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On normed semialgebras

by

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1. Preliminaries. In this paper we shall use the term *semiring* in the sense stated by the author in an earlier one [1]. For the sake of completeness, we repeat that a semiring is a system consisting of a set S and two binary operations in S called *addition* and *multiplication* such that

(a) S together with addition is a semigroup;

(b) S together with multiplication is a semigroup;

(c) the left- and right-hand distributive laws $a(b+c) = ab+ac$ and $(b+c)a = ba+ca$ hold.

Semigroup is used in the sense of a closed associative system. We shall assume that the additive semigroup is commutative and that S possesses a zero element 0, $0+s = s$ and $0s = 0s = 0$, for every s in S. If both semigroups of a semiring are commutative, we say that the semiring is *commutative*. Following Słowikowski and Zawadowski [15], we state a commutative semiring S is *positive*, if S possesses a unit element e and $e+s$ has an inverse in S, for every s in S.

In the body of this paper we shall make use of some facts about maximal ideals in semirings. Since the reference [6] is not readily available, we shall take the liberty of repeating some of the pertinent theorems and their proofs.

There are many suggested definitions for an ideal in a semiring. In fact, in his thesis, Bugenhagen [6] concerned himself mainly with "a comparison of three definitions of ideals in semirings". For the record, we shall use our own definition given in reference [1], cited above, and which is the better one from the point of view of structure theory, our main interest. We repeat that *a right ideal of S is a subset I of S closed under addition, such that $is \in I$, for any $i \in I$ and any $s \in S$.* When the term ideal occurs in this paper, we are using it to mean a two-sided ideal. Also, $a \equiv b(I)$, I an ideal of S if and only if there exist elements $i_1, i_2 \in I$, such that $i_1+a = i_2+b$ [1]. This equivalence relation partitions S into classes $C_a, \ldots,$ where $C_a = \{x | x \equiv a(I)\}$. Relative to the usual definitions of addition and multiplication of classes, $C_a, \ldots$ form a semiring, symbolized by S/I. *The big difference here is that C_a cannot be necessarily written as $a+I$.* However, *in the case that addition is commutative I is contained in a congruence class which is denoted by C_I. C_I is an ideal and $S/C_I = S/I$.* This fact points up the importance of the assumption of commutativity of the additive semigroup of the semiring S. As in the ring case, we call the semiring of equivalence classes $C_a, \ldots$ the quotient semiring determined by I and symbolize it by S/I. As per usual an ideal is called *maximal* if it is not properly contained in any proper ideal of S. Here, either $C_M = S$ or $C_M = M$. A division semiring is a semiring in which the elements $\neq 0$, form a multiplicative group [2].

Definition 1. A *semifield* is a commutative division semiring.

Since Bugenhagen's thesis is not easily accessible, we now proceed to give the statements and proofs of some theorems pertinent to our theory.

THEOREM. *If S is a commutative semiring with zero, M a maximal ideal for which $x^2 \epsilon M$ implies $x \epsilon M$, then S/M is a semifield.*

Proof. Either $C_M = S$ or $C_M = M$. In the case $C_M = S$ the result is trivial. Let us suppose $C_M \neq S$ and thus $C_M = M \subset S$. Let a be a particular element of S, $a \notin M$, then $a^2 \notin M$ and $Sa \not\subset M$. The commutativity of S implies that Sa is an ideal and furthermore $M+Sa$ is an ideal. Trivially, $M \subseteq M+Sa$, but $a^2 \notin M$ and $a^2 = 0+a^2 \epsilon M+Sa$ and thus $M \subset M+ +Sa$. Since M is maximal then $M+Sa = S$ and $s = m+ta$, for any $s \epsilon S$ and some $m \epsilon M$ and $t \epsilon S$. Set $s = m'+b$, $b \epsilon S$ and $m' \epsilon M$, then $m'+b = m+ta$. Therefore $ta \equiv b(M)$ and $C_t C_a = C_b$ in S/M. According to Huntington [11], the elements $\neq C_M$ form a multiplicative group.

