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On representation by hermitian forms

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§ 1. Introduction. Let S be an m-rowed positive-definite symmetric matrix with rational integers as elements and T an n-rowed rational integral symmetric matrix ($n \leqslant m$). If $A(S, T)$ denotes the number of rational integral (m, n) matrices G satisfying $G'SG = T$, then we know from the researches of Siegel on the analytic theory of quadratic forms that a certain "weighted average" $A_0(S, T)$ of $A(S, T)$ over the classes in the 'genus' of S is equal to the infinite product of the p-adic densities of representation of T by S, p running over all the finite primes and the infinite prime spot. If the 'genus' of S contains only one class (this not being true, in general), then this gives us an 'exact' formula for $A(S, T)$ itself, in terms of the 'local' densities of representation. On the other hand, from the work of Hardy-Ramanujan, Hecke and Petersson, we have for $n = 1$ and $m > 4$, an 'asymptotic' formula for $A(S, T)$ in which the principal term is precisely $A_0(S, T)$. A generalization of this was obtained recently in [8], for $n \geqslant 1$ and $m > 2n+2$.

If S is now an m-rowed non-singular symmetric rational integral indefinite matrix and T, an n-rowed rational integral symmetric matrix, then Siegel [11] has shown 'with suitable mild restrictions on n' that $\mu(S, T)$ (the "measure of the representations of T by S") is equal to the infinite product of the p-adic densities of representation of T by S, extended over the rational primes p. Thus $\mu(S, T)$ is a 'genus-invariant' of S while for $S > 0$, $A(S, T)$ is not so. It is interesting to observe that whereas for $S > 0$, we have only an 'asymptotic' formula for $A(S, T)$, we have in the indefinite case, an 'exact' formula for $\mu(S, T)$.

Our object, in this paper, is to seek analogues of the foregoing, in the case of representation of hermitian matrices with elements which are integers in an imaginary quadratic field $k = \Gamma(\sqrt{d})$ of discriminant $d < 0$, over the field Γ of rational numbers. The methods are complicated by the fact that one has now to consider representation by singular matrices too.

For this purpose, we need to carry over Siegel's 'generalized Farey dissection' [11] to the space $\mathfrak{H}_n$ of n-rowed complex square matrices Z

with $i^{-1}(Z-\bar{Z}')>0$, with the hermitian modular group of degree n acting on $\mathfrak{H}_n$. This is done in § 5. We follow the pattern set by Siegel in [11] and also use some important results of H. Braun [2].

The main ideas of the paper may be set forth as follows. Let S be an m-rowed hermitian matrix of signature (p, q) and having elements which are integers in k and let H be a 'majorant' (see § 3) of S. Further let Z_1, Z_2 be two complex matrices such that $Z_1, -Z_2 \in \mathfrak{H}_n$. With S, H, Z_1 and Z_2 we associate a 'theta-series' $f(S, H, Z_1, Z_2, 0)$ (see § 3) and obtain its behaviour when Z_1, Z_2 are subjected to a general hermitian modular transformation of degree n (Theorem 1). We then specialize Z_2 to be $\bar{Z}_1'$. In case $S \geqslant 0$, we see that $f(S, H, Z_1, \bar{Z}_1', 0)$ is complex analytic in Z_1 and actually a hermitian modular form $f(S, Z_1)$ of degree n and dimension $-p$. If $pq>0$, then $f(S, H, Z_1, \bar{Z}_1')$ is no longer complex analytic in Z_1. However, in each case, we associate with $f(S, H, Z_1, \bar{Z}_1', 0)$, for $r=p+q>2n$, another function $\varphi(S, Z_1)$ (a generalized Eisenstein series) which 'mimics' $f(S, H, Z_1, \bar{Z}_1', 0)$ under a class of hermitian modular transformations on Z_1. The Fourier coefficients of this Eisenstein series are the so-called 'singular series'. Further on, we split our discussion into two parts, according as $S \geqslant 0$ or S is indefinite.

For $S \geqslant 0$, making use of the Farey dissection in $\mathfrak{H}_n$, we obtain, for the Fourier coefficients $c(T)$ of $f(S, Z_1)-\varphi(S, Z_1)$, for $T>0$ and $|T|>$ (a constant), the estimate

$$|c(T)| \leqslant \text{const}\cdot\{(\min T^{-1})^{(n-1)(n-r-1)}+|T|^{r-n}(\min T)^{(2n-r)/2}\}$$

where $\min R$ for $R=\bar{R}'>0$ denotes the 'minimum' of R (see § 2). Such an estimate, as it stands, is not useful, since as $|T|$ tends to infinity, $\min T$ may remain fixed. Thus, as in [8], in order to make it worthwhile, we impose on T the condition "$\min T \geqslant c|T|^{1/n}$" for a fixed constant $c>0$; for 'reduced' T this means that $|T|^{-1/n}T$ belongs to a compact set in the space of positive hermitian matrices of determinant 1. Our estimates will therefore involve this compact set. Writing $f(S, Z_1) = \varphi(S, Z_1)+f(S, Z_1)-\varphi(S, Z_1)$, we obtain under this condition on T the required 'asymptotic' formula for $A(S, T)$, the number of 'reduced' representations of T by S, in Theorem 5.

If S is indefinite, employing Farey dissection in a cube of 'height' ε (>0) in $\mathfrak{H}_n$, we prove Theorem 6 which, in particular, implies that under the stated conditions, S represents T 'in the large', if and only if S represents T 'locally' for every prime p. To obtain a quantitative refinement of this fact, we have to carry out an integration over a fundamental region in the 'majorant-space' of S, for the 'reduced unit-group' of S. The necessary preliminaries are given in § 6. We then define, after Siegel [11] and Ramanathan [9] the measure $\mu(S)$ of the 'reduced unit group' of S and the measure $\mu(S, G)$ of a 'reduced' representation G of $T=S[G]$ by S and obtain formulae (93) and (96) connecting them with integrals over certain fundamental regions in the 'majorant-space' of S. Defining $M(S, T)$, the 'measure of representation of T by S' by $M(S, T)=\sum_G \mu(S, G)$ (G running over a complete set of 'reduced' representations of T by S, of rank n and not associated on the left with respect to the 'reduced unit group' of S) and proceeding as in [11], we obtain for $p, q \geqslant n$ and $p+q>2n$, the formula

$$\frac{M(S, T)}{\mu(S)} = \prod_p a_p(S, T).$$

The right-hand side is the infinite product of the p-adic densities $a_p(S, T)$ of representation of T by S, over all rational primes p (see (54), for definition of $a_p(S, T)$).

In § 7, we go back to our 'asymptotic' formula for $A(S, T)$ for $S \geqslant 0$ and prove an analogue of a theorem of Tartakowsky's [14].

In § 8, we show that the considerations in §§ 3-5 may be generalized taking hermitian modular forms of degree n instead of the special theta-series. Finally using the Siegel operator Φ on hermitian modular forms [3], we obtain an asymptotic formula for $A(S, T)$, for $S \geqslant 0$, $T \geqslant 0$ and T not necessarily non-singular (Theorem 11).

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§ 2. Notation and generalities. By $A^{(m,n)}$ we mean a complex matrix of m rows and n columns or briefly, an (m, n) matrix. If $m=n$, then $A^{(m,n)}$ is denoted, for brevity, as $A^{(m)}$. The m-rowed identity matrix is always denoted by $E^{(m)}$ and a zero matrix by 0. For a given $A=(a_{kl})$, $\bar{A}$ is the *conjugate* $(\bar{a}_{kl})$, A' is the *transpose* of A and $\tilde{A}$ is the *conjugate transpose* of A. By $r(A)$ and $\delta(A)$, we mean respectively the *rank* and the *discriminant* of A. If B is a square matrix, then the *trace*, the *determinant* and the *absolute value of the determinant* of B are denoted respectively by $\sigma(B)$, $|B|$ and $\|B\|$. For a given square matrix C, $\eta(C)$ is an abbreviation for $e^{2\pi i\sigma(C)}$, where $i=\sqrt{-1}$. If $A=\tilde{A}$, then A is called *hermitian*. We abbreviate $\tilde{B}AB$ as $A[B]$.

