then Tg is continuous and

$$e^{2\pi it(\sigma)} = g(t(\sigma)) = \Phi_{t(\sigma)}(g)$$
$$= \lambda^{-1}(T^*\Phi_\sigma)(g) = \lambda^{-1}\Phi_\sigma(Tg) = \lambda^{-1}Tg(\sigma), \quad -\tfrac{1}{2} < \sigma < +\tfrac{1}{2}.$$

It is now clear that t must satisfy either (4.6) or (4.7) since it is one-one continuous and satisfies (4.8).

We are now able to complete the proof of Theorem 4.1. Suppose that the mapping t satisfies (4.6). Then if f is any function in $\operatorname{lip}\alpha$,

$$Tf(\sigma) = \Phi_\sigma(Tf) = (T^*\Phi_\sigma)(f)$$
$$= \lambda\Phi_{t(\sigma)}(f) = \lambda f(t(\sigma)) = \lambda f(\varrho+\sigma), \quad -\tfrac{1}{2} < \sigma < +\tfrac{1}{2},$$

and as a consequence,

$$Tf(\sigma) = \lambda f(\varrho+\sigma), \quad \sigma \in R,$$

for all f in $\operatorname{lip}\alpha$.

Similarly, if the mapping t satisfies (4.7), then

$$Tf(\sigma) = \lambda f(\varrho-\sigma), \quad \sigma \in R,$$

for all f in $\operatorname{lip}\alpha$.

Bibliography

[1] N. Bourbaki, *Integration*, Éléments de Mathématique, XIII, Book VI, Paris 1952.

[2] N. Dunford and J. Schwartz, *Linear Operators*, Part I, New York 1958.

[3] H. Mirkil, *Continuous translation of Hölder and Lipschitz functions*, Canadian Journal of Mathematics 12 (1960), p. 674-685.

[4] A. Zygmund, *Trigonometric Series*, Cambridge, 1959.

Reçu par la Rédaction le 14. 11. 1960

STUDIA MATHEMATICA, T. XXI. (1961)

A remark on an imbedding theorem of Kondrashev type

by

P. SZEPTYCKI (Pretoria)

1. The present note may be considered as the second part of Paper [1]. An approach developed there in order to obtain an elementary proof of complete continuity of the imbedding of the space $W_m^p(\Omega)$ in $C(\Omega)$ for m large enough (see the definition below) is applied here to study the similar property of the imbedding of $W_m^p(\Omega)$ into the space of functions integrable to the power p over a sufficiently smooth variety contained in Ω, and of a dimension smaller than that of Ω. An elementary proof of the Kondrashev theorem is obtained under conditions imposed on the variety under consideration, which differ from the original ones as presented in [4]. To prove the continuity of the imbedding mentioned, it is natural to impose the geometric conditions I invented by Ehrling; for its complete continuity, the more stringent conditions II seem to be necessary.

Several papers have been published recently in connection with simplifications of imbedding theorems (cf. for references [2]).

In what follows Ω will denote a fixed bounded domain in N-dimensional Euclidean space with points $x, y, \ldots$ and corresponding volume elements $dx, dy, \ldots$; $C(\Omega)$ will denote the space of functions continuous on Ω, $C^\infty(\Omega)$ the space of functions with continuous derivatives of all orders on Ω. In $C^\infty(\Omega)$ we introduce the norm

$$\|f\|_m = \Big(\sum_a \int_\Omega |D_a f|^p\,dx\Big)^{1/p}, \quad p > 1,$$

where the summation is extended over all derivatives of f of order not larger than

$$m\left(D_a f = \frac{\partial^a f}{\partial x_1^{a_1}\ldots\partial x_N^{a_N}}, \quad |a| = a_1+\ldots+a_N\right).$$

By completion of $C^\infty(\Omega)$ in the norm $\|\ \|_m$ we obtain a Banach space $W_m^p(\Omega)$ of all functions of $L^p(\Omega)$ whose generalised derivatives up to order m all belong to $L^p(\Omega)$. In the occurrence of other norms, we shall indicate the space concerned, e. g. $f_{W^p_m(\Sigma)}$ denotes the norm of f in $W^p_m(\Sigma)$. The simplified notation $|Df|^p = \sum |\partial f/\partial x_j|^p$ is used throughout this paper (the summation is extended over all the derivatives concerned).