THEOREM. *If S is a commutative semiring with zero 0 and unit e, and M is a maximal ideal of S, then $m^2 \epsilon M$ implies $m \epsilon M$.*

Proof. Let $m^2 \epsilon M$, $m \notin M$ and (m) the principal ideal generated by m [1]. Now $M \subseteq M+(m)$, but $m = 0+me \epsilon M+(m)$, $m \notin M$, thus $M \subset M+ +(m)$. Since M is maximal in S, then $M+(m) = S$. Therefore, for any $s \epsilon S$, $s = m'+tm$ for some $t \epsilon S$ and $m' \epsilon M$. Specifically, $e = m'+tm$. On multiplying both sides of this equation by m we have that $m = mm'+ +tm^2$. Since $m^2 \epsilon M$, then $m \epsilon M$, which contradicts our assumption.

Thus, in the case S has a unit e, the assumption that the maximal ideal M containing the square of an element automatically contains the element itself is redundant. Hence, as in the ring case for a commutative semiring with zero and unit, M maximal in S implies that S/M is a semifield.

Henriksen [10] pointed out that there is little loss of generality in assuming that a semiring has a zero. In [9] he called attention to the fact that if S has a unit, an ideal in a semiring sense is also an ideal in a ring sense and conversely.

Definition 2. A *halfring* H is a semiring which is embeddable in a ring.

Zassenhaus [17] gave an equivalent definition in his classic monograph on groups. Since addition is commutative in our semiring S, a necessary and sufficient condition for S to be a halfring is that the additive semigroup of S be cancellative. Examples of halfrings are the non-negative integers P^+ and the non-negative rationals Q^+.

Let H be a halfring. Following [4], we construct the ring $\mathfrak{R}$ in which H is embedded. The product set $H \times H$ again forms a halfring according to the laws of addition and multiplication:

$$(1) \qquad \begin{aligned} (i_1, j_1)+(i_2, j_2) &= (i_1+i_2, j_1+j_2), \\ (i_1, j_1)(i_2, j_2) &= (i_1 i_2+j_1 j_2, i_1 j_2+i_2 j_1). \end{aligned}$$

The diagonal $\Delta = \{(x, x) | x \epsilon H\}$ of H is an ideal $H \times H$.

We define the following equivalence modulo Δ: $(i_1, j_1) \equiv (i_2, j_2)(\Delta)$ if and only if there exist elements (x, x) and (y, y) in Δ such that

$$(2) \qquad (i_1, j_1)+(x, x) = (i_2, j_2)+(y, y).$$

The quotient ring $\mathfrak{R} = H \times H/\Delta$ is called the *ring generated by H*. Let ν denote the natural homomorphism of $H \times H$ onto $\mathfrak{R}$, then the halfring H is embedded in the ring $\mathfrak{R}$, for the mapping $h \leftrightarrow \nu(h+a, a)$, for any a, is an isomorphism of H into $\mathfrak{R}$. We designate by $\nu(H)$ this isomorphic image of H in $\mathfrak{R}$ and by $\nu(s, t)$ the equivalence class of (s, t). In order to construct the ring $\mathfrak{R}$ generated by H, it is not necessary to assume that H possess a zero, for $\mathfrak{R}$ automatically acquires a zero, the class Δ.

As in [3] we give

Definition 3. A *topological semiring* is a semiring S together with a Hausdorff topology on S under which the semiring operations are continuous.

If in the above definition S is a halfring, we refer to it as a topological halfring.

Definition 4. A *halffield* H is a semifield which is embeddable in a field.

Examples of halffields are the non-negative rationals Q^+ and the non-negative reals R^+.

2. Introduction. Gelfand [8] defined a commutative real normed ring $\mathfrak{R}$ as a set $x, y, \ldots$ satisfying the following conditions:

(a) $\mathfrak{R}$ is a commutative algebra over the field R of real numbers.

(b) $\mathfrak{R}$ as a vector space gotten by considering only the operations addition and multiplication by scalars is a Banach space.

(c) $\mathfrak{R}$ has an identity e with respect to multiplication.

