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# Mean-value estimations for the Möbius function I

by

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**1.** This paper and its continuation will be concerned with some improvements of my earlier results [2], [3] in the distribution of values of the Möbius function. Writing

$$M(x) = \sum_{n \leqslant x} \mu(n),$$

where $\mu(n)$ denotes the Möbius function, I have proved ([2]) the following:

*Suppose*

$$(1.1) \qquad \int_1^T \left(\frac{M(x)}{x}\right)^2 dx < a \log T, \qquad (T \geqslant 1,\ a \textit{ independent of } T).$$

*Then*

$$(1.2) \qquad \max_{1 \leqslant x \leqslant T} |M(x)| > T^{1/2} \exp\left(-\frac{\log T}{\sqrt{\log\log T}}\right)$$

*for*

$$T > \max(c_1, \exp 300a)\ ^{(1)}.$$

The implication (1.1) ⇒ (1.2) has been refined in a way in [3]. The result is:

*Suppose*

$$(1.3) \qquad \int_1^T \frac{|M(x)|}{x} dx < aT^{1/2}, \qquad (T \geqslant 1,\ a \textit{ independent of } T).$$

*Then*

$$(1.4) \qquad \int_1^T \frac{|M(x)|}{x} dx > T^{1/2} \exp\left(-\frac{\log T}{\sqrt{\log\log T}}\right)$$

*for*

$$T > \max(c_2, e^a).$$

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(1) Throughout this paper $c_1, c_2, c_3, \ldots$ stand for positive, numerical constants.

The proofs of both these results base on the observation that (1.1) resp. (1.3) would imply what follows:

(1.5) *All the complex $\zeta$-zeros have $\sigma = \frac{1}{2}$ (the Riemann hypothesis).*

(1.6) *All $\zeta$-zeros are simple.*

(1.7) *Let $\varrho_1\ \varrho_2$, be different complex $\zeta$-zeros. Then*

$$|\varrho_1-\varrho_2| \geqslant \frac{1}{15\sqrt{a}(\max\limits_{i=1,2}|\varrho_i|)^4}, \quad \textit{resp.} \quad |\varrho_1-\varrho_2| \geqslant \frac{1}{8a(\max\limits_{i=1,2}|\varrho_i|)^4}.$$

One might improve a little the results of [2], [3] by deriving estimates (1.2) resp. (1.4), holding however only in some finite $T$-interval, from (1.5), (1.6), (1.7) assumed true also in a suitable finite range. But this would be of no deeper significance.

The aim of this work is to improve the results of [2], [3] in two directions. First, it will be shown that (1.3) implies a much better inequality than (1.4); I shall in fact prove

THEOREM I. *Suppose* (1.3). *Then for*

$$T \geqslant (a+2)^{c_3} \overset{\text{def}}{=} H$$

*we have*

$$\int\limits_{\frac{1}{H}T}^{T} \frac{|M(x)|}{x}\,dx > \frac{1}{H}T^{1/2} \tag{1.8}$$

($c_3$ *can be numerically calculated*).

Secondly, as it will be seen, for the proofs of (1.2), (1.4) we can dispense with the condition (1.7) altogether. We have, namely,

THEOREM II. *Suppose that all the zeros of $\zeta(s)$ in the rectangle*

$$0<\sigma<1, \quad |t| \leqslant \omega$$

*are simple and have $\sigma = \frac{1}{2}$. In that case we have*

$$\int\limits_{X}^{T} \frac{|M(x)|}{x}\,dx > T^{1/2}\exp\Big(-12\frac{\log T}{\log\log T}\log\log\log T\Big) \tag{1.9}$$

*with*

$$X = T\exp\Big(-100\frac{\log T}{\log\log T}\log\log\log T\Big)$$

*for*

$$c_4 \leqslant T \leqslant \exp(\omega^{10}) \tag{1.10}$$

($c_4$ *can be numerically calculated*).