In the following, k will denote a fixed imaginary quadratic field of discriminant $d<0$ over Γ, the field of rational numbers. By an *integer* we shall always mean an algebraic integer in k; the ring of integers in k is represented by $\mathfrak{O}$. If $\mathfrak{A} \subset k$, then $\{\mathfrak{A}\}_{m,n}$ denotes the set of all (m, n) matrices with elements in $\mathfrak{A}$. If $\mathfrak{A}$ is an ideal in $\mathfrak{O}$, then $A^{(m,n)}$, $B^{(m,n)}$ are said to be *congruent modulo* $\mathfrak{A}$ (in symbols, $A \equiv B \pmod{\mathfrak{A}}$), if $A-B \in \{\mathfrak{A}\}_{m,n}$. If $A \in \{\mathfrak{O}\}_{m,n}$, we say A is *integral*. If $A^{(n)}=(a_{kl})$ and if a_{kk}, $\sqrt{d}\,a_{kl} \in \mathfrak{O}$ for $1 \leqslant k, l \leqslant n$, then A is said to be *semi-integral*. If $A^{(m,n)}=B+GC$

with $G \in \{\mathfrak{O}\}_{m,n}$, then A is said to be *congruent to B modulo C*. An integral matrix U is *unimodular*, if it has an integral inverse. We denote the group of (n, n) unimodular matrices by $\Omega_n(k)$ or Ω_n, briefly.

If $A = \tilde{A}$, then we denote A being *positive-definite* or *non-negative definite*, in symbols, by $A > 0$ or $A \geqslant 0$ respectively. If $A = \tilde{A}$ has p positive and q negative characteristic roots, we say that the *signature* of A is (p, q). If $T^{(n)} > 0$, then, by $\min T$ (the *minimum* of T) we mean the smallest non zero real number of the form $\tilde{X} T X$ as X runs over n-rowed integral columns.

For an (n, n) matrix Z, we denote $\frac{1}{2}(Z+\tilde{Z})$ by $R(Z)$ and $\frac{1}{2i}(Z-\tilde{Z})$ by $I(Z)$. By $\mathfrak{H}_n$, we mean the space of (n, n) complex matrices with $I(Z) > 0$. The space of all (n, n) hermitian matrices H is denoted by $\mathcal{H}_n$. Now any $X \in \mathcal{H}_n$ can be written as $X_1 + iX_2$ with real $X_1 = (x_{ij}^{(1)})$ and real $X_2 = (x_{kl}^{(2)})$. By dX, we mean the volume element $\prod_{1 \leqslant i \leqslant j \leqslant n} dx_{ij}^{(1)} \prod_{1 \leqslant k < l \leqslant n} dx_{kl}^{(2)}$ in $\mathcal{H}_n$.

For $\alpha \in k$, $N(\alpha)$ and $Tr(\alpha)$ denote the *norm* and the *trace* of α over Γ respectively. For an ideal $\mathfrak{A}$ in k, $N(\mathfrak{A})$ stands for its norm over Γ. For $\mathfrak{O}$, we know $(1, \omega)$, where $\omega = (d + \sqrt{d})/2$, is an integral basis.

A pair (CD) of matrices in $\{k\}_{n,n}$ is said to be a *hermitian pair* if $C\tilde{D} = D\tilde{C}$. If, *in addition*, there exist $X, Y \in \{\mathfrak{O}\}_{n,n}$ such that $CX + DY = E^{(n)}$, then it is called a *coprime pair*.

If $\mathfrak{P}$ is the space of (n, n) complex matrices $P > 0$, then for $U \in \Omega_n$, the mapping $P \to P[U]$ is a mapping of $\mathfrak{P}$ onto itself. If we agree to identify U and ϱU for any root of unity ϱ in k, this gives a faithful representation of Ω_n as a discontinuous group of mappings of $\mathfrak{P}$ onto itself. Humbert [6] constructed for Ω_n a fundamental region $\mathfrak{T}^{(n)}$ consisting of all the so-called 'reduced' matrices in $\mathfrak{P}$. Further there exist finitely many (n, n) non-singular integral matrices $A_1, \ldots, A_\varkappa$ ($\varkappa$ depending only on k and n) such that for $P \in \mathfrak{T}^{(n)}$, there exists an A_i among these for which $P[A_i] = P_0 = (p_{lm})$ satisfies the conditions

$$(1) \qquad 0 < p_{ll} \leqslant c_1 p_{mm}, \qquad |p_{lm}| \leqslant c_1 p_{mm} \ (l \leqslant m), \qquad p_{11} \ldots p_{nn} \leqslant c_1 |P_0|$$

where c_1 is a constant depending only on k and n. We call the set of all matrices in $\mathfrak{P}$ for which (1) is satisfied, the *Humbert domain* and denote it by $\mathfrak{T}_0^{(n)}$. The matrices $A_1, \ldots, A_\varkappa$ may be called the *Humbert matrices of order n*.

From (1), it is trivial to deduce that for $P \in \mathfrak{P}$,

$$(2) \qquad 0 < \min P \leqslant \mu_n |P|^{1/n}$$

where again μ_n is a constant depending only on k and n.

The *congruence hermitian modular group* $\mathfrak{M}_n(s)$ *of degree n and stufe s* (relative to k) is defined as the group of $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \{\mathfrak{O}\}_{2n,2n}$ such that

$$M I \tilde{M} = I = \begin{pmatrix} 0 & E^{(n)} \\ -E^{(n)} & 0 \end{pmatrix}, \qquad M \equiv E^{(2n)} \ (\mathrm{mod}\,(s))$$

s being a positive rational integer and $A, B, C, D \in \{\mathfrak{O}\}_{n,n}$. For $s = 1$, $\mathfrak{M}_n(1) = \mathfrak{M}_n$ is the hermitian modular group of degree n.

For each $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{M}_n$, the mapping $Z \to M\langle Z \rangle = (AZ+B) \times (CZ+D)^{-1}$ is an analytic homeomorphism of $\mathfrak{H}_n$ onto itself. This gives a representation of $\mathfrak{M}_n(s)$ as a discontinuous group of mappings of $\mathfrak{H}_n$ onto itself. From [2, III], one can see the following set $\mathfrak{F}_n$ is a fundamental region for $\mathfrak{M}_n$ in $\mathfrak{H}_n$. $\mathfrak{F}_n$ is the set of $Z \in \mathfrak{H}_n$ for which

i) $\|CZ+D\| \geqslant 1$ for every n-rowed coprime pair (CD),

ii) $I(Z)$ is reduced in the sense of Humbert,

iii) if $X = R(Z) = \dot{X} + \omega \ddot{X}$ with real $\dot{X} = (\dot{x}_{ij})$ and real $\ddot{X} = (\ddot{x}_{kl})$ then $0 \leqslant \dot{x}_{ij}, \ddot{x}_{kl} \leqslant 1$ for $1 \leqslant i \leqslant j \leqslant n$ and $1 \leqslant k < l \leqslant n$.

It may be verified without difficulty that $\mathfrak{F}_n$ has the following two important properties, viz.

a) any compact set in $\mathfrak{H}_n$ is intersected at most by finitely many images of $\mathfrak{F}_n$ under hermitian modular transformations, and

b) there exists a constant γ_n depending only on k and n such that for $Z \in \mathfrak{F}_n$, we have $I(Z) > \gamma_n E^{(n)}$ and moreover $\min I(Z) > \gamma_n'$ for a constant γ_n' depending only on k and n.

A complex-valued function $f(Z)$ defined on $\mathfrak{H}_n$ is a *hermitian modular form of degree n, stufe s, dimension $-r$, belonging to a multiplier-system* $\{v(M) | M \in \mathfrak{M}_n(s)\}$ (in symbols, $f(Z) \in \{n, s, -r, v\}$) if

i) $f(Z)$ is regular in $\mathfrak{H}_n$ in the n^2 variables z_{kl} constituting $Z \in \mathfrak{H}_n$ and

ii) for every $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{M}_n(s)$, $f(Z)$ satisfies

$$f(Z)|M \equiv f(M\langle Z \rangle)|CZ+D|^{-r} = v(M) f(Z).$$

For $n = 1$, we require in addition that

iii) for all $M \in \mathfrak{M}_n$, $f(Z)/M$ is bounded in $\mathfrak{F}_n$.

We shall suppose that the multipliers $v(M)$ satisfy the condition $|v(M)| = 1$ for all $M \in \mathfrak{M}_n(s)$ and furthermore, that for $M = P^{-1}\begin{pmatrix} E^{(n)} & S \\ 0 & E \end{pmatrix} P \in \mathfrak{M}_n(s)$ with arbitrary $P \in \mathfrak{M}_n$, we have $v(M) = 1$. It is easily seen that under these conditions again, Satz 1 and 2 of H. Braun ([3], p. 138) continue to be valid. Thus, for $f(Z) \in \{n, s, -r, v\}$ and $M \in \mathfrak{M}_n$, we have the Fourier expansion

$$f(Z)/M = \sum_{T \geqslant 0} a(T, M)\eta(s^{-1}TZ)$$

the summation being over all (n, n) semi-integral $T \geqslant 0$.