(d) x^2 is quasi-regular for every $x \epsilon \mathfrak{R}$.

By embedding $\mathfrak{R}$ in its complexification R, he showed that *the quotient ring of a commutative real normed ring $\mathfrak{R}$ by a maximal ideal is the field of real numbers* and that *each commutative real normed ring $\mathfrak{R}$ can be mapped homomorphically into some ring of real-valued continuous functions on a compact space, so that the kernel of the homomorphism is the radical of the ring* [8]. In the proofs of these theorems, Gelfand made use of the Mazur [14] basic theorem for normed rings, that *every complete normed division ring is isomorphic to the field of complex numbers*.

Tornheim [16] removed the necessity of completeness in his proof that every normed field over the real field R is either the real field R or the complex field C.

In section 3 we introduce the concept of a normed halfring H over the halffield of non-negative reals R^+ and show that H is embeddable in a normed ring $\mathfrak{R}$ over the real field R. In section 4 we extend the Mazur theorem to read that *every positive normed halffield H, in which $(s_1^2+s_2^2, 2s_1s_2)$ is semi-regular for any $s_1, s_2 \epsilon H$ is isomorphic to the halffield of non-negative reals R^+*. The above mentioned Gelfand theorem is extended to read that *each positive normed halfring H, in which $(s_1^2+s_2^2, 2s_1s_2)$ is semi-regular for any $s_1, s_2 \epsilon H$, can be mapped homomorphically into some halfring of real-valued non-negative continuous functions on a compact space.*

3. Normed halfrings. In agreement with Iizuka [12] we give.

Definition 5. A commutative semigroup S with zero is called a *left Σ-semimodule* if and only if Σ is a semiring and a law of composition $\Sigma \times S$ into S is defined, which, for $\sigma, \tau \epsilon \Sigma$ and $s, t \epsilon S$ satisfies (a) $\sigma(s+t) = \sigma s + \sigma t$, (b) $(\sigma+\tau)s = \sigma s + \tau s$, (c) $(\sigma\tau)s = \sigma(\tau s)$.

If Σ has a unit 1 and $1s = s$ for all $s \epsilon S$, then S is called a *unital* left Σ-semimodule.

In the following, we introduce the concept of a semialgebra.

Definition 6. A semiring S is said to be a *semialgebra*, over a commutative semiring Σ with unit, if a law of composition $(\sigma, s) \to \sigma s$ of the product set $\Sigma \times S$ is defined such that

(i) $(S, +)$ is a unital left Σ-semimodule relative to the composition $(\sigma, s) \to \sigma s$,

(ii) for all $\sigma \epsilon \Sigma$ and $s, t \epsilon S$, $\sigma(st) = (\sigma s)t = s(\sigma t)$.

In the case that S is an arbitrary ring and Σ is a commutative ring with unit, then our concept of a semialgebra coincides with the concept of an algebra given by Jacobson [13]. As is with algebras, when we wish to apply the notion of homomorphism we restrict our semialgebras to semialgebras over the same commutative semiring Σ. Thus this notion will be a mapping which is both a semiring homomorphism and a Σ-semimodule homomorphism T, i. e., if S and S' are Σ-semimodules, then

(i) T is a semigroup homomorphism of $(S, +)$ into $(S', +)$,

(ii) T is homogeneous on S.

We indicate this briefly by referring to T as a Σ-homomorphism. The terms Σ-isomorphism, Σ-endomorphism, and Σ-automorphism are similarly defined.

Definition 7. A *semivector space* is a semialgebra over a semifield.

Definition 8. A *semilinear space* S is a semivector space over the halffield of non-negative reals R^+.

Definition 9. A *metric* for a semilinear space S is a function d defined on $S \times S$ to R^+ satisfying for $s, t, u \epsilon S$ and $\varrho \epsilon R^+$

(1) $$d(s, t) = d(t, s),$$

(2) $$d\varrho(s, t) = \varrho d(s, t),$$

(3) $$d(s, u) \leqslant d(s, t) + d(t, u),$$

(4) $$d(s, t) = 0 \text{ if and only if } s = t.$$

In this case S is said to be a *metric* semilinear space.