In the present paper I shall prove only Theorem I; the proof of Theorem II is postponed to the second part of this paper.

The essential idea of the proof of Theorem I is due to A. E. Ingham. Actually, Ingham was concerned with the function $\Delta(x) = \psi(x)-x$ $\big(\psi(x) = \sum\limits_{n\leqslant x}\Lambda(n)\big)$ and found some lower bound for its absolute value ([2]). His idea is, roughly speaking, the following: on multiplying both sides of the explicit formula for $\psi(x)$ by a certain suitable factor, on integrating it further and observing that the $\zeta$-zero with minimal positive imaginary part is "very remote" from the remaining $\zeta$-zeros, one can reject those remaining ones and carry out the desired estimation by using only *one* (numerically known) $\zeta$-zero.

I hear from Mr. Ingham that he had no intention of publishing his result. It seems therefore that the best form of my acknowledgement would be, with Mr. Ingham's consent, the reference to this letter to Professor Turán.

**2.** In what follows we assume (1.3) (and have therefore as its consequence (1.5) and (1.6)). Let $\{T_\nu\}$ be the sequence of numbers such that

$$\nu \leqslant T_\nu \leqslant \nu+1 \quad (\nu = 1, 2, \ldots),$$

and

$$\left|\frac{1}{\zeta(s)}\right| \leqslant c_8 t^{1/20}, \quad s = \sigma+it, \quad \tfrac{1}{2} \leqslant \sigma \leqslant 2, \quad t = T_\nu.$$

The existence of $\{T_\nu\}$ follows by (1.5) (see [4], p. **303**). Write

$$M_1(x) \overset{\text{def}}{=} \lim_{\nu\to\infty} \sum_{|\gamma|<T_\nu} \frac{x^\varrho}{\varrho\zeta'(\varrho)}, \quad 0<x<\infty,$$

where $\varrho = \beta+i\gamma$ ($\beta = \frac{1}{2}$ by (1.5)) runs through the $\zeta$-zeros. As is well-known (see [4], p. 318) $M_1(x)$ is defined correctly (i.e. the limit $\lim\limits_{\nu\to\infty}$ exists) and furthermore

$$M_1(x) = \frac{M(x-0)+M(x+0)}{2} + 2 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2\pi/x)^{2n}}{(2n)!\,n\zeta(2n+1)}. \tag{2.1}$$

LEMMA 1. *We have, under* (1.3), *for $x \geqslant \frac{3}{4}$, $\nu = 1, 2, \ldots$*

$$\Big|M_1(x) - \sum_{|\gamma|<T_\nu} \frac{x^\varrho}{\varrho\zeta'(\varrho)}\Big| < c_9\frac{x^3}{\nu^{1/2}} + \vartheta_\nu(x), \tag{2.2}$$

*where*

$$\vartheta_\nu(x) = \begin{cases} 0 & \textit{for} \quad \min\limits_{n=1,2,\ldots}|x-n| \geqslant \dfrac{1}{\nu^{1/2}}, \\ 3 & \textit{for} \quad \min\limits_{n=1,2,\ldots}|x-n| < \dfrac{1}{\nu^{1/2}}. \end{cases}$$

([2]) I have known the result, together with a sketchy proof, through the medium of Professor Turán who had been informed about it by Mr. Ingham in a letter.

Proof. We consider the integral

$$\frac{1}{2\pi i}\int_{2-iT_\nu}^{2+iT_\nu}\frac{x^s}{s}\cdot\frac{ds}{\zeta(s)} = F_\nu(x)$$

and get by the theorem of residues (see [4], p. 318)

$$(2.3)\qquad F_\nu(x) = \sum_{|\gamma|<T_\nu}\frac{x^\varrho}{\varrho\zeta'(\varrho)} - 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2\pi/x)^{2n}}{(2n)!\,n\zeta(2n+1)} + O\left(\frac{x^2}{\nu^{19/20}}\right),$$

the constant involved with $O(\ldots)$ being numerical. On the other hand

$$F_\nu(x) = \frac{1}{2\pi i}\int_{2-iT_\nu}^{2+iT_\nu}\frac{x^s}{s}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}ds = \sum_{n=1}^{\infty}\frac{\mu(n)}{2\pi i}\int_{2-iT_\nu}^{2+iT_\nu}\left(\frac{x}{n}\right)^s\frac{ds}{s}.$$