In integration over varieties of dimension less than that of Ω we denote the surface element of the variety by the same letter as the variety itself.

In some of the estimates which follow, constants which are generally different, will be denoted by the same symbols, but in each case it will be adequately described.

2. We consider an s dimensional variety S_s contained in Ω. S_s will be subjected to one of the following conditions.

I. S_s can be divided into a finite number of subvarieties V_s having the following properties (Ehrling [2]):

(i) In a suitable system of coordinates V_s can be represented by the system of equations $y_{s+1} = \varphi_{s+1}(\overset{*}{y}), \ldots, y_N = \varphi_N(\overset{*}{y})$, where $\overset{*}{y} = (y_1, \ldots, y_s)$ varies over an s-dimensional domain Ω_s and the functions $\varphi_{s+1}, \ldots, \varphi_N$ satisfy the Lipschitz condition in Ω.

(ii) There exists a fixed $N-s$ dimensional spherical segment Σ with radius and solid angle both positive, and such that for every point $y = (\overset{*}{y}, \varphi(\overset{*}{y}))$ of V_s there exists an isometric image Σ_y of Σ with vertex at y, situated in the $y_{s+1}, \ldots, y_N$-space and entirely contained in Ω.

In connection with the above and the following definitions we introduce the following notation. If Σ_x with vertex at x is an isometric image of the fixed spherical segment Σ, and r is a positive number not larger than the radius of Σ, then we denote by $\Sigma_x(r)$ the spherical segment obtained from Σ_x by shortening its radius to r.

We also introduce the stronger condition II.

II. For any fixed positive δ, S_s can be divided into a finite number of subvarieties V_s, with the following properties:

(i) The variation over V_s of the angle between the tangent plane to V_s and any fixed vector does not exceed $\delta/2$.

(ii) For every sufficiently short vector k the variety $V_s + k$ obtained from V_s by shifting each of its points by k is also contained in Ω.

(ii) For every vector k, which forms with the tangent to V_s at a certain point an angle not less than (and therefore with the tangent at every point of V_s and angle not less than $\delta/2$) the variety V_s can be represented in a system of coordinates $y_1, \ldots, y_N$, in which the vector k lies in the subspace $y_{s+1}, \ldots, y_N$, in the form

$$y_{s+1} = \varphi_{s+1}(\overset{*}{y}), \ldots, y_N = \varphi_N(\overset{*}{y})$$

where $\overset{*}{y}$ varies over an s-dimensional domain Ω_s and $\varphi_{s+1}, \ldots, \varphi_N$ satisfy the Lipschitz condition on Ω_s.

(iv) There exists a fixed $N-s$ dimensional spherical segment Σ with positive radius and solid angle such that: For every sufficiently short vector k forming with the tangent to V_s at a certain point an angle not less than δ and for every x from V_s there exist segments Σ_{x+k} and Σ_x, with vertices respectively at $x+k$ and x, situated in $y_{s+1}, \ldots, y_N$-space, contained in Ω and such that the common part $A_{x,k}$ of $\Sigma_x(|k|)$ and $\Sigma_{x+k}(|k|)$ (cf. notation introduced above) has a volume not less than $a|k|^{N-s}$, where the constant $a > 0$ does not depend on x and k, but might depend on δ.

We are not going to discuss the possible interdependence of these conditions; we mention only that the set of conditions II is stronger than the set I. The relation between these two sets is similar to that between the cone property of Ehrling and the strong cone property of Nirenberg of [2] and [3]. If the variety S_s is situated inside $\overline{\Omega}$ and is smooth enough, e. g. of class C^2, then both I and II are satisfied.

