the conjugate space of $L_{M(s,t)}$), whose support $e_a = \{t : \bar{a}(t) \neq 0\}$ is contained in e.

Putting $N(s,t) = \bar{a}(t)s$, we have $L_{M(s,t)} \subseteq L_{N(s,t)}$ and by (5.1)

(5.3) $s|\bar{a}(t)| \leqslant \delta M(s,t) + c(t)$

for all $s \geqslant 0$ and a. e. $t \epsilon [0,1]$, where $c(t) \epsilon L_1\ [0,1]$.

From (5.3) it follows that

$$\varliminf_{s\to\infty} \frac{M(s,t)}{s} > 0 \quad \text{for a. e.} \quad t \epsilon e_a.$$

Since e is an arbitrary measurable subset of $[0,1]$, we have (5.2).

Conversely let (5.2) be true. Then we put

(5.4) $$F(t) = \mathrm{Min}\left(1, \varliminf_{s\to\infty} \frac{M(s,t)}{s}\right).$$

$F(t)$ is a finite-valued measurable function, as can easily be seen on $[0,1]$ with $F(t)>0$ for a. e. $t\in[0,1]$. Let E_n be the set

$$E_n=\left\{t: t\in[0,1], \frac{M(m,t)}{m}>\tfrac{1}{2}F(t) \text{ for all } m\geqslant n\right\}.$$

Then $E_n\uparrow_n [0,1]$ and

$$\frac{M(s,t)}{s}>\tfrac{1}{4}F(t) \quad \text{for all} \quad s\geqslant n \text{ and } t\in E_n. \tag{5.5}$$

For the arbitrary measurable set e ($m_{es}(e)>0$) in $[0,1]$, there exists E_{n_0} with $m_{es}(e\cap E_{n_0})>0$. Now we put for every $0\leqslant x(t)\in L_{M(s,t)}$,

$$f(x)=\int\limits_{e\cap E_{n_0}} F(t)\cdot x(t)\,dt. \tag{5.6}$$

Then by (5.5) we obtain

$$\int\limits_{e\cap E_{n_0}} F(t)x(t)\,dt\leqslant n_0 m(e\cap E_{n_0})+4\int\limits_0^1 M\big(x(t),t\big)\,dt.$$

This shows that $f(x)$ defined by (5.6) is a universally continuous linear functional on $L_{M(s,t)}$. Since e is arbitrary, $L_{M(s,t)}$ is semi-regular.

Let $L_{M(s,t)}$ be a finite quasi-modular space with the quasi-modular

$$\varrho(x)=\int\limits_0^1 M\big(|x(t)|,t\big)\,dt.$$

We can find positive numbers ε, γ such that

$$\varrho(2x)\leqslant\gamma\varrho(x) \quad \text{for all} \quad x\in R \text{ with } \varrho(x)\geqslant\varepsilon,$$

by virtue of Theorem 4.3. Hence, by a similar method applied to the proof of Theorem 5.1, we can prove

THEOREM 5.2. *In order that $L_{M(s,t)}$ be finite, it is necessary and sufficient that there exist a positive number γ and a function $a_0(t)\in L_1[0,1]$ such that*

$$M(2s,t)\leqslant\gamma M(s,t)+a_0(t) \tag{5.7}$$

for all $s\geqslant 0$ and a. e. $t\in[0,1]$.

This theorem is considered as a generalization of that of [9] concerning the so-called Δ_2-condition.

Let $\Phi(s,t)$, $(s,t)\in[0,+\infty)\times[0,1]$ satisfy the following:

(Φ) $\Phi(s,t)$ is a convex function of $s\geqslant 0$ which is not identical to zero for a. e. $t\in[0,1]$,

in addition to (1.1)-(1.3). Such a function $\Phi(s,t)$ is called a *modular function* and $L_{\Phi(s,t)}$ is called a *modular function space*.

From Theorem 3.2, 5.1, 5.2 and Radon-Nikodym's Theorem we can infer

THEOREM 5.3. *In order that $L_{M(s,t)}$ (where $M(s,t)$ satisfies (5.2)) become a Banach space, it is necessary and sufficient that there exist a modular function $\Phi_0(s,t)$, $\gamma>0$ and $a_0(t)$ belonging to $L_1[0,1]$ such that*

$$\Phi_0(s,t)\leqslant M(s,t)\leqslant\Phi_0(\gamma s,t)+a_0(t) \tag{5.8}$$

for all $s\geqslant 0$ and a. e. $t\in[0,1]$.

We may choose as $\Phi_0(s,t)$ the maximal modular function Φ_0 among those Φ for which $\Phi(s,t)\leqslant M(s,t)$ for all $s\geqslant 0$ and a. e. $t\in[0,1]$.

Finally we comment shortly on a sequence space which can be considered as an example of a discrete quasi-modular space.

Let $M_n(\xi)$ $(n=1,2,\ldots)$ be the sequence of non-decreasing functions of $\xi\geqslant 0$. The sequence space $l(M_n)$ is the totality of all sequences $x=\{\xi_n\}$ $(n\geqslant 1)$ with

$$\varrho(ax)=\sum_{n=1}^{\infty}M_n(a|\xi_n|)<+\infty \quad \text{for some } a>0.$$

We easily see that $l(M_n)$ is a quasi-modular space with the quasi-modular $\varrho(x)=\sum_{n=1}^{\infty}M_n(|\xi_n|)$ for $x=\{\xi_n\}\in l(M_n)$.