Suppose $\min\limits_{n=1,2,\ldots}|x-n| \geqslant 1/\nu^{1/2}$ and use [4], p. 53, Lemma 3.12. This gives

$$F_\nu(x) = \frac{M(x-0)+M(x+0)}{2} + O\left(\frac{x^2}{\nu}\right) + O\left(\frac{x}{\nu\min\limits_{n=1,2,\ldots}|x-n|}\right),$$

whence and by (2.3) the result follows in this case.

Introduce now the following notation:

$$I(y) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{y^s}{s}ds, \qquad I(y, T) = \frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{y^s}{s}ds,$$
$$\Delta(y, T) = I(y) - I(y, T).$$

As is well known (see [1], p. 75)

$$|\Delta(y, T)| < \frac{y^2}{\pi T|\log y|} \quad \text{if} \quad y \neq 1,\ y > 0,$$
$$|\Delta(y, T)| < y^2 \quad \text{if} \quad y > 0.$$
$$I(y) = \begin{cases} 1 & \text{if} \quad y > 1, \\ \frac{1}{2} & \text{if} \quad y = 0, \\ 0 & \text{if} \quad 0 < y < 1. \end{cases}$$

Hence

$$(2.4)\qquad F_\nu(x) = \sum_{n=1}^{\infty}\mu(n)I\left(\frac{x}{n}, T_\nu\right) = \sum_{n=1}^{\infty}\mu(n)I\left(\frac{x}{n}\right) - \sum_{n=1}^{\infty}\mu(n)\Delta\left(\frac{x}{n}, T_\nu\right)$$
$$= \frac{M(x-0)+M(x+0)}{2} + O\left(\sum_{n=1}^{\infty}\left|\Delta\left(\frac{x}{n}, T_\nu\right)\right|\right).$$



Let $m = m(x)$ be the integer defined by $m-\frac{1}{2} < x \leqslant m+\frac{1}{2}$. We have (compare [1], 79)

$$(2.5)\qquad \left|\Delta\left(\frac{x}{n}, T_\nu\right)\right| < \begin{cases} \left(\frac{x}{n}\right)^2\frac{1}{\pi\nu}\cdot\frac{1}{\left|\log\frac{x}{n}\right|} < \left(\frac{x}{n}\right)^2\frac{1}{\pi\nu}\cdot\frac{n+x}{|n-x|} & \text{if} \quad n \neq m, \\ \left(\frac{x}{m}\right)^2 < 3 \quad \text{if} \quad n = m, \end{cases}$$

so that

$$\sum_{n=1}^{m-1}\left|\Delta\left(\frac{x}{n}, T_\nu\right)\right| < \frac{x^2}{\pi\nu}\sum_{n=1}^{m-1}\frac{4x}{n^2} < c_{10}\frac{x^3}{\nu},$$
$$\sum_{n=m+1}^{2([x]+1)}\left|\Delta\left(\frac{x}{n}, T_\nu\right)\right| < \frac{x^2}{\pi\nu}\sum_{n=m+1}^{2([x]+1)}\frac{4n}{n^2} < c_{11}\frac{x^2}{\nu}\log\big(2(x+1)\big),$$
$$\sum_{n=2([x]+1)}^{\infty}\left|\Delta\left(\frac{x}{n}, T_\nu\right)\right| < \frac{x^2}{\pi\nu}\sum_{u=2([x]+1)}^{\infty}\frac{(3/2)^n}{n^2(1/2)^n} < c_{12}\frac{x}{\nu}.$$

This together with (2.3), (2.4) and (2.5) completes the proof of Lemma 1.