We say $f(Z) \in \{n, s, -r, v\}$ is a *cusp form* if, for every $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in \{k\}_{n,n}$ and $RI\tilde{R} = I = \begin{pmatrix} 0 & E^{(n)} \\ -E^{(n)} & 0 \end{pmatrix}$ we have

$$\lim_{\lambda\to\infty} f\left(\left(A\begin{pmatrix} i\lambda & 0 \\ 0' & Z_1 \end{pmatrix} + B\right)\left(C\begin{pmatrix} i\lambda & 0 \\ 0' & Z_1 \end{pmatrix} + D\right)^{-1}\right)\left|C\begin{pmatrix} i\lambda & 0 \\ 0' & Z_1 \end{pmatrix} + D\right|^{-r} = 0$$

for $Z_1 \in \mathfrak{H}_{n-1}$ (see [3]). For $k = \Gamma$, this process is just the well-known *Siegel operator* Φ taking modular forms of degree n into modular forms of degree $n-1$ ([13]). If $f(Z) \in \{n, s, -r, v\}$ is a cusp form, then for $M \in \mathfrak{M}_n$, we have the Fourier expansion

$$f(Z)/M = \sum_{T > 0} a(T, M)\eta(s^{-1}TZ)$$

where T runs over all semi-integral positive-definite (n, n) matrices.

Two matrices $F, G \in \{k\}_{m,n}$ $(m \leqslant n)$ are said to *lie in a class*, if there exists a non-singular $K \in \{k\}_{m,m}$ such that $F = KG$. Using a lemma of Siegel ([10], Lemma 5, p. 219) it is easy to prove that given $F \in \{k\}_{m,n}$ of rank m, there exists $G \in \{\mathfrak{O}\}_{m,n}$ in the class of F such that $\delta(G)$ belongs to a *fixed* finite set of integral ideals in k and there exists a rational integer c_2 (depending only on k and n) such that

$$N(\delta(G)) \leqslant c_2 \tag{3}$$

(if $m = n$, this is quite trivial, since G may be chosen to be $E^{(m)}$). Further it is easy to show that there exists a rational integer c_3 (depending only on k and n) such that G can be completed to a non-singular matrix $H \in \{\mathfrak{O}\}_{n,n}$ for which

$$\|H\| \leqslant c_3 . \tag{4}$$

Given any $C \in \{k\}_{n,n}$ of rank r, there exists by [10] (Lemma 9, p. 223), a matrix $\check{O} \in \{\mathfrak{O}\}_{n,n}$ which is again of rank r, is idempotent (i.e. $\check{O}\check{O} = \check{O}$) and further satisfies $C\check{O} = C$. Such a matrix $\check{O}$ is known as a *right unit* (Rechtseinheit) of C or briefly a *r-unit* of C and is referred to as a *right-idem* by Braun [2, I]. If $r = n$, necessarily $\check{O} = E^{(n)}$. *Left-units* or briefly *l*-units are similarly defined.

Using (3) and (4), it is easy to prove the following, viz. for a given $C \in \{k\}_{n,n}$ of rank r and with a r-unit $\check{O}$, there exists a non-singular $Q \in \{\mathfrak{O}\}_{n,n}$ such that

$$Q = \begin{pmatrix} A^{(r,n)} \\ * \end{pmatrix}, \quad Q\check{O}Q^{-1} = \begin{pmatrix} E^{(r)} & 0 \\ * & 0 \end{pmatrix}, \quad N(\delta(A)) \leqslant c_2, \quad \|Q\| \leqslant c_3 . \tag{5}$$

Now, from $C\check{O} = C$, we have

$$CQ^{-1} = \begin{pmatrix} C_1^{(r)} & 0 \\ * & 0 \end{pmatrix}. \tag{6}$$

For $r = n$, Q may be taken to be $E^{(n)}$.

Let (CD) be an n-rowed coprime pair. We need, in the sequel a 'canonical decomposition' depending on $t = r(C)$ for the pair (CD), similar to what Siegel has obtained in the case of the rational number field (see [13], (12), p. 624). If $t = 0$, then $(CD) = (0U)$ with $U \in \Omega_n$, and we need nothing more. Let then $t \geqslant 1$. Then, by (6), there exists $Q \in \{\mathfrak{O}\}_{n,n}$, $0 < \|Q\| \leqslant c_3$ such that $CQ^{-1} = (C_0^{(n,t)} 0)$. Further we can find $U \in \Omega_n$ such that the first column of $|Q|UCQ^{-1}$ is of the form $(\alpha\beta 0 \dots 0)'$ with $\alpha, \beta \in \mathfrak{O}$. Let $\mathfrak{A}$ be the ideal (α, β) generated by α, β in $\mathfrak{O}$. In each of the h ideal classes of k, let us choose a fixed integral ideal $\mathfrak{A}_i$ (say, of minimum norm) with integral basis (α_i, β_i) $(1 \leqslant i \leqslant h)$. Corresponding to $\mathfrak{A}_i$, define $P_i = \begin{pmatrix} \delta_i & -\gamma_i \\ -\beta_i & \alpha_i \end{pmatrix}$ where $\gamma_i, \delta_i \in \mathfrak{O}_i$ such that $|P_i| = N(\mathfrak{A}_i)$. Now $\mathfrak{A} = (\mu)\mathfrak{A}_i$ for some i and $\mu \in k$. Then the first column of $|Q|\begin{pmatrix} P_i & 0 \\ 0 & E^{(n-2)} \end{pmatrix}UCQ^{-1}$ is $(\mu N(\mathfrak{A}_i)\, 0 \dots 0)'$. Applying induction on t, one can show that there exists a rational integer c_4 depending on k and n and $P \in \{\mathfrak{O}\}_{n,n}$ with $0 < \|P\| < c_4$ such that

$$PCQ^{-1} = \begin{pmatrix} C_1^{(t)} & 0 \\ 0' & 0 \end{pmatrix}, \quad C_1 \in \{\mathfrak{O}\}_{t,t}, \quad |C_1| \neq 0$$

and further such that c_4P^{-1}, c_4Q^{-1}, $c_4|Q|^{-1}$ are integral. Now writing $PD\tilde{Q} = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$, $D_1 = D_1^{(t)}$, we see that $D_3 = 0$; further $|D_4| \neq 0$ and D_4, $c_4^2D_4^{-1}$ are integral. Choosing $\begin{pmatrix} E^{(t)} & -D_2D_4^{-1} \\ 0 & D_4^{-1} \end{pmatrix}P$ instead of P, we obtain

LEMMA 1. *For any n-rowed coprime pair (CD) with $r(C) = t \geqslant 1$, there exist $P \in \{k\}_{n,n}$ and $Q \in \{\mathfrak{O}\}_{n,n}$ with $|P| \neq 0$ such that*

$$P(CQ^{-1}D\tilde{Q}) = \begin{pmatrix} C_1^{(t)} & 0 & D_1^{(t)} & 0 \\ 0 & 0 & 0 & E^{(n-t)} \end{pmatrix}, \quad C_1 \in \{\mathfrak{O}\}_{t,t}, \quad |C_1| \neq 0 \tag{7}$$

and further there exists a rational integer c_5 depending only on k and n such that c_5P, c_5P^{-1} and $c_5Q^{-1} \in \{\mathfrak{O}\}_{n,n}$. Further, if $Q = \begin{pmatrix} G_0^{(t,n)} \\ * \end{pmatrix}$, then $N(\delta(G_0)) \leqslant c_3$, and $c_5\delta(G_0)^{-1} \subset \mathfrak{O}$.

This is a sharper version of a lemma of H. Braun ([2, I], p. 830).

Let $Z \in \mathfrak{H}_n$. Using the form (7) of (CD), define

$$L = \begin{pmatrix} C_1^{-1}D_1 & 0 \\ 0' & 0 \end{pmatrix}[\widetilde{Q}^{-1}], \qquad P_1 = P^{-1}\begin{pmatrix} C_1^{(t)} & 0 \\ 0 & E \end{pmatrix}. \tag{8}$$

Then $|CZ+D| = |P_1||\widetilde{Q}^{-1}||G_0(Z+L)\widetilde{G}_0|$. Now, since $L = \widetilde{L}$, L could be written as $A^{-1}B$ with (AB) being an n-rowed coprime pair and $|A| \neq 0$. Further (AB) can be chosen such that $|A|$ is real and positive. We call A, a *denominator* of L and $|A|$ is denoted $d(L)$. It is easy to verify that $d(L)$ is uniquely defined. From [2, I] (Lemma 7, p. 844), we have

$$|P_1||\widetilde{Q}^{-1}|N(\delta(G_0)) = \varepsilon d(L) \tag{9}$$

where ε is a root of unity in k. Thus we have

$$|CZ+D| = \varepsilon d(L)N(\delta(G_0))^{-1}|G_0(Z+L)\widetilde{G}_0|. \tag{9*}$$

Let now $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{M}_n$. If $t = r(C) = 0$, then $M = \begin{pmatrix} \widetilde{U} & \widetilde{U}S \\ 0 & U^{-1} \end{pmatrix}$ with $U \in \Omega_n$ and $S = \widetilde{S} \in \{\mathfrak{O}\}_{n,n}$. Such M form a subgroup $\mathfrak{A}_n$ of $\mathfrak{M}_n$. We denote by $\mathfrak{G}_n$ the set of $A_0\langle Z\rangle$ for $Z \in \mathfrak{F}_n$ and $A_0 \in \mathfrak{A}_n$. From property b) of $\mathfrak{F}_n$, it is trivial to see that for $Z \in \mathfrak{G}_n$, $\min I(Z) > \gamma_n'$.