3. We are now going to prove the following

THEOREM (Kondrashev, cf. [4]). *Let $f \in W^p_m(\Omega)$ for $m \geqslant [(N-s)/p] + 1$* [1] *and S_s be an s-dimensional variety in Ω satisfying conditions* I. *Then:* (i) *f restricted to S_s if of class $L^p(S_s)$ and* (ii) *the imbedding $W^p_m(\Omega) \to L^p(S_s)$ defined by restriction of functions from $W^p_m(\Omega)$ to S_s (which is justified by the previous statement* (i)) *is continuous.* (iii) *If S_s satisfies conditions* II *then this imbedding is completely continuous.*

The proof is based on the following lemmas.

LEMMA 1. *If Σ is an n-dimensional spherical segment with vertex at* 0, *and β and p are fixed numbers, $0 < \beta < n$, $p > 1$, then there exists a constant C depending on p, β, and Σ, and such that for every $f \in C^\infty(\Sigma)$*

$$\int_\Sigma |x|^{-\beta} |f(x)|^p dx \leqslant C \|f\|^p_{W^p_m(\Sigma)}$$

if only $m \geqslant [(m-\beta)/p]$.

LEMMA 2 (Nirenberg [3]). *If Σ is an n-dimensional spherical segment with radius h and vertex at* 0, *and β is a number such that $0 < a = 1-(N-\beta)/p < 1$, then there exists a constant C which depends on p, β and the solid angle of Σ and such that for every $f \in C^\infty(\Sigma)$*

$$\int_\Sigma |f(x) - f(0)|\, dx \leqslant C h^{n+a} \Big(\int_\Sigma |x|^{-\beta} |Df|^p dx\Big)^{1/p}.$$

For futher comments on these two lemmas we refer to the Paper [1].

---

[1] $[q]$ denotes the entire part of q.

LEMMA 3. *Let V_s be an s-dimensional variety situated in Ω and satisfying conditions* II. *Let k be a vector which forms with the tangent to V_s at every point an angle not less than $\delta/2$, where δ is a fixed constant, and such that V_s+k is also contained in Ω. Then there exists positive constants C and γ such that for every function $f\in C^\infty(\ )$ and $m\geqslant[(N-s)/p]+1$*

$$\int_{V_s}|f(x+k)-f(x)|^p\,dV\leqslant C\|f\|^p_{W^p_m(\Sigma)}|k|^\gamma .$$

Note that in this inequality C does not depend on f.

Proof. We use the properties (iii) and (iv) of II. We represent V_s in the form $y_{s+1}=\varphi_{s+1}(\overset{*}{y}),\ldots,y_N=\varphi_N(\overset{*}{y})$ and to every point $x=(\overset{*}{x},\varphi(\overset{*}{x}))$ and $x+k=(\overset{*}{x},\varphi(\overset{*}{x})+k)$ of V_s and V_s+k we attach $N-s$-dimensional segments Σ_x and Σ_{x+k} with vertices at x, and $x+k$ isometric with the fixed segment Σ and situated in the $y_{s+1},\ldots,y_N$-space. Also from the condition (iv) we can assume that the volume of the common part of these two segments shortened to $|k|$ viz. $\Sigma_k(|k|)\cap\Sigma_{x+k}(|k|)=\Lambda_{x,k}$ is not less than $\alpha|k|^{n-s}$, where $\alpha>0$ is a constant independent of x.

We have for a fixed x and every $y\in\Lambda_{x,k}$

$$\int|f(x+k)-f(x)|\leqslant|f(x+k)-f(y)|+|f(x)-f(y)|.$$

The integration of this inequality over $\Lambda_{x,k}$ yields

$$\begin{aligned}\alpha|k|^{N-s}|f(x+k)-f(x)|&\leqslant\int_{\Lambda_{x,k}}\big(|f(x+k)-f(y)|+|f(x)-f(y)|\big)\,dy\\&\leqslant\int_{\Sigma_{x+k}(|k|)}|f(x+k)-f(y)|\,dy+\int_{\Sigma_x(|k|)}|f(x)-f(y)|\,dy\end{aligned}$$