LEMMA 2. *We have, under* (1.3), *for* $0 < x \leqslant \frac{3}{4}$

$$M_1(x) = \frac{i}{2\pi}\int_{(1/4)}\frac{x^s}{s}\frac{ds}{\zeta(s)}.$$

*Also*

$$(2.6)\qquad \left|\sum_{|\gamma|<T_\nu}\frac{x^\varrho}{\varrho\zeta'(\varrho)} - M_1(x)\right| \leqslant \frac{c_{13}}{\nu^{1/5}}$$

*and*

$$(2.7)\qquad |M_1(x)| \leqslant c_{14} \quad \textit{if} \quad 0 < x \leqslant 1.$$

Proof: We have ([4], p. 53)

$$(2.8)\qquad 0 = \frac{1}{2\pi i}\int_{2-iT_\nu}^{2+iT_\nu}\frac{x^s}{s}\cdot\frac{ds}{\zeta(s)} + O\left(\frac{1}{\nu}\right)$$
$$= \sum_{|\gamma|<T_\nu}\frac{x^\varrho}{\varrho\zeta'(\varrho)} + \frac{1}{2\pi i}\int_{1/4-iT_\nu}^{1/4+iT_\nu}\frac{x^s}{s}\cdot\frac{ds}{\zeta(s)} + O\left(\frac{1}{\nu^{19/20}}\right).$$

Hence, passing with $\nu$ to $\infty$,

$$M_1(x) = \frac{i}{2\pi}\int_{(1/4)}\frac{x^s}{s}\cdot\frac{ds}{\zeta(s)}.$$

This gives in particular (2.7) for $0 < x \leqslant \frac{3}{4}$. For $\frac{3}{4} < x \leqslant 1$ (2.7) follows straightaway from (2.1). Further

$$\left|\frac{1}{2\pi i}\int\limits_{(1/4)}\frac{x^s}{s}\cdot\frac{ds}{\zeta(s)}-\frac{1}{2\pi i}\int\limits_{1/4-iT_\nu}^{1/4+iT_\nu}\frac{x^s}{s}\cdot\frac{ds}{\zeta(s)}\right| \leqslant c_{15}\int\limits_{\nu}^{\infty}\frac{dt}{t^{1+1/5}} = \frac{5c_{15}}{\nu^{1/5}},$$

whence and by (2.8) we obtain (2.6).

LEMMA 3. *Let us assume* (1.3). *Then*

$$\int\limits_1^T \frac{|M_1(x)|}{x}\,dx < (a+c_{16})\,T^{1/2}, \qquad (T \geqslant 1). \tag{2.9}$$

Proof. We have by (2.1)

$$|M_1(x)| \leqslant |M(x)| + c_{17}, \qquad (x \geqslant 1),$$

whence

$$\int\limits_1^T \frac{|M_1(x)|}{x}\,dx < (a+2c_{17})\,T^{1/2}.$$

**3.** We turn to the proof of Theorem I. Supposing $T \geqslant (a+2)^{c_3}$ we write

$$M_1(e^u) = \lim_{\nu\to\infty}\sum_{|\gamma|<T_\nu}\frac{e^{u\varrho}}{\varrho\zeta'(\varrho)}, \qquad (-\infty < u < +\infty),$$

and multiply it by $e^{-\varrho_1 u-\delta(u-\omega)^2}$, where $\delta = c_{18}/\log(2+a)$ $(<1;\ c_{18}$ sufficiently small—to be determined later), $\varrho_1 = \frac{1}{2}+i\,14.13\ldots$ is the $\zeta$-zero with minimal positive imaginary part, $\omega = \log T\varphi$, $\varphi = (a+2)^{-\frac{1}{2}c_{19}}$ ($c_{19}$ will be determined after $c_{18}$ is fixed; for the time being we assume only $\varphi < \dfrac{1}{e}$ and $c_3 > c_{19}$ so that writing $\theta = \dfrac{\log\varphi^{-1}}{\log T\varphi}$ we have $0 < \theta < 1$). Next we integrate the resulting expression in $-\infty < u < +\infty$