Suppose $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{M}_n$ with $t = r(C) \geqslant 1$. Using the form (7) of (CD), we see that

$$P^{-1}(AQ^{-1}B\widetilde{Q}) = \begin{pmatrix} A_1^{(t)} & 0 & B_1^{(t)} & B_2 \\ A_3 & E^{(n-t)} & B_3 & B_4 \end{pmatrix}.$$

Now, if $c_6 = c_5^2$, then clearly c_6A_1, c_6A_3 and c_6B_3 are all integral. Let $Q^* = \begin{pmatrix} E^{(t)} & 0 \\ c_6A_3 & c_6E \end{pmatrix}Q$ and $P^* = \begin{pmatrix} c_6^2E^{(t)} & 0 \\ 0 & c_6^{-1}E \end{pmatrix}P$. Then it is easy to verify that

$$M^{-1} = \begin{pmatrix} \widetilde{D} & -\widetilde{B} \\ -\widetilde{C} & \widetilde{A} \end{pmatrix} = \begin{pmatrix} Q^{*-1} & 0 \\ 0 & \widetilde{Q}^* \end{pmatrix}\begin{pmatrix} \widetilde{D}_1^* & 0 & -\widetilde{B}_1^* & -\widetilde{B}_3^* \\ A_3^*\widetilde{D}_1^* & E & -\widetilde{B}_2^* & -\widetilde{B}_4^* \\ -\widetilde{C}_1^* & 0 & \widetilde{A}_1^* & 0 \\ 0 & 0 & 0 & E \end{pmatrix}\begin{pmatrix} \widetilde{P}^{*-1} & 0 \\ 0 & P^* \end{pmatrix}, \tag{10}$$

where $A_1^* = c_6^{-2}A_1$, $B_1^* = c_6^{-2}B_1$, $C_1^* = c_6^2C_1$, $D_1^* = c_6^2D_1$, $A_3^* = c_6A_3$, $B_3^* = c_6B_3$, $B_2^* = c_6^{-1}(B_1\widetilde{A}_3+B_2)$ and $B_4^* = c_6^2(B_3A_3+B_4)$. We shall use the reduction (10) of M^{-1} later, when we study the 'transformation formulae'

of 'generalized theta-series' in § 3. The following lemma whose proof, being completely similar to a lemma of Siegel ([11], Lemma 8, p. 585), we shall omit, shall also be needed in the same connection.

Lemma 2. *For $Z \in \mathfrak{H}_n$, $M \in \mathfrak{M}_n$ and $Z^* = M^{-1}\langle Z\rangle$, let $Z[P^{*-1}] = \begin{pmatrix} Z_a^{(t)} & Z_b \\ Z_c & Z_d \end{pmatrix}$, $Z_e = Z_a - \widetilde{C}_1^{*-1}\widetilde{A}_1^*$ using the form (10) of M^{-1}. Then*

$$Z^*[\widetilde{Q}^*] = \begin{pmatrix} -\widetilde{D}_1^*\widetilde{C}_1^{*-1} & -\widetilde{B}_3^* \\ -B_3^* & Z_d - \widetilde{B}_4^* \end{pmatrix} - \begin{pmatrix} C_1^{*-1} \\ Z_c \end{pmatrix} Z_e^{-1}(\widetilde{C}_1^{*-1}Z_b). \tag{11}$$

Two n-rowed coprime pairs (CD), (C_1D_1) are said to be *in the same class*, if $(CD) = U(C_1D_1)$ for $U \in \Omega_n$. We denote the class of (CD) by $\{C, D\}$.

Let $\mathfrak{L}^{(t,n)}$ $(t < n)$ denote a complete set of (t, n) integral matrices of rank t such that for no two elements $G_i, G_j \in \mathfrak{L}^{(t,n)}$, we have $G_i = RG_j$ with $R \in \{k\}_{n,n}$, $|R| \neq 0$. In view of (3), we may suppose $G_0 \in \mathfrak{L}^{(t,n)}$ to have been so chosen that $N(\delta(G_0)) \leqslant c_2$ and to each $G_0 \in \mathfrak{L}^{(t,n)}$ we assign a fixed complement G_0^* so that $Q = \begin{pmatrix} G_0 \\ G_0^* \end{pmatrix} \in \{\mathfrak{O}\}_{n,n}$ and $0 < \|Q\| \leqslant c_3$. In the form (7) of (CD), we may thus suppose $G_0 \in \mathfrak{L}^{(t,n)}$ and Q to be the non-singular matrix associated with G_0 as above.

With the notation as in Lemma 1, let $\check{O}_1 = Q^{-1}\begin{pmatrix} E^{(t)} & 0 \\ 0 & 0 \end{pmatrix}Q$. Then, for any r-unit $\check{O}$ of C, we have $\check{O}_1\check{O} = \check{O}_1$ and $\check{O}\check{O}_1 = \check{O}$. Although $\check{O}_1$ is not necessarily integral, we may look upon $\check{O}_1$ as a *'generalized r-unit'* of C, viz. satisfying $C\check{O}_1 = C$, $r(\check{O}_1) = t$ and $\check{O}_1\check{O}_1 = \check{O}_1$. In the notation of H. Braun [2, I], both $\check{O}$ and $\check{O}_1$ determine the same "type" of right idems. By Lemma 5 of H. Braun ([2, I], p. 843), the classes $\{C, D\}$ with $r(C) = t$ stand in one-one correspondence with the "types" of right idems $\check{O}_1$ of rank t and the matrices L defined by (8) and satisfying $\check{O}_1L = L$. On the other hand, it is easy to prove that the "types" of right idems $\check{O}$ of rank t are in one-one correspondence with $G_0 \in \mathfrak{L}^{(t,n)}$ under the correspondence that to the "type" of $\check{O} = Q^{-1}\begin{pmatrix} E^{(t)} & 0 \\ 0 & 0 \end{pmatrix}Q$ with $Q = \begin{pmatrix} G_0^{(t,n)} \\ G_0^* \end{pmatrix}$, we associate $G_0 \in \mathfrak{L}^{(t,n)}$. Thus we have, as an analogue of a lemma of Siegel ([11], Lemma 5, p. 584),

Lemma 3. *The classes $\{C, D\}$ of n-rowed coprime pairs (CD) with $t = r(C) \geqslant 1$ are in one-one correspondence with $G_0 \in \mathfrak{L}^{(t,n)}$ and $L = \widetilde{L} \in \{k\}_{n,n}$ for which $Q^{-1}\begin{pmatrix} E^{(t)} & 0 \\ 0 & 0 \end{pmatrix}QL = L$, where $Q = \begin{pmatrix} G_0 \\ G_0^* \end{pmatrix}$ is associated with $G_0 \in \mathfrak{L}^{(t,n)}$.*

For $t = n$, $\mathfrak{L}^{(t,n)}$ is taken to consist just of $E^{(n)}$ and then the associated Q is equal to $E^{(n)}$.

In symbols, we denote the correspondence referred to in Lemma 3, by $\{C, D\} \leftrightarrow [G_0, L]$.

We note finally that the classes $\{A, B\}$ of n-rowed coprime pairs with $r(A) = n$ are in one-one correspondence with hermitian matrices in $\{k\}_{n,n}$ under the assignment $\{A, B\} \to A^{-1}B$ ([2, I]).

§ 3. Generalized theta-series and transformation-formulae. Let $S = \tilde{S}$ be an m-rowed integral matrix of signature (p, q) with $p, q \geqslant 0$ and $p+q = r = r(S)$. We say that S *represents* $T^{(n)} = \tilde{T}$ *integrally* if there exists $G \in \{\mathfrak{O}\}_{m,n}$ such that $S[G] = T$; G is called an *integral representation* of T by S. Hereafter, we shall mean by a 'representation', an integral representation always. In order to make a quantitative study of the representations of T by S, we have to consider a so-called 'generalized theta-series associated with S'. In the case when S is definite and $r = m$, for example, the number of representations of T by S is finite and the theta-series is merely a 'generating function' for the number of representations of T by S.