Definition 10. A metric for a semilinear space S is said to be *invariant* if and only if $d(s+x, t+x) = d(s, t)$ for all $s, t, x \epsilon S$.

Lemma 1. *A semilinear space S with an invariant metric is a topological semiring.*

Proof. Let $s_n \to s$, $t_n \to t$, then $d(s_n, t) \to 0$ and $d(t_n, t) \to 0$. Now $d(s_n+t_n, s+t) \leqslant d(s_n+t_n, s+t_n) + d(s+t_n, s+t) = d(s_n, s) + d(t_n, t)$, for d is invariant. Hence, $d(s_n+t_n, s+t) \to 0$ and $s_n+t_n \to s+t$.

Similarly, $d(s_nt_n, st) \leqslant d(s_nt_n, st_n) + d(st_n, st) = d(s_n, s) + d(t_n, t)$. Again, $d(s_nt_n, st) \to 0$ and $s_nt_n \to st$. Also, $d(\varrho s_n, \varrho s) = \varrho d(s_n, s)$ and $\varrho s_n \to \varrho s$, $\varrho \epsilon R^+$.

Definition 11. A *norm* for a semilinear space S is a non-negative real-valued function $\|s\|$ satisfying for $s, t \epsilon S$ and $\varrho \epsilon R^+$

(i) $\|s\| \geqslant 0$,

(ii) $\|s\| = 0$ if and only if $s = 0$,

(iii) $\|\varrho s\| = \varrho \|s\|$,

(iv) $\|s+t\| \leqslant \|s\| + \|t\|$.

In this case, S is said to be a *normed* semilinear space.

Lemma 2. *A semilinear space with an invariant metric is a normed semilinear space.*

Proof. We define $\|s\| = d(s, 0)$. Conditions (i)-(iii) for a norm are obviously fulfilled. Now $\|s+t\| = d(s+t, 0) \leqslant d(s+t, t) + d(t, 0) = d(s, 0) + d(t, 0) = \|s\| + \|t\|$, for d is invariant.

In the case of a normed linear space a topology on the space is defined by its norm. However, this is not the case for a normed semilinear space, for the norm does not define an invariant metric on the space. Hence, we shall confine our study to semilinear spaces with an invariant metric.

Definition 12. A set S of elements $s, t, \ldots$ is a *normed semiring* lif and only if

(1) S is a semialgebra over the halffield of non-negative reals R^+.
(2) S is a semilinear space with an invariant metric d.
(3) $\|st\| \leqslant \|s\|\|t\|$, where $\|s\| = d(s, 0)$, for any $s, t \in S$.
(4) If S possesses a unit e, then $\|e\| = 1$.

If in definition 12, the semiring S is a halfring H, we shall refer to it as a normed halfring.

Examples of normed halfrings are the following:

(1) Let $C^+(T)$ be the halfring of all non-negative real-valued functions $x(t), y(t), \ldots$ on a compact space T, with the usual operations of addition and multiplication. We define the invariant metric $d(x, y)$ on $C^+(T)$ by the formula $d(x, y) = \sup_{t \in T} |x(t) - y(t)|$ and the norm $\|x\| = d(x, 0)$.

(2) Let W^+ be the halfring of convergent series

$$x(t) = \sum_{n=0}^{\infty} \varrho_n e^{\mathrm{int}}, \quad y(t) = \sum_{n=0}^{\infty} \sigma_n e^{\mathrm{int}}, \quad \varrho_n, \sigma_n \in R^+,$$

with addition, multiplication, and scalar multiplication as the operations on $x(t)$. We give W^+ the invariant metric

$$d(x, y) = \sum_{n=0}^{\infty} |\varrho_n - \sigma_n|$$

and norm it with $\|x\| = d(x, 0)$.