$$\int\limits_{-\infty}^{+\infty}\frac{M_1(e^u)}{e^{\varrho_1 u+\delta(u-\omega)^2}}\,du = \int\limits_{-\infty}^{+\infty}\lim_{\nu\to\infty}\sum_{|\gamma|<T_\nu}\frac{e^{u(\varrho-\varrho_1)-\delta(u-\omega)^2}}{\varrho\zeta'(\varrho)}\,du. \tag{3.1}$$

We shall show in § 4 on the right-hand side of (3.1) $\int\limits_{-\infty}^{+\infty}$ and $\lim\limits_{\nu\to\infty}$ can be inverted. We put first

$$f(t) = \int\limits_1^t \frac{|M_1(u)|}{u}\,du$$



whence by (2.9) $f(t) \leqslant (a+c_{16})t^{1/2}$) and have

$$\begin{aligned}\int\limits_{\omega(1+\theta)}^{\infty}\frac{|M_1(e^u)|}{e^{\frac12 u+\delta(u-\omega)^2}}\,du &= \int\limits_{\omega(1+\theta)}^{\infty}\frac{\frac{d}{du}f(e^u)}{e^{\frac12 u+\delta(u-\omega)^2}}\,du \\ &= \frac{f(e^u)}{e^{\frac12 u+\delta(u-\omega)^2}}\Bigg|_{\omega(1+\theta)}^{\infty} + \int\limits_{\omega(1+\theta)}^{\infty}\frac{f(e^u)\{\frac12+2\delta(u-\omega)\}}{e^{\frac12 u+\delta(u-\omega)^2}}\,du \\ &< \frac12\int\limits_{\omega(1+\theta)}^{\infty}\frac{a+c_{16}}{e^{\delta(u-\omega)^2}}\,du + 2\delta\int\limits_{\omega(1+\theta)}^{\infty}\frac{(u-\omega)(a+c_{16})}{e^{\delta(u-\omega)^2}}\,du \\ &< (a+c_{16})(\tfrac12+2\delta)\int\limits_{\delta\omega^2\theta^2}^{\infty}\frac{1}{2\delta}\frac{dz}{e^z} \\ &= \frac{(a+c_{16})(\frac12+2\delta)}{2\delta}\,e^{-\delta\omega^2\theta^2}.\end{aligned}$$

After having fixed $c_{18}$ (or, what is the same, $\delta$) we choose $c_{19}$ in such a way as to make the last expression

$$< \frac{\sqrt{\pi}}{8}\cdot\frac{1}{|\varrho_1\zeta'(\varrho_1)|}.$$

Further

$$\begin{aligned}\int\limits_{-\infty}^{\omega(1-\theta)}\frac{|M_1(e^u)|}{e^{\frac12 u+\delta(u-\omega)^2}}\,du &\leqslant \int\limits_{-\infty}^{0}\frac{c_{14}\,du}{e^{\frac12 u+\delta(u-\omega)^2}} + \int\limits_{0}^{\omega(1-\theta)}\frac{|M_1(e^u)|}{e^{\frac12 u+\delta(u-\omega)^2}}\,du \\ &= \int\limits_{-\infty}^{0}\frac{c_{14}\,du}{e^{\frac12 u+\delta(u-\omega)^2}} + \frac{f(e^u)}{e^{\frac12 u+\delta(u-\omega)^2}}\Bigg|_{0}^{\omega(1-\theta)} + \int\limits_{0}^{\omega(1-\theta)}\frac{f(e^u)\{\frac12+2\delta(u-\omega)\}}{e^{\frac12 u+\delta(u-\omega)^2}}\,du.\end{aligned}$$