Let E_S be a *fixed* r-unit of S. If $r = m$, then $E_S = E^{(m)}$. By a lemma of Siegel ([10], Lemma 30, p. 232), there exists $Q \in \{\mathfrak{O}\}_{m,m}$ with $|Q| \neq 0$ such that

$$(12) \qquad QE_SQ^{-1} = \begin{pmatrix} E^{(r)} & 0 \\ 0' & 0 \end{pmatrix}.$$

From $SE_S = S = \tilde{S}$, it follows that

$$(13) \qquad S[Q^{-1}] = \begin{pmatrix} S_1^{(r)} & 0 \\ 0' & 0 \end{pmatrix}, \qquad |S_1| \neq 0.$$

Writing $Q = \begin{pmatrix} A^{(r,m)} \\ A^* \end{pmatrix}$, we have

$$(14) \qquad S = S_1[A], \qquad AE_S = A.$$

Now, by [10], there exists a unique inverse A^{-1} of A satisfying

$$(15) \qquad A^{-1}A = E_S, \qquad AA^{-1} = E^{(r)}$$

and then

$$(16) \qquad S_1 = S[A^{-1}].$$

A representation G of T by S is called E_S-*reduced* (or briefly, *reduced*) if $E_SG = G$.

We remark, in passing, that if $S \geqslant 0$ and $S[G] = T$, $E_SG = G$, then for any r-unit E_T, of T, we have *necessarily* $GE_T = G$; for, $S[GE_T - G] = 0$ and $E_SG = G$ together imply that $GE_T - G = 0$.

Let $\mathfrak{P}(S)$ denote the set of all m-rowed complex $H = \tilde{H}$ satisfying

$$(17) \qquad HS^{-1}H = S, \qquad HE_S = H \geqslant 0,$$

where S^{-1} is the unique inverse of S determined by

$$S^{-1}S = E_S, \qquad SS^{-1} = \tilde{E}_S.$$

It is easy to verify that the definition (17) of $\mathfrak{P}(S)$ is independent of the choice of the r-unit E_S of S.

From $HE_S = H$, we have similar to (13)

$$(18) \qquad H[Q^{-1}] = \begin{pmatrix} H_1^{(r)} & 0 \\ 0 & 0 \end{pmatrix}, \qquad H_1 = \tilde{H}_1 > 0$$

and again similar to (14), (16) we have

$$(19) \qquad H = H_1[A], \qquad H_1 = H[A^{-1}],$$

where A^{-1} satisfies (15). Furthermore, H_1 satisfies

$$(17') \qquad H_1S_1^{-1}H_1 = S_1, \qquad H_1 > 0.$$

The space of H_1 for which (17′) is true is merely the space $\mathfrak{P}(S_1)$ attached to S_1 and is known as the *majorant-space* of S_1 or the *symmetric Riemannian space associated with* S_1 (see [9], § 9). When S_1 is definite, then $\mathfrak{P}(S_1)$ consists just of S_1 or $-S_1$. The space $\mathfrak{P}(S)$ and the majorant-space $\mathfrak{P}(S_1)$ are homeomorphic under the correspondence (19).

Let now $Z_1, -Z_2 \in \mathfrak{H}_n$. Corresponding to $S, H \in \mathfrak{P}(S)$ and $V^{(m,n)}$, an arbitrary complex matrix, we define the *generalized theta-series* $f(S, H, Z_1, Z_2, V)$ by

$$(20) \qquad f(S, H, Z_1, Z_2, V) = \sum_{E_SG = G \in \{\mathfrak{O}\}_{m,n}} \eta\left(\tfrac{1}{2}H[G+V](Z_1 - Z_2) + \tfrac{1}{2}S[G+V](Z_1+Z_2)\right).$$

When G runs over all (m, n) integral matrices with $E_SG = G$, we see that $G_1 = AG$ covers a lattice $\mathfrak{N}$ of rank $2rn$ over the field of real numbers. Writing $V_1 = AV$, we obtain

$$(20') \qquad f(S, H, Z_1, Z_2, V) = \sum_{G_1 \in \mathfrak{N}} \eta\left(\tfrac{1}{2}H_1[G_1+V_1](Z_1 - Z_2) + \tfrac{1}{2}S_1[G_1+V_1](Z_1+Z_2)\right).$$

There exists a complex non-singular matrix C such that $H_1[C] = E^{(r)}$ and $S_1[C]$ is a diagonal matrix, with $d_1, \ldots, d_r$ as diagonal elements. In view of (17′), $d_j = \pm 1$, necessarily and we may suppose $d_1 = \ldots = d_p = +1$, without loss of generality. It is easy to prove that, for $I(Z_1) > \mu E^{(n)}$, $-I(Z_2) > \mu E^{(n)}$ $(\mu > 0)$, the series (20′) is majorized by $\sum_{G_1 \in \mathfrak{N}} \eta(i\mu H_1[G_1 + V_1])$ which is clearly absolutely convergent. Thus the series (20′) converges absolutely and uniformly over compact sets in $\mathfrak{H}_n$ to which $Z_1, -Z_2$ belong. Therefore, $f(S, H, Z_1, Z_2, V)$ represents an analytic function of Z_1 and Z_2 for $Z_1, -Z_2 \in \mathfrak{H}_n$.

Our object now will be to study the behaviour of $f(S, H, Z_1, Z_2, 0)$ under the simultaneous transformations $Z_1 \to M\langle Z_1\rangle$, $Z_2 \to M\langle Z_2\rangle$ for $M \in \mathfrak{M}_n$, with a view to get certain inequalities concerning $f(S, H, Z_1, \tilde{Z}_1, 0)$ necessary in § 5. As a first step in this direction, we have the *theta-transformation formula* (viz. when $M = \begin{pmatrix} 0 & E^{(n)} \\ -E^{(n)} & 0 \end{pmatrix}$) i.e.

$$(21)\quad f(S, H, -Z_1^{-1}, -Z_2^{-1}, V) = |\delta(S)|^{-n}|d|^{-rn/2}|-iZ_1|^p|iZ_2|^q \times$$
$$\times \sum_{\tilde{E}_S G = G \in \{(\sqrt{d})^{-1}\}_{m,n}} \eta\left(\tfrac{1}{2}H^{-1}[G](Z_1 - Z_2) + \tfrac{1}{2}S^{-1}[G](Z_1+Z_2) + \tilde{G}V + \tilde{V}G\right)$$

the summation on the right-hand side being over all $G \in \{(\sqrt{d})^{-1}\}_{m,n}$ for which $\tilde{E}_S G = G$ and further H^{-1} is the unique inverse of H for which $HH^{-1} = \tilde{E}_S$, $H^{-1}H = E_S$. Formula (21) is a direct consequence of the well-known *Poisson summation formula* in several variables. Since $\mathfrak{N}$ is a lattice of (maximal) rank $u = 2rn$ over the reals, there exist generators $C_1, C_2, \ldots, C_u$ of $\mathfrak{N}$ such that $G_1 \in \mathfrak{N}$ if and only if $G_1 = g_1 C_1 + \ldots + g_u C_u$ with rational integers g_i. Writing $V_1 = v_1 C_1 + \ldots + v_u C_u$ with real numbers $v_1, \ldots, v_u$, we see that

$$(22)\quad f(S, H, -Z_1^{-1}, -Z_2^{-1}, V)$$
$$= \sum_{g_i=-\infty}^{\infty} \eta\left(-\tfrac{1}{2}H_1\Big[\sum_{j=1}^{u}(g_j+v_j)C_j\Big](Z_1^{-1} - Z_2^{-1}) - \tfrac{1}{2}S_1\Big[\sum_{j=1}^{u}(g_j+v_j)C_j\Big](Z_1^{-1}+Z_2^{-1})\right).$$

Applying the Poisson summation formula to (22) and noticing that the lattice $\mathfrak{N}^*$ 'complementary' to $\mathfrak{N}$ is the set of $G_1^* \in \{k\}_{r,n}$ for which $T_r(\sigma(\tilde{G}_1^* G_1))$ is a rational integer for every $G_1 \in \mathfrak{N}$, we obtain

$$f(S, H, -Z_1^{-1}, -Z_2^{-1}, V) = |\delta(S)|^{-n}|d|^{-rn/2}|-iZ_1|^p|iZ_2|^q \times$$
$$\times \sum_{G_1^* \in \mathfrak{N}^*} \eta\left(+\tfrac{1}{2}H^{-1}[\tilde{A}G_1^*](Z_1 - Z_2) + \tfrac{1}{2}S^{-1}[\tilde{A}G_1^*](Z_1+Z_2) + \tilde{G}_1^* A V + \tilde{V}\tilde{A}G_1^*\right).$$

Now it is easily seen that $G_1^* \in \mathfrak{N}^*$ if and only if $G^* = \tilde{A}G_1^* \in \{(\sqrt{d})^{-1}\}_{m,n}$ and further $\tilde{E}_S G^* = G^*$. Thus (21) is an immediate consequence.