LEMMA 3. *If H is a normed halfring, then the halfring $H \times H$ is a normed halfring over R^+ with invariant metric $D\big((s_1, s_2), (t_1, t_2)\big) = d(s_1, t_1) + d(s_2, t_2)$ and $\|(s_1, s_2)\| = \|s_1\| + \|s_2\|$.*

Proof. A straightforward verification.

The ideal Δ in $H \times H$ is a closed set in the product topology [7].

LEMMA 4. *The ring $\mathfrak{R}$ generated by the normed halfring H is a normed ring over the real field R, with norm*

$$\|\nu(s_1, s_2)\| = \inf_{(u,v) \in \nu^{-1}\nu(s_1, s_2)} \|u, v)\|. \tag{3}$$

Proof. We verify that the conditions for a norm are fulfilled by $\|\nu(s_1, s_2)\|$

(i) $\|v(s_1, s_2)\| \geqslant 0$.

(ii) $\|\Delta\| = 0$. Let $\|\nu(s_1, s_2)\| = 0$, then there exist sequences $s_{in} \to 0$, $i = 1, 2$, such that $(s_{1n}, s_{2n}) \equiv (s_1, s_2)(\Delta)$. The latter condition implies that there exist elements (x_n, x_n), (y_n, y_n) such that $(s_{1n}, s_{2n}) + (x_n, x_n) = (s_1, s_2) + (y_n, y_n)$, or equivalently $s_{n1} + s_2 = s_{2n} + s_1$. The continuity of addition in H yields that $s_1 = s_2$ and $\nu(s_1, s_2) = \nu(s_1, s_1) = \Delta$.

(iii) Since H contains e, we may identify ϱe in H with ϱ in R^+. Now $\|\varrho\nu(s_1, s_2)\| = \|\nu(a, \varrho e + a)\nu(s_1, s_2)\| = \|\nu(\varrho s_2 + b, \varrho s_1 + b)\| = \|\nu(\varrho s_2, \varrho s_1\| = -\varrho \inf_{(u,v) \in \nu^{-1}\nu(s_1, s_2)} = -\varrho\|\nu(s_1, s_2)\|$, for $\varrho \in R^-$. Similarly for $\varrho \in R^+$. Thus, $\|\varrho\nu(s_1, s_2)\| = |\varrho|\,\|v(s_1, s_2)\|$, $\varrho \in R$.

(iv) For every $\varepsilon > 0$, there exist $(u_1, u_2) \in \nu^{-1}\nu(s_1, s_2)$ and $(\nu_1, \nu_2) \in \nu^{-1}\nu(t_1, t_2)$ such that $\|(u_1, u_2)\| \leqslant \|\nu(s_1, s_2\| + \varepsilon$ and $\|(v_1, v_2)\| \leqslant \|\nu(t_1, t_2)\| + \varepsilon$. Then $\|\nu(s_1, s_2) + \nu(t_1, t_2)\| = \|\nu(s_1 + t_1, s_2 + t_2)\| \leqslant \|(u_1 + u_2, v_1 + v_2)\| = \|u_1 + u_2\| + \|v_1 + v_2\| \leqslant \|u_1\| + \|u_2\| + \|v_1\| + \|v_2\| = \|(u_1, u_2)\| + \|(v_1, v_2)\|$. Therefore, $\|\nu(s_1, s_2) + \nu(t_1, t_2) \leqslant \|\nu(s_1, s_2)\| + \|\nu(t_1, t_2)\| + 2\varepsilon$. Since ε is arbitrary, $\|\nu(s_1, s_2) + \nu(t_1, t_2)\| \leqslant \|\nu(s_1, s_2)\| + \|\nu(t_1, t_2)\|$.