Making $c_{19} > \dfrac{1}{2c_{18}}$ we obtain

$$\begin{aligned}\frac12+2\delta(u-\omega) &< \frac12-\frac{2c_{18}}{\log(2+a)}\,\theta\omega = \frac12-\frac{2c_{18}}{\log(2+a)}\log\varphi^{-1} \\ &= \frac12-\frac{2c_{18}}{\log(2+a)}\cdot\frac12\log(2+a)^{c_{19}} = \frac12-c_{18}c_{19} < 0,\end{aligned}$$

whence the last integral is $< 0$ and thus

$$\int\limits_{-\infty}^{\omega(1-\theta)}\frac{|M_1(e^u)|}{e^{\frac12 u+\delta(u-\omega)^2}}\,du \leqslant \int\limits_{-\infty}^{0}\frac{c_{14}\,du}{e^{\frac12 u+\delta(u-\omega)^2}} + \frac{a}{e^{\delta(\omega\theta)^2}} < c_{14}\int\limits_{-\infty}^{0}\frac{du}{e^{\delta u^2+\delta\omega^2}} + \frac{a}{e^{\delta(\omega\theta)^2}}.$$

This will be in turn

$$\leqslant \frac{c_{20}}{\delta^{1/2}e^{\delta\omega^2}} + \frac{a}{e^{\delta(\omega\theta)^2}} \overset{\text{def}}{=} \Omega$$

if only

$$2\omega\delta = 2\frac{c_{18}}{\log(a+2)}\cdot\log(T\varphi) \geqslant \frac{1}{2}$$

(which will be satisfied if $c_3 > \frac{1}{4c_{18}} + \frac{c_{19}}{2}$).

After having fixed $c_{18}$ (which will be done in the next section) we can make $\Omega$ again

$$< \frac{\sqrt{\pi}}{8}\cdot\frac{1}{|\varrho_1\zeta'(\varrho_1)|}.$$

It follows that

(3.2) $$\left|\int_{-\infty}^{+\infty}\frac{M_1(e^u)}{e^{\varrho_1 u+\delta(u-\omega)^2}}du - \int_{\omega(1-\theta)}^{\omega(1+\theta)}\frac{M_1(e^u)}{e^{\varrho_1 u+\delta(u-\omega)^2}}du\right| < \frac{\sqrt{\pi}}{4}\cdot\frac{1}{|\varrho_1\zeta'(\varrho_1)|}.$$

**4.** In order to justify the inversion of $\int\limits_{-\infty}^{+\infty}$ and $\lim\limits_{\nu\to\infty}$ in (3.1) we use (2.2) and (2.6) which yield

$$\left|\int_{-\infty}^{+\infty}\Big\{M_1(e^u) - \sum_{|\gamma|<T_\nu}\frac{e^{\varrho u}}{\varrho\zeta'(\varrho)}\Big\}e^{-\varrho_1 u-\delta(u-\omega)^2}du\right|$$

$$\leqslant \frac{c_9}{\nu^{1/2}}\int_{\log 3/4}^{\infty}\frac{e^{3u}}{e^{\frac{1}{2}u+\delta(u-\omega)^2}}du + \int_{\log 3/4}^{\infty}\frac{\vartheta_\nu(e^u)}{e^{\frac{1}{2}u}}du + \frac{c_{13}}{\nu^{1/5}}\int_{-\infty}^{\log 3/4}\frac{du}{e^{\frac{1}{2}u+\delta(u-\omega)^2}}$$

$$< \frac{c(\delta,\omega)}{\nu^{1/5}} + \int_{3/4}^{\infty}\frac{\vartheta_\nu(x)}{x^{3/2}}dx < \frac{c(\delta,\omega)}{\nu^{1/5}} + 3\frac{2}{\nu^{1/5}}\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)^{3/2}.$$