Let now $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{M}_n$. If $t = r(C) = 0$, then $M \in \mathfrak{A}_n$ and it is trivial to verify that $f(S, H, M\langle Z_1\rangle, M\langle Z_2\rangle, 0) = f(S, H, Z_1, Z_2, 0)$. Let then $t \geqslant 1$.

Referring to the decomposition (10) of M^{-1}, let us assume that $I(Z_1[P^{*-1}]) = E$, $I(Z_2[P^{*-1}]) = -E$. More explicitly, let

$$(23)\quad \begin{gathered} Z_1[P^{*-1}] = \begin{pmatrix} Z_{1a}^{(t)} & Z_{1b} \\ \tilde{Z}_{1b} & Z_{1c} \end{pmatrix}, \quad Z_2[P^{*-1}] = \begin{pmatrix} Z_{2a}^{(t)} & Z_{2b} \\ \tilde{Z}_{2b} & Z_{2c} \end{pmatrix} \\ I(Z_{1a}) = E^{(t)} = -I(Z_{2a}), \quad I(Z_{1c}) = E^{(n-t)} = -I(Z_{2c}), \\ Z_{1e} = Z_{1a} - \tilde{C}_1^{*-1}\tilde{A}_1^*, \quad Z_{2e} = Z_{2a} - \tilde{C}_1^{*-1}\tilde{A}_1^*. \end{gathered}$$

Let $R = (R_1^{(m,t)} R_2^{(m,n-t)})$ run through a complete set of modulo (1) incongruent (m, n) matrices of the form GQ^{*-1} with $G \in \{\mathfrak{D}\}_{m,n}$ satisfying $E_S G = G$. This set of R is finite, in view of the fact that $c_6^2 GQ^{*-1}$ is integral for all $G \in \{\mathfrak{D}\}_{m,n}$. Splitting up G as $(G_1^{(m,t)} G_2^{(m,n-t)})$ we have, for $Z_1^* = M^{-1}\langle Z_1\rangle$, $Z_2^* = M^{-1}\langle Z_2\rangle$ that

$$(24)\quad f(S, H, Z_1^*, Z_2^*, 0)$$
$$= \sum_{\substack{E_S G = G = (G_1 G_2) \in \{\mathfrak{D}\}_{m,n} \\ R = (R_1 R_2)}} \eta\left(\tfrac{1}{2}(S+H)[G_1+R_1\ \ G_2+R_2]Z_1^*[\tilde{Q}^*] + \right.$$
$$\left. + \tfrac{1}{2}(S-H)[G_1+R_1\ \ G_2+R_2]Z_2^*[\tilde{Q}^*]\right)$$

where $G = (G_1 G_2)$ runs over all elements of $\{\mathfrak{D}\}_{m,n}$ with $E_S G = G$ and $R = (R_1 R_2)$ over the system described above. In view of Lemma 2, with (11) applied to $Z_1^*[\tilde{Q}^*]$ and $Z_2^*[\tilde{Q}^*]$, the summand in (24) is precisely

$$(25)\quad \eta(S[G_1+R_1\ \ G_2+R_2]F) \times$$
$$\times \eta\left(\tfrac{1}{2}(S+H)[G_2+R_2]Z_{1c} + \tfrac{1}{2}(S-H)[G_2+R_2]Z_{2c}\right) \times$$
$$\times \eta\left(-\tfrac{1}{2}(S+H)[(G_1+R_1)C_1^{*-1} + (G_2+R_2)\tilde{Z}_{1b}]Z_{1e}^{-1} - \right.$$
$$\left. - \tfrac{1}{2}(S-H)[(G_1+R_1)C_1^{*-1} + (G_2+R_2)\tilde{Z}_{2b}]Z_{2e}^{-1}\right),$$

where $F = \begin{pmatrix} -\tilde{D}_1^*\tilde{C}_1^{*-1} & -\tilde{B}_3^* \\ -B_3^* & -\tilde{B}_4^* \end{pmatrix}$. Using the relations $\frac{1}{2}(S \pm H)S^{-1}(S \pm H) = S \pm H$, $(S+H)S^{-1}(S-H) = 0$, we may rewrite the last factor in (25) as $\eta\left(-\frac{1}{2}(S+H)[(G_1+R_1)C_1^{*-1} + U]Z_{1e}^{-1} - \frac{1}{2}(S-H)[(G_1+R_1)C_1^{*-1} + U]Z_{2e}^{-1}\right)$ where $U = \frac{1}{2}S^{-1}(S+H)(G_2+R_2)\tilde{Z}_{1b} + \frac{1}{2}S^{-1}(S-H)(G_2+R_2)\tilde{Z}_{2b}$. We write now $G_1 = P_1 + G_1^* C_1^*$; clearly, if P_1 runs over a full system of matrices in $\{\mathfrak{D}\}_{m,t}$ incongruent modulo C_1^* and satisfying $E_S P_1 = P_1$ and G_1^* covers all elements of $\{\mathfrak{D}\}_{m,t}$ with $E_S G_1^* = G_1^*$, then G_1 covers exactly once all matrices in $\{\mathfrak{D}\}_{m,t}$ with $E_S G_1 = G_1$. Since $C_1^*, D_1^* \in \{(c_6^2)\}_{t,t}$ and further $c_6^2 R_1$ and $\widetilde{C}_1^* \widetilde{G}_1^* S R_2 B_3^*$ are integral, we see that $\eta(S[P_1 + G_1^* C_1^* + R_1 G_2 + R_2]F)$ is independent of G_1^*. As a consequence, we have

$$(26)\quad f(S, H, Z_1^*, Z_2^*, 0) = \sum_{\substack{R=(R_1,R_2),\, E_S G_2 = G_2 \in \{\mathfrak{O}\}_{m,n-t} \\ E_S P_1 = P_1^{(m,t)} \bmod C_1^*}} \eta(S[P_1 + R_1 G_2 + R_2]F) \times$$
$$\times \eta\left(\tfrac{1}{2}(S+H)[G_2+R_2]Z_{1c} + \tfrac{1}{2}(S-H)[G_2+R_2]Z_{2c}\right) \times$$
$$\times \sum_{E_S G_1 = G_1 \in \{\mathfrak{O}\}_{m,t}} \eta\left(-\tfrac{1}{2}(S+H)[G_1 + (P_1+R_1)C_1^{*-1} + U]Z_{1e}^{-1} - \right.$$
$$\left. -\tfrac{1}{2}(S-H)[G_1 + (P_1+R_1)C_1^{*-1} + U]Z_{2e}^{-1}\right).$$

Applying (21) to the inner sum in (26), we see that it is equal to

$$(27)\quad |\delta(S)|^{-t}|d|^{-rt/2}|-iZ_{1e}|^p|iZ_{2e}|^q \sum_{E_S G_1 = G_1 \in \{(\sqrt{d})^{-1}\}_{m,t}} \eta\left(\tfrac{1}{2}H^{-1}[G_1](Z_{1e} - Z_{2e}) + \right.$$
$$\left. + \tfrac{1}{2}S^{-1}[G_1](Z_{1e}+Z_{2e}) + \widetilde{G}_1 F_1 + \widetilde{F}_1 G_1\right)$$

where $F_1 = (P_1 + R_1)C_1^{*-1} + U$.