and applying Lemmas 2 and 1 we get

$$\begin{aligned}&|k|^{N-s}|f(x+k)-f(x)|\\&\leqslant C|k|^{N-s+\gamma/p}\Big[\int_{\Sigma_{x+k}(|k|)}|Df(y)^p|x-k-y|^{-\beta}\,dy+\int_{\Sigma_x(|k|)}|x-y|^{-\beta}|Df(y)|^p\,dy\Big]^{1/p}\\&\leqslant C|k|^{N-s+\gamma/p}\Big[\int_{\Sigma_{x+k}}|y-x-k|^{-\beta}|Df(y)|^p\,dy+\int_{\Sigma_x}|y-x|^{-\beta}|Df(y)|^p\,dy\Big]^{1/p}\\&\leqslant C|k|^{N-s+\gamma}\big(\|f\|^p_{W^p_m(\Sigma_{x+k})}+\|f\|^p_{W^p_m(\Sigma_x)}\big)^{1/p}.\end{aligned}$$

Integrating the above inequality to the power p over V_s (i. e. in effect the integration is extended over two N-dimensional domains — the first composed of the segments Σ_x arranged on V_s and the second composed of the segments Σ_{x+k} arranged on V_s+k) we get

$$\int_{V_s}|f(x+k)-f(x)|^p\,dV\leqslant C|k|^\gamma\|f\|^p_m$$

as asserted.

Proof of the theorem. We prove first that the imbedding $W^p_m(\Omega)\to\to L^p(S_s)$ is defined and continuous if S_s satisfies the conditions of I. It is sufficient to prove this fact for any one of the finite number of subvarieties V_s of S_s. For a fixed segment Σ, as described in the condition (ii) of I, let us denote by $V_s\times\Sigma$ the N-dimensional domain obtained by attaching to each point x of V_s the corresponding segment Σ_x with vector at x.

For every function $f\in C^\infty(\Omega)$ and for every fixed $x\in V_s$ and $y\in\Sigma_x$ we have

$$|f(x)|\leqslant|f(x)-f(y)|+|f(y)|.$$

The integration of this inequality over Σ_x yields

$$\operatorname{vol}(\Sigma)|f(x)|\leqslant\int_{\Sigma_x}|f(x)-f(y)|\,dy+\int_{\Sigma_x}|f(y)|\,dy$$

which from Lemmas 1 and 2 and the Schwartz inequality gives for $m\geqslant[(N-s)/p]+1$

$$|f(x)|\leqslant C\|f\|W^p_m(\Sigma_x).$$

Integrating this to the power p over V_s we get

$$\int_{V_s}|f(x)|^p\,dV\leqslant C\int_{V_S+\Sigma}\sum_{|\alpha|\leqslant m}|D_\alpha f|^p\,dx\leqslant C\|f\|^p_m .$$

We can extend this estimate by continuity to the whole of $W^p_m(\Omega)$ and the first part of the theorem follows.

To prove the complete continuity of the imbedding we must verify that the image under this imbedding of a bounded set in $W^p_m(\Omega)$ is compact in $L^p(S_s)$. We are going to use the known sufficient condition for the compactness of a set in L^p: A set in $L^p(S_s)$ is compact if it is bounded and if

$$\lim_{|k|\to0}\int_{S_s}|f(x+k)-f(x)|^p\,ds$$

is 0 uniformly with respect to f in this set. The first condition is automatically satisfied as a consequence of the continuity of the imbedding; to prove the second condition we reason as follows. Let $f\in C^\infty(\Omega)$ and for a fixed number $\delta>0$ let us divide S_s into subvarieties as described in II. If for a fixed subvariety V_s the angle between the tangent to V_s and the vector k is not less that $\delta/2$, then according to Lemma 3 we can find a constant C, which may depend on δ, such that

$$\int_{V_s} |f(x+k)-f(x)|^p\,dV \leqslant C\|f\|_m^p\,|k|^\gamma$$

where γ is a positive constant.

If the angle between k and the tangent plane to V_s is less than $\delta/2$ then we can find another vector k_1 of length $|k_1| = |k|$ and such that the angles between $k-k_1$ and the tangent to V_s+k, and between k_1, and the tangent to V_s, are not less than $\delta/2$ (if V_s has the properties described in II then V_s+k has these properties also provided that k is short enough).