Hence

$$\int_{-\infty}^{+\infty}\frac{M_1(e^u)}{e^{\varrho_1 u+\delta(u-\omega)^2}}du = \lim_{\nu\to\infty}\sum_{|\gamma|<T_\nu}\frac{1}{\varrho\zeta'(\varrho)}\int_{-\infty}^{+\infty}e^{u(\varrho-\varrho_1)-\delta(u-\omega)^2}du.$$

But

$$\int_{-\infty}^{+\infty}e^{u(\varrho-\varrho_1)-\delta(u-\omega)^2}du$$

$$= e^{(\varrho-\varrho_1)\omega+\frac{(\varrho-\varrho_1)^2}{4\delta}}\int_{-\infty}^{+\infty}e^{-\left\{\sqrt{\delta}(u-\omega)-\frac{\varrho-\varrho_1}{2\sqrt{\delta}}\right\}^2}du = e^{(\varrho-\varrho_1)\omega+\frac{(\varrho-\varrho_1)^2}{4\delta}}\cdot\frac{\sqrt{\pi}}{\delta^{1/2}},$$

whence

(4.1) $$\int_{-\infty}^{+\infty}\frac{M_1(e^u)}{e^{\varrho_1 u+\delta(u-\omega)^2}}du = \frac{\sqrt{\pi}}{\delta^{1/2}}\lim_{\nu\to\infty}\sum_{|\gamma|<T_\nu}\frac{1}{\varrho\zeta'(\varrho)}e^{(\varrho-\varrho_1)\omega+\frac{(\varrho-\varrho_1)^2}{4\delta}}.$$



Since ([3], p. 385)

$$\left|\frac{1}{\zeta'(\varrho)}\right| \leqslant \frac{a|\varrho|}{2},$$

we have

$$\left|\frac{1}{\varrho\zeta'(\varrho)}e^{(\varrho-\varrho_1)\omega+\frac{(\varrho-\varrho_1)^2}{4\delta}}\right| \leqslant \frac{a}{2}e^{-\frac{(\gamma-\gamma_1)^2}{4\delta}}$$

$$(\varrho = \tfrac{1}{2}+i\gamma,\ \ \varrho_1 = \tfrac{1}{2}+i\gamma_1 = \tfrac{1}{2}+i\,14.13\ldots),$$

so that by (4.1)

(4.2) $$\left|\int_{-\infty}^{+\infty}\frac{M_1(e^u)}{e^{\varrho_1 u+\delta(u-\omega)^2}}du\right| \geqslant \frac{\sqrt{\pi}}{\delta^{1/2}}\left(\frac{1}{|\varrho_1\zeta'(\varrho_1)|} - \frac{a}{2}\sum_{\gamma\neq\gamma_1}e^{-\frac{(\gamma-\gamma_1)^2}{4\delta}}\right).$$

Now it may be observed that denoting, as usually, by $N(x)$ the number of $\zeta$-zeros in $0<\sigma<1$, $0<t\leqslant x$, we obtain

$$\sum_{\gamma>\gamma_1}e^{-\frac{(\gamma-\gamma_1)^2}{4\delta}}$$

$$= \int_{20}^{\infty}e^{-\frac{(x-\gamma_1)^2}{4\delta}}dN(x) = N(x)e^{-\frac{(x-\gamma_1)^2}{4\delta}}\Big|_{20}^{\infty} + \int_{20}^{\infty}N(x)e^{-\frac{(x-\gamma_1)^2}{4\delta}}\left(\frac{x-\gamma_1}{2\delta}\right)dx,$$

which, in view of the (rough) inequality

$$N(x) \leqslant c_{21}x^2, \quad (x \geqslant 20),$$

gives

$$\sum_{\gamma>\gamma_1}e^{-\frac{(\gamma-\gamma_1)^2}{4\delta}} < \frac{c_{21}}{2\delta}\int_{20}^{\infty}x^2(x-\gamma_1)e^{-\frac{(x-\gamma_1)^2}{4\delta}}dx < c_{22}\int_{6/\delta}^{\infty}te^{-t}dt < \frac{c_{23}}{\delta}\cdot e^{-\frac{6}{\delta}}.$$