Let $Q_1^{(m,t)}$ run over a full system of modulo (1) incongruent matrices of the form $(\sqrt{d})^{-1}S^{-1}G_1$ with $G_1 \in \{\mathfrak{O}\}_{m,t}$ and $\widetilde{E}_S G_1 = G_1$. Clearly $E_S Q_1 = Q_1$. Moreover, when G_1^* runs over all elements in $\{\mathfrak{O}\}_{m,t}$ with $E_S G_1^* = G_1^*$ and Q_1 over the above-mentioned system, then $G_1^* + Q_1$ runs over all matrices of the form $(\sqrt{d})^{-1}S^{-1}G_1$ with $G_1 \in \{\mathfrak{O}\}_{m,t}$ and $E_S G_1 = G_1$. Noting that $H^{-1}[G_1] = H[(\sqrt{d})^{-1}S^{-1}\sqrt{d}\,G_1]$ and $S^{-1}[G_1] = S[(\sqrt{d})^{-1}S^{-1}\sqrt{d}\,G_1]$, we find that (27) is exactly

$$|\delta(S)|^{-t}|d|^{-rt/2}|-iZ_{1e}|^p|iZ_{2e}|^q \sum_{\substack{E_S G_1 = G_1 \in \{\mathfrak{O}\}_{m,t} \\ Q_1}} \eta\left(\tfrac{1}{2}H[G_1+Q_1](Z_{1e}-Z_{2e}) + \right.$$
$$\left. + \tfrac{1}{2}S[G_1+Q_1](Z_{1e}+Z_{2e}) + (\widetilde{G_1+Q_1})SF_1 + \widetilde{F}_1 S(G_1+Q_1)\right)$$

where G_1 runs over all elements of $\{\mathfrak{O}\}_{m,t}$ with $E_S G_1 = G_1$ and Q_1 over the system described above. It is easy to see that

$$|-iZ_{1e}|^p = |-i(Z_{1a} - \widetilde{C}_1^{*-1}\widetilde{A}_1^*)|^p = e^{\pi itp/2}|\widetilde{C}_1^*|^{-p}|\widetilde{Q}^*|^{-p}|-\widetilde{C}Z_1 + \widetilde{A}|^p|P^*|^{-p}.$$

Referring to the decomposition (7) of (CD) and the definition (8) of L, we have from (9)

$$(9')\qquad |P|^{-1}|C_1||\widetilde{Q}|^{-1}N(\delta(G_0)) = \varepsilon d(L)$$

ε being a root of unity in k. But

$$(28)\qquad |P^*|^{-1}|\widetilde{Q}^*|^{-1}|C_1^*| = |P|^{-1}|\widetilde{Q}|^{-1}|C_1|.$$

Therefore

$$|-iZ_{1e}|^p = \varepsilon^p e^{\pi itp/2}\|C_1^*\|^{-2p}N(\delta(G_0))^{-p}d(L)^p|-\widetilde{C}Z_1 + \widetilde{A}|^p$$

and in a similar way, we have

$$|iZ_{2e}|^q = \varepsilon^q e^{-\pi itq/2}\|C_1^*\|^{-2q}N(\delta(G_0))^{-q}d(L)^q|-\widetilde{C}Z_2 + \widetilde{A}|^q.$$

Thus

$$(29)\quad f(S, H, Z_1^*, Z_2^*, 0)|-\widetilde{C}Z_1 + \widetilde{A}|^{-p}|-\widetilde{C}Z_2 + \widetilde{A}|^{-q}$$
$$= \varepsilon^r e^{(\pi it/2)(2p-r)}|\delta(S)|^{-t}|d|^{-rt/2}\|C_1^*\|^{-2r}d(L)^r N(\delta(G_0))^{-r} \times$$
$$\times \sum_{\substack{E_S G = G = (G_1 G_2) \\ E_S Q_1 = Q_1,\, E_S R = R = (R_1 R_2)}} \eta\left(\tfrac{1}{2}(S+H)[G_1+Q_1\ \ G_2+R_2]Z_1[P^{*-1}] + \right.$$
$$\left. + \tfrac{1}{2}(S-H)[G_1+Q_1\ \ G_2+R_2]Z_2[P^{*-1}]\right) \times$$
$$\times \sum_{E_S P_1 = P_1 \bmod C_1^*} \eta\left(-S[G_1+Q_1]\widetilde{C}_1^{*-1}\widetilde{A}_1^* + (\widetilde{G_1+Q_1})SF_2 + \widetilde{F}_2 S(G_1+Q_1)\right) \times$$
$$\times \eta(S[P_1 + R_1\ \ G_2 + R_2]F),$$

where $F_2 = (P_1+R_1)\widetilde{C}_1^{*-1}$. Now $(G_1+Q_1\ \ G_2+R_2)\widetilde{P}^{*-1} = \gamma^{-1}G^*$ for $G^* \in \{\mathfrak{O}\}_{m,n}$ with $E_S G^* = G^*$, where $\gamma = c_6^5|\delta(S)||d|^r$ is a rational integer which is clearly independent of G_1, G_2, Q_1, R_2, P^* and depends only on k, n and S. For such G^*, we define the generalized 'Gauss sum' $\lambda(G^*, M^{-1})$ corresponding to $M^{-1} \in \mathfrak{M}_n$ by

$$(30)\quad \lambda(G^*, M^{-1}) = \varepsilon^r e^{\pi it(2p-r)/2}|\delta(S)|^{-t}|d|^{-rt/2}\|C_1^*\|^{-2r}d(L)^r N(\delta(G_0))^{-r} \times$$
$$\times \sum_{\substack{E_S P_1 = P_1 \bmod C_1^* \\ E_S R_1 = R_1}} \eta\left(-S[G_1+Q_1]\widetilde{C}_1^{*-1}\widetilde{A}_1^* + (\widetilde{G_1+Q_1})SF_2 + \widetilde{F}_2 S(G_1+Q_1)\right) \times$$
$$\times \eta(S[P_1+R_1\ \ G_2+R_2]F),$$

where

$$F_2 = (P_1+R_1)\widetilde{C}_1^{*-1}, \qquad F = \begin{pmatrix} -\widetilde{D}_1^*\widetilde{C}_1^{*-1} & -\widetilde{B}_3^* \\ -B_3^* & -\widetilde{B}_4^* \end{pmatrix}$$

and further, in the summation in (30), P_1 runs over a complete set of elements in $\{\mathfrak{O}\}_{m,t}$ with $E_S P_1 = P_1$ and incongruent modulo C_1^* and R_1 runs over a set of (m, t) matrices for which there is an $R = (R_1 R_2)$ in the system described in (24).

We have proved (29) only under the assumptions (23). Now both sides of (29) represent analytic functions of Z_1 and Z_2 for $Z_1, -Z_2 \in \mathfrak{H}_n$. But $Z_1, -Z_2 \in \mathfrak{H}_n$ if and only if $W_1 = (w_{ij}^{(1)}) = Z_1[P^{*-1}]$ and $-W_2 = -Z_2[P^{*-1}] = (w_{kl}^{(2)})$ are in $\mathfrak{H}_n$. Since both sides of (29) which are analytic in $w_{ij}^{(1)}, w_{kl}^{(2)}$ for $1 \leqslant i, j, k, l \leqslant n$, coincide on the domain

$$w_{ij}^{(1)} = \overline{w_{ji}^{(1)}}, \quad w_{kl}^{(2)} = \overline{w_{lk}^{(2)}}, \quad R(w_{ij}^{(1)}) \text{ and } R(w_{kl}^{(2)}) \text{ arbitrary, } i \neq j,\ k \neq l,$$
$$I(w_{ii}^{(1)}) = I(w_{jj}^{(2)}) = 1, \quad R(w_{ii}^{(1)}) \text{ and } R(w_{jj}^{(2)}) \text{ arbitrary,}$$

they coincide, by analytic continuation, for all $w_{ij}^{(1)}, w_{kl}^{(2)}$ for which $W_1, -W_2 \in \mathfrak{H}_n$. Thus (29) is valid for *all* $Z_1, -Z_2 \in \mathfrak{H}_n$ and we have

THEOREM 1. *Under the transformation* $Z_1 \to Z_1^* = M^{-1}\langle Z_1\rangle$, $Z_2 \to Z_2^* = M^{-1}\langle Z_2\rangle$, *where* $M = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \mathfrak{M}_n$, $t = r(C) \geqslant 1$ *and* M^{-1} *has the decomposition* (10), *the behaviour of* $f(S, H, Z_1, Z_2, 0)$ *is given by*

$$f(S, H, Z_1^*, Z_2^*, 0) = |-\widetilde{C}Z_1 + \widetilde{A}|^p |-\widetilde{C}Z_2 + \widetilde{A}|^q \times$$
$$\times \sum_{G_1, G_2, Q_1, R_2} \lambda(G^*, M^{-1})\,\eta\Big(\frac{i}{2\gamma^2}(S+H)[G^*]Z_1 + \frac{1}{2\gamma^2}(S-H)[G^*]Z_2\Big)$$

where, in the summation, G_1 *and* G_2 *run over all matrices in* $\{\mathfrak{O}\}_{m,t}$ *and* $\{\mathfrak{O}\}_{m,n-t}$ *respectively with* $E_S G_1 = G_1$ *and* $E_S G_2 = G_2$, Q_1, R_2 *run over the finite systems described earlier,* $G^* = (G_1 + Q_1\ G_2 + R_2)\widetilde{P}^{*-1}$ *and* $\lambda(G^*, M^{-1})$ *is defined by* (30).