We verify that $\|\nu(s_1, s_2)\nu(t_1, t_2)\| \leqslant \|\nu(s_1, s_2)\|\|\nu(t_1, t_2)\|$. Let (u_1, u_2) and (v_1, v_2) be defined as in (iv), then $\|\nu(s_1, s_2)\nu(t_1, t_2)\| \leqslant \|(u_1, u_2)(v_1, v_2)\| = \|(u_1v_1 + u_2v_2, u_1v_2 + u_2v_1)\| \leqslant \|(u_1v_1, u_1v_2)\| + \|(u_2v_2, u_2v_1)\| \leqslant \|u_1\|\|v_1\| + \|u_1\|\|v_2\| + \|u_2\|\|v_2\| + \|u_2\|\|v_1\| = (\|u_1\| + \|u_2\|)(\|v_1\| + \|v_2\|) = \|(u_1, u_2)\|\|(v_1, v_2)\|$. Therefore, $\|\nu(s_1, s_2)\nu(t_1, t_2)\| \leqslant \|\nu(s_1, s_2)\|\|\nu(t_1, t_2) + \varepsilon\big(\|\nu(s_1, s_2)\| + \|\nu(t_1, t_2)\|\big) + \varepsilon^2$. Since ε is arbitrary, then $\|\nu(s_1, s_2)\nu(t_1, t_2)\| \leqslant \|\nu(s_1, s_2)\|\|\nu(t_1, t_2)\|$.

As in [7], we prove

LEMMA 5. *The mapping ν of $H \times H$ onto $\mathfrak{R}$ is open, and $\mathfrak{R}$ is a topological ring.*

Proof. Let $\nu(s_{1n}, s_{2n}) \to \nu(s_1, s_2)$. Then

$$\|\nu(s_2 + s_{1n}, s_1 + s_{2n})\| = \inf_{(u,v) \in \nu^{-1}\nu(s_2 + s_{1n}, s_1 + s_{2n})} \|u, v)\| \to 0,$$

yields sequences $v_{1n}, v_{2n} \in H$, such that $\|v_{1n}\| \to 0$, $\|v_{2n}\| \to 0$ and $(v_{1n}, v_{2n}) \equiv (s_2 + s_{1n}, s_1 + s_{2n})(\Delta)$, or equivalently $\underline{v_{1n} + s_1} + s_{2n} = \underline{v_{2n} + s_2} + s_{1n}$. Let $u_{1n} = v_{1n} + s_1$ and $u_{2n} = v_{2n} + s_2$, then $(u_{1n}, u_{2n}) \equiv (s_{1n}, s_{2n})(\Delta)$ and $(u_{1n}, u_{2n}) \to (s_1, s_2)$. This implies that the mapping ν is open.

We introduce in $\mathfrak{R}$ the quotient topology, that is, the largest topology for $\mathfrak{R}$ such that the projection (quotient map) ν is a continuous mapping of $H \times H$ onto $\mathfrak{R}$. Since ν is open, the operations in H continuous and the topology in H is Hausdorff, then $\mathfrak{R}$ is a topological ring [7, theorem 4]. As a consequence of lemmas 4 and 5, we have

THEOREM 1. *A normed halfring H over the non-negative reals R^+ is embeddable with preservation of norm in the normed ring $\mathfrak{R}$ over the reals R.*

4. Positive halfrings.

Following Bourne [1], we state

Definition 13. A pair of elements (s_1, s_2) of the halfring H is said to be *semi-regular* if there exists a pair of elements (t_1, t_2) in H such that

$$s_1 + t_1 + s_1t_1 + s_2t_2 = s_2 + t_2 + s_2t_1 + s_1t_2. \tag{4}$$

LEMMA 6. *If H is commutative and $(s_1^2+s_2^2, 2s_1s_2)$ is semi-regular, for any $s_1, s_2 \in H$ the square of any element of the ring $\mathfrak{R}$ is quasi-regular.*

Proof. Since $(s_1^2+s_2^2, 2s_1s_2)$ is semi-regular in H, we have that (t_1, t_2) exists in H such that

$$(s_1^2+s_2^2)+t_1+(s_1^2+s_2^2)t_1+2s_1s_2t_2 = 2s_1s_2+t_2+2s_1s_2t_1+(s_1^2+s_2^2)t_2.$$

This implies that

$$(s_1, s_2)^2+(t_1, t_2)+(s_1, s_2)^2(t_1, t_2)+(x, x) = (y, y)$$

and

$$[\nu(s_1, s_2)]^2+\nu(t_1, t_2)+[\nu(s_1, s_2)]^2[\nu(t_1, t_2)] = 0.$$

Hence the square of the element of $\mathfrak{R}$ is quasi-regular.