Consequently the sum of the series in (4.2) will not exceed

$$\frac{2c_{23}}{\delta}e^{-\frac{6}{\delta}},$$

whence fixing now $c_{18}$ at a sufficiently small value we get from (4.2)

$$\left|\int_{-\infty}^{+\infty}\frac{M_1(e^u)}{e^{\varrho_1 u+\delta(u-\omega)^2}}du\right| \geqslant \frac{\sqrt{\pi}}{\delta^{1/2}}\cdot\frac{1}{2}\cdot\frac{1}{|\varrho_1\zeta'(\varrho_1)|} > \frac{\sqrt{\pi}}{2}\cdot\frac{1}{|\varrho_1\zeta'(\varrho_1)|}.$$

This and (3.2) give

$$\int_{\omega(1-\theta)}^{\omega(1+\theta)}\left|\frac{M_1(e^u)}{e^{\varrho_1 u+\delta(u-\omega)^2}}\right|du \geqslant \frac{\sqrt{\pi}}{4}\cdot\frac{1}{|\varrho_1\zeta'(\varrho_1)|},$$

i.e.

$$\int_{T\varphi^2}^{T}\frac{|M_1(x)|}{x^{3/2}}dx \geqslant c_{24},$$

whence

$$\int_{T\varphi^2}^{T} \frac{|M(x)|}{x^{3/2}} dx \geqslant c_{25}$$

and finally

$$\int_{T\varphi^2}^{T} \frac{|M(x)|}{x} dx \geqslant c_{26}\varphi T^{1/2}.$$

This clearly implies (1.8).

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# On the zeros of Hecke's $L$-functions II

by

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### Introduction

**1.** In the first paper (see [1]) it has been proved in particular that the Hecke-Landau function $\zeta(s,\chi)$ of the field $K$ of degree $n \geqslant 1$ with a complex character $\chi$ modulo $\mathfrak{f}$ has no zero in a rectangle

$$1-A_0/\log D \leqslant \sigma \leqslant 1, \quad |t| \leqslant D^2$$

(where $D = |\Delta| N\mathfrak{f} \geqslant D_0 > 1$, $\Delta$ denotes the discriminant of the field and $A_0 > 0$ depends only on $n$). For at most one real $\chi$ in that rectangle may be a simple zero $\beta' = 1-\delta$ of $\zeta(s,\chi)$; it is real and, if $D_0$ is large enough, then

(1) $$\delta > D^{-2n}.$$

$\beta'$, if it exists, is called the "exceptional" zero. The corresponding character $\chi = \chi'$ and function $\zeta(s,\chi')$ also are called the "exceptional" ones. Consider, that $\chi'$ is a real character, not necessarily different from the principal one.

In this paper we shall prove the following

THEOREM. *There is an absolute constant $A > 0$ (which depends only on $n$) such that for*

$$\delta_0 = \begin{cases} \delta & \text{if} \quad \delta \leqslant A/\log D, \\ A/\log D & \text{otherwise}, \end{cases}$$

$$\lambda_0 = A \log \frac{eA}{\delta_0 \log D} \in [A, \tfrac{1}{2}\log D]$$

*in the rectangle* $(1-\lambda_0/\log D \leqslant \sigma \leqslant 1, |t| \leqslant D)$ *there is no zero of the function* $Z(s) = \prod_{\chi} \zeta(s,\chi)$ *with at most one exception* $\beta'$.

We may suppose that the exceptional zero exists. If it does not, then this theorem (with $\delta_0 = A_0/\log D$) is a simple consequence of that proved in [1]. And so it is (with $\delta_0 = \frac{1}{6}A_0/\log D$) if $\delta_0 \in [\frac{1}{6}A_0/\log D, A_0/\log D]$. Hence, in what follows we suppose that

$$\delta < \tfrac{1}{6}A_0/\log D.$$