Corresponding to Lemma 3, we have the following

LEMMA 4. *$l(M_n)\subseteq l(N_n)$ if and only if there exist positive numbers ε, γ and an integer ν and a sequence of positive numbers $\{a_n\}$ $(n\geqslant 1)$ with $\sum_{n=1}^{\infty}a_n<+\infty$ such that*

$$N_n\left(\frac{1}{\nu}\xi\right)\leqslant\gamma M_n(\xi)+a_n \quad \textit{for} \quad M_n(\xi)<\varepsilon. \tag{5.9}$$

To avoid repetition, we do not prove the above lemma. We also have

THEOREM 5.4. *$l(M_n)$ is finite if and only if there exist positive numbers ε, γ and a sequence of positive numbers $\{a_n\}$ $(n\geqslant 1)$ with $\sum_{n=1}^{\infty}a_n<+\infty$ such that*

$$M_n(2\xi)\leqslant\gamma M_n(\xi)+a_n \quad \textit{for} \quad M_n(\xi)<\varepsilon; \tag{5.10}$$

$$M_n(\xi)<+\infty \quad \textit{for each} \quad \xi\geqslant 0 \textit{ and } n\geqslant 1. \tag{5.11}$$

If $l(M_n)$ has a normal basis, we can see by Theorem 3.2 that $l(M_n)$ is a modular space with a modular $m_\varrho(x)$ and

$$m_\varrho(x) = \sum_{n=1}^{\infty} M'_n(\xi), \tag{5.12}$$

where $M'_n(\xi)$ is a convex function of $\xi \geqslant 0$ for every $n \geqslant 1$.

THEOREM 5.5. *Let $l(M_n)$ have a normal basis; then $l(M_n)$ is a Banach space if and only if there exists a sequence of convex function $\{M'_n\}$ $(n \geqslant 1)$ satisfying the following conditions:*

(5.13) $M'_n(\xi) \leqslant M_n(\xi)$ *for* $M'_n(\xi) \leqslant \varepsilon$;

(5.14) *there exist positive numbers ε, α, γ and a sequence of positive numbers a_n, $\sum_{n=1}^{\infty} a_n < +\infty$ such that*

$$M_n(\alpha\xi) \leqslant \gamma M'_n(\xi) + a_n \quad \textit{for} \quad M'_n(\xi) < \varepsilon.$$

We cannot find an explicit condition for the existence of a normal basis, but there exist many examples of quasi-modular spaces which have no normal basis. For instance: $(s) = \{$the space of all sequences$\}$ and $(d) = \{$the space of sequences whose coordinates are 0 except finite members$\}$ have no normal basis. (s) is not considered as a normed space, yet (d) is considered as a normed space. On (d), however, *no perfect norm can be defined*. In the case of l_p $(0 < p < 1)$, m_ϱ which is defined in Theorem 3.2 is l_1-type, i. e. $m_\varrho(x) = \sum_{i=1}^{\infty} |\xi_i|$ for $x = \{\xi_i\}_{(i \geqslant 1)}$ with $|\xi_i| \leqslant 1$ for all $i \geqslant 1$. Hence, l_1-space is the best possible, if we consider l_1 as a normed space $(0 < p < 1)$. Here l_p $(0 < p < 1)$ is a complete semi-normal manifold of l_1.

References

[1] I. Amemiya, *A generalization of the Theorem of Orlicz and Birnbaum*, Jour. Fac. Sci. Hokkaido Univ. 13, No. 2 (1956), p. 60-64.

[2] J. Ishii, *On equivalence of Modular Function Spaces*, Proc. Japan Acad. 35 (1959), p. 551-556.

[3] T. Mori, I. Amemiya and H. Nakano, *On the reflexivity of semi-continuous norms*, Proc. Japan Acad. 31, No. 10 (1955), p. 684-685.

[4] H. Nakano, *Modulared semi-ordered linear spaces*, Tokyo, 1950.

[5] — *Modulared linear spaces*, Jour. Fac. Sci. Tokyo Univ. 6 (1951), p. 85-131.

[6] — *Concave modulars*, Journal Math. Soc. Japan 5, No. 1 (1953), p. 29-49.

[7] S. Mazur and W. Orlicz, *On some classes of linear metric spaces*, Studia Math. 17 (1958), p. 97-119.

[8] J. Musielak and W. Orlicz, *On modular spaces*, ibidem 18 (1959), p. 49-65.

[9] Z. B. Birnbaum and W. Orlicz, *Ueber die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen*, ibidem 3 (1931), p. 1-67.

[10] S. Rolewicz, *Some remarks on the space $N(L)$ and $N(l)$*, ibidem 18 (1959), p. 1-9.

[11] T. Shimogaki, *A generalization of Vainberg's Theorem* 1, Proc. Japan Acad. 34 (1958), p. 518-523.

[12] T. Shimogaki, *Note on Orlicz-Birnbaum-Amemiya's Theorem*, ibidem 33 (1957), p. 676-680.

DEPARTMENT OF MATHEMATICS
HOKKAIDÔ UNIVERSITY, SAPPORO

Reçu par la Rédaction le 10. 5. 1960.