We get the vector k_1

$$\Big(\int_{V_s} |f(x+k)-f(x)|^p\,dV\Big)^{1/p} \leqslant \Big(\int_{V_s} |f(x+k)-f(x+k_1)|^p\,dV\Big)^{1/p}$$

$$\leqslant \Big(\int_{V_s} |f(x+k_1)-f(x)|^p\,dV\Big)^{1/p}$$

$$= \Big(\int_{V_{s+k}} |f(x)-f(x+k-k_1)|^p\,dV\Big)^{1/p} + \Big(\int_{V_s} |f(x)-f(x+k_1)|^p\,dV\Big)^{1/p}$$

each of two above integrals can be estimated by means of Lemma 3. Thus in both cases the integral under consideration tends uniformly to 0 and the proof is completed; the estimate of lemma 3 can obviously be extended by continuity to the whole of $W_m^p(\Omega)$.

References

[1] E. Ehrling, *On a type of eigenvalue problems for certain elliptic differential operators*, Mathematica Scandinavica 2 (1954), p. 267-287.

[2] K. K. Golovkin, DAN SSSR 134, No. 1 (1960), p. 19-22.

[3] L. Nirenberg, *Remarks on strongly elliptic partial differential equations*, Communications on Pure and Applied Mathematics 8 (1955), p. 649-675.

[4] S. L. Sobolev, *Some applications of functional analysis in mathematical physics* (in Russian), Moscov 1950.

[5] P. Szeptycki, *A simple proof of an embedding theorem of the Kondrashev type*, Bull. Acad. Pol. Sci., Série des Sci. math., astr. et phys., VI, No. 9 (1958), p. 561-564.

Reçu par la Rédaction le 14. 12. 1960

STUDIA MATHEMATICA, T. XXI. (1961)

Approximation des opérateurs de J. Mikusiński par des fonctions continues

par

C. FOIAŞ (Bucureşti)

Soit $C[0,\infty)$ l'espace des fonctions continues définies sur $[0,\infty)$ muni de la topologie de la convergence uniforme sur tout compact de $[0,\infty)$. Par rapport à la convolution, $C[0,\infty)$ devient un domaine d'intégrité dont le corps $\mathfrak{M}$ des quotients formels $\frac{f}{g}$, $f, g \in C[0,\infty)$, est l'espace des opérateurs de J. Mikusiński (¹). Une suite $\left\{\frac{f_n}{g_n}\right\}$ converge vers $\frac{f}{g}$ (dans $\mathfrak{M}$), s'il existe des représentations $\frac{f_n}{g_n} = \frac{f_n'}{g_n'}$, $\frac{f}{g} = \frac{f'}{g'}$ telles que $f_n' \to f'$ (dans $C[0,\infty)$) (²).

Un problème naturel est de savoir si tout opérateur $\frac{f}{g} \in \mathfrak{M}$ peut être approché par une suite $\{k_n\}$, $k_n \in C[0,\infty)$. Il est évident que si g est nulle dans un voisinage de l'origine, alors cela n'est pas toujours vrai. Dans cette note nous montrons que la propriété ci-dessus a lieu dès que g n'est pas identiquement nulle au voisinage de l'origine.

Je tiens à remercier M. J. Mikusiński qui a bien voulu me proposer ce problème (au cours du Colloque d'Analyse Numérique tenu à Cluj, décembre 1960).

Nous commençons par établir une proposition préliminaire dont notre résultat sera une conséquence immédiate.

LEMME. *Soient f, g des fonctions sommables dans $[0,T]$, g non nulle presque partout au voisinage de 0; alors il existe une suite de fonctions k_n continues dans $[0,T]$ telle que $\{g*k_n\}$ converge en moyenne vers f.*

Démonstration. Dans le cas contraire, en vertu du théorème de Banach et de Hahn et du théorème sur la représentation des fonctionnelles linéaires continues sur l'espace des fonctions sommables sur $[0,T]$,

(¹) Voir J. Mikusiński, *Operational calculus* 1959.

(²) Op. cit., p. 144.