From here on, H is isomorphically and homeomorphically embedded in $\mathfrak{R}$. For the sake of simplification of notation, we shall write s for $\nu(s_1, s_2)$, and when s in H, rather than write $\nu(s+a, a)$ we shall simply state this fact. The condition that s^2 shall be quasi-regular in $\mathfrak{R}$ is precisely the one Gelfand [8] assumed and makes $\mathfrak{R}$ a commutative real normed ring in the sense of Gelfand. We recall that *the quotient ring of a commutative real normed ring by a maximal ideal is the field of real numbers* [8]. Let $\mathfrak{M}_{\mathfrak{R}}$ be the set of maximal ideals of the ring $\mathfrak{R}$. We denote the natural homomorphism of $\mathfrak{R}$ onto $\mathfrak{R}/M$, $M \in \mathfrak{M}_{\mathfrak{R}}$, by φ_M. If we hold s fixed and let M vary over $\mathfrak{M}_{\mathfrak{R}}$, we obtain a real-valued function $f_s(M) = \varphi_M(s)$, defined on $\mathfrak{M}_{\mathfrak{R}}$. *The mapping $s \to f_s$ is an $\mathfrak{R}$-homomorphism of the commutative real normed ring $\mathfrak{R}$ into a ring of real-valued continuous functions on a compact space so that the kernel of the homomorphism is the radical of the ring* [8]. It is the canonical mapping of the Gelfand theory. We shall now prove the following basic result:

THEOREM 2. *If H is a normed commutative positive halfring, in which $(s_1^2+s_2^2, 2s_1s_2)$ is semi-regular for any $s_1, s_2 \in H$, then the quotient semiring of H by a maximal ideal is the halffield of non-negative reals R^+.*

Proof. Let $\mathfrak{R}$, $\mathfrak{M}_{\mathfrak{R}}$, and $f_s(M)$ be defined as above. If s is an element of H, then the positive nature of H implies that $1+f_s(M) \neq 0$, for $\mathfrak{R}/M$ is the real field. If $f_s(M) = -\varrho$, ϱ a positive real number, then $f_{s/\varrho}(M) = -1$, a contradiction. Thus, if $s \in H$, then $f_s(M)$ is non-negative.

For each $M \in \mathfrak{M}_{\mathfrak{R}}$ there exists a proper homomorphism of H into the halffield of non-negative real numbers R^+, which in turn determines a maximal ideal $M^+ \in \mathfrak{M}_H$, the set of maximal ideals of the halfring H [6], such that $f_s(M^+) = f_s(M)$, for $s \in M$. If $s \in M^+$, then $0 = f_s(M^+) = f_s(M)$, which implies that $s \in M$. Hence, $M^+ = H \cap M$.

We now show that every ideal of $\mathfrak{M}_H$ is obtained in this fashion. Let $M^+ \in \mathfrak{M}_H$ and M be the ideal of $\mathfrak{R}$ generated by M^+. M consists of all differences m_1-m_2, with $m_1, m_2 \in M^+$. M is a maximal ideal for the mapping which associates to each element $s_1-s_2 \in \mathfrak{R}$ the real number $f_{s_1-s_2}(M^+) = f_{s_1}(M^+)-f_{s_2}(M^+)$ defines a proper homomorphism of $\mathfrak{R}$ into R, with M as kernel. Hence, $H \cap M = M^+$. Since M is the minimal such ideal of $\mathfrak{R}$, M^+ is contained in no other ideal of $\mathfrak{M}_{\mathfrak{R}}$.

We have set up a 1-1 correspondence between the sets $\mathfrak{M}_H$ and $\mathfrak{M}_{\mathfrak{R}}$, such that $f_s(M^+) = f_s(M)$, for any $s \in H$. Since $f_s(M)$, $s \in H$, is non-negative, the quotient semiring H/M^+ is the halffield of non-negative reals R^+.

We topologize $\mathfrak{M}_H$ after the manner of Gelfand [8]. It is the weakest topology in which the functions $f_s(M^+)$ are continuous, and $\mathfrak{M}_H$ is a compact Hausdorff space. Since the halfring H generates the ring $\mathfrak{R}$, $\mathfrak{M}_H \leftrightarrow \mathfrak{M}_{\mathfrak{R}}$, and $f_s(M^+) = f_s(M)$, $M^+ \leftrightarrow M$, $s \in H$, the topology of $\mathfrak{M}_H$ is the same as that of $\mathfrak{M}_{\mathfrak{R}}$. Hence, we have

THEOREM 3. *If H is a normed positive halfring, in which $(s_1^2+s_2^2, 2s_1s_2)$ is semi-regular for any $s_1, s_2 \in H$, then there exists a homomorphism of H into the halfring of real-valued non-negative continuous functions on a compact Hausdorff space.*

Definition 14. The *spectral norm* $|s|$, $s \in H$, is given by the formula

$$|s| = \sup_{M^+ \in \mathfrak{M}_H} f_s(M^+).$$

Since $f_s(M^+) = f_s(M)$, $s \in H$ and $M \in \mathfrak{M}_{\mathfrak{R}}$, it follows that

$$|s| = \sup_{M \in \mathfrak{M}_{\mathfrak{R}}} f_s(M) = \lim_{n\to\infty} \|s^n\|^{1/n}.$$

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Banach spaces of Lipschitz functions

by

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§ 1. Introduction. If $0<\alpha<1$, $\mathrm{Lip}\,\alpha$ is the space of all complex valued continuous functions on the real line R of period 1 with

$$\sup_{\sigma\in R}|f(\sigma+\tau)-f(\sigma)| = O(|\tau|^{\alpha}) \quad \text{as} \quad \tau\to 0.$$

$\mathrm{lip}\,\alpha$ is the subset of $\mathrm{Lip}\,\alpha$ consisting of those f with

$$\sup_{\sigma\in R}|f(\sigma+\tau)-f(\sigma)| = o(|\tau|^{\alpha}) \quad \text{as} \quad \tau\to 0.$$

Supplied with the norm $\|\cdot\|_{\alpha}$ defined by

$$\|f\|_{\alpha} = \sup_{\varrho,\sigma,\tau}\left\{|f(\varrho)|,\ \frac{|f(\sigma+\tau)-f(\sigma)|}{|\tau|^{\alpha}}\right\},$$

$\mathrm{Lip}\,\alpha$ is a Banach space and $\mathrm{lip}\,\alpha$ is a closed linear subspace (¹).

We show in § 2 that the Banach space $\mathrm{Lip}\,\alpha$ is canonically isomorphic and isometric to the second dual space of the Banach space $\mathrm{lip}\,\alpha$. In § 3 we identify the extreme points of the unit sphere of the dual of $\mathrm{lip}\,\alpha$ and obtain as a consequence in § 4 the fact that $\mathrm{lip}\,\alpha$ has no isometries in addition to the expected ones.

§ 2. $\mathrm{Lip}\,\alpha$ is the second dual of $\mathrm{lip}\,\alpha$. Two definitions are necessary before we are able to state the main result of this section. For each σ in R, we define the functional Φ_{σ} in the dual space $(\mathrm{lip}\,\alpha)^*$ of $\mathrm{lip}\,\alpha$ by

$$\Phi_{\sigma}(f) = f(\sigma), \qquad f\in\mathrm{lip}\,\alpha.$$

For each functional F in the dual space $(\mathrm{lip}\,\alpha)^{**}$ of $(\mathrm{lip}\,\alpha)^*$, we define the function $\hat{F}$ on R by

$$\hat{F}(\sigma) = F(\Phi_{\sigma}), \qquad \sigma\in R.$$

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(¹) In [3] it is shown that $\mathrm{lip}\,\alpha$ is the closed linear subspace of $\mathrm{Lip}\,\alpha$ spanned by trigonometric polynomials.