Before we proceed further, we ought to know more about $\lambda(G^*, M^{-1})$. The sum in (30) is nothing but

$$\|C_1^*\|^{+2r} a^{-2rt} \times$$
$$\times \sum_{E_S P_1 = P_1 \bmod (a),\, R_1} \eta\big(-S[G_1+Q_1]\widetilde{C}_1^{*-1}\widetilde{A}_1^* + (\widetilde{G_1+Q_1})SF_2 + \widetilde{F}_2 S(G_1+Q_1)\big) \times$$
$$\times \eta(S[P_1 + R_1\ \ G_2 + R_2]F)$$

where $a > 0$ is a rational integer divisible by $c_6^5 |d|^t |\delta(S)|^2 |C_1^*|$ and now P_1 runs over a complete system of modulo (a) incongruent integral (m, t) matrices with $E_S P_1 = P_1$ and R_1 runs over the system described earlier. This is a consequence of applying a lemma of Siegel ([10], Lemma 7, p. 220). Now proceeding by the usual methods ([11], p. 593, Lemma 16; [2, II], (28), p. 98), we can prove that the sum in (30) is majorized by $\|C_1^*\|^r |\delta(S)|^t |d|^{rt/2} c_6^{9rt}$. We thus obtain

$$|\lambda(G^*, M^{-1})| \leqslant c_6^{9rt} \|C_1^*\|^{-r} d(L)^r N\big(\delta(G_0)\big)^{-r}.$$

From (9′) and (28), $\|C_1^*\|^{-1} d(L) N\big(\delta(G_0)\big)^{-1} = \|P^*\|^{-1}\|Q^*\|^{-1}$. Since $c_6^2 Q^{*-1}$, $c_6^3 P^{*-1}$ are integral, it follows that for a constant c_7 depending only on k and n, we have $\|P^{*-1}\|\,\|Q^*\|^{-1} \leqslant c_7$. This leads to

LEMMA 4. *For a constant* c_8 *depending only on* k, r *and* n

$$|\lambda(G^*, M^{-1})| \leqslant c_8 . \tag{31}$$

This is the analogue of Lemma 16 of Siegel in [11].

The sums $\lambda(0, M^{-1})$ are simple in nature and can be identified with certain other sums occurring in the work of H. Braun [2, II]. We have in this direction

LEMMA 5.

$$\lambda(0, M^{-1}) = \varepsilon^r e^{\pi i t(2p-r)/2} |\delta(S)|^{-t} |d|^{-rt/2} d(L)^{-r} \sum_{\substack{G \bmod N \\ E_S G = G = G\check{O}}} \eta(-S[G]L). \tag{32}$$

(Note. In (32), $M = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \mathfrak{M}_n$, $t = r(C) \geqslant 1$, M^{-1} has the decomposition (10), L, Q, ε are defined relative to (CD) by (7), (8) and (9), $\check{O}$ is a r-unit of C, N is a denominator of L in the sense described on p. 40, $d(L) = |N|$ and G runs over a complete set of (m, n) integral matrices incongruent modulo N and satisfying $E_S G = G = G\check{O}$.)

Proof of Lemma 5. It is obvious that (32) is independent of the particular r-unit $\check{O}$ of C chosen. For, if $\check{O}_1$ is any other (generalized) r-unit of C, and if $G\check{O} = G$ then $G\check{O}_1 = G\check{O}\check{O}_1 = G\check{O} = G$. For our subsequent discussion, we may take $\check{O}$ to be $Q^{-1}\begin{pmatrix} E^{(t)} & 0 \\ 0 & 0\end{pmatrix} Q$, without loss of generality. Define $Q_2 = \begin{pmatrix} E^{(t)} & 0 \\ 0 & c_6 E\end{pmatrix} Q$. Then, with our earlier notation (p. 40) we have $Q^* = \begin{pmatrix} E & 0 \\ c_6 A_3 & E\end{pmatrix} Q_2$, $L = \begin{pmatrix} C_1^{*-1} D_1^* & 0 \\ 0 & 0\end{pmatrix} [\widetilde{Q}_2^{-1}]$ and $Q_2 \check{O} Q_2^{-1} = \begin{pmatrix} E^{(t)} & 0 \\ 0 & 0\end{pmatrix}$. Since $c_6 A_3$ is integral, we have $GQ^{*-1} \equiv (R_1 0)\ \big(\mathrm{mod}\,(1)\big)$ if and only if $GQ_2^{-1} \equiv (R_1 0)\ \big(\mathrm{mod}\,(1)\big)$. Now

$$\lambda(0, M^{-1}) = \alpha(S, M^{-1}) \sum_{R_1, P_1 \bmod C_1^*} \eta(-S[P_1 + R_1] C_1^{*-1} D_1^*), \tag{33}$$

where $\alpha(S, M^{-1}) = \varepsilon^r e^{\pi i t(2p-r)/2} |\delta(S)|^{-t} |d|^{-rt/2} d(L)^r \|C_1^*\|^{-2r} N\big(\delta(G_0)\big)^{-r}$ and where P_1 runs over a complete set of modulo C_1^* incongruent (m, t) integral matrices with $E_S P_1 = P_1$ and $R_1^{(m,t)}$ over a full set of modulo $\mathfrak{O}$ incongruent matrices for which there exists $G \in \{\mathfrak{O}\}_{m,n}$ with $GQ_2^{-1} = (R_1 0)\ (\mathrm{mod}\,\mathfrak{O})$ and $E_S G = G$. From $Q_2 \check{O} Q_2^{-1} = \begin{pmatrix} E^{(t)} & 0 \\ 0 & 0\end{pmatrix}$, it is clear that $GQ_2^{-1} = (R_1 0)\ (\mathrm{mod}\,\mathfrak{O})$ if and only if $G\check{O} = G$. Further, obviously the summations over P_1 and R_1 in (33) can be welded into the single summation over a complete set of modulo $\begin{pmatrix} C_1^* & 0 \\ 0 & 0\end{pmatrix} Q_2$ incongruent matrices G in $\{\mathfrak{O}\}_{m,n}$ for which $G\check{O} = G = E_S G$ and $(P_1 + R_1\ \ 0)$ may be replaced by GQ_2^{-1} for G in this system. Moreover, $\sigma(S[P_1 + R_1] C_1^{*-1} D_1^*) = \sigma(S[G]L)$. On the other hand, we can show that

$$a^{-2rt} \sum_{E_S G = G\check{O} = G (\bmod (a))} \eta(-S[G]L) = d(L)^{-2r} \sum_{E_S G = G\check{O} = G \bmod N} \eta(-S[G]L) \tag{34}$$

(cf. [2, II], (21), p. 97) where a is a rational integer divisible by $c_6^5|d|^t|\delta(S)|^2|C_1^*|$. For proving formula (34), one has essentially to observe first that there exist $V_1, V_2 \in \Omega_n$ such that $GV_2 \equiv (G_1^{(m,t)}\ 0) \pmod{(a)}, L[\tilde{V}_2^{-1}] \equiv \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} \pmod{(1)}$ and $V_1 N V_2 = \begin{pmatrix} K & 0 \\ 0 & E \end{pmatrix}$ where K is a 'denominator' of $W \in \{k\}_{t,t}$ and then use Lemma 6 of [10]. By using the same lemma again, it could be shown that

$$(35)\qquad \sum_{E_S G = G\check{O} = G \bmod \begin{pmatrix} C_1^* & 0 \\ 0 & 0 \end{pmatrix} Q_2} \eta(-S[G]L) = N(\delta(G_0))^r \|C_1^*\|^{2r} a^{-2rt} \sum_{E_S G = G\check{O} = G \pmod{(a)}} \eta(-S[G]L)\,.$$

Now (34) and (35) together give (32).

Remark. In the case when $S \geqslant 0$, we have $H = S$ and then $\varepsilon^{-r}\lambda(0, M^{-1})$ coincides with the quantity $v(H, C, D)$ defined by H. Braun ([2, II], p. 104). The concerned root of unity ε in k is equal to $|P|^{-1}|\tilde{Q}|^{-1}|C_1|N(\delta(G_0))\,d(L)^{-1}$.

Let S be definite and, in fact, of signature $(r, 0)$ without loss of generality. (Else, we could take $-S$, instead!). Then $H = S$, and for $Z \in \mathfrak{H}_n$,

$$(36)\qquad f(S, H, Z, \tilde{Z}, 0) = \sum_{E_S G = G \in \{\mathfrak{O}\}_{m,n}} \eta(S[G]Z) = \sum_{T = \tilde{T} \geqslant 0} A(S, T)\eta(TZ)$$

where, for a given $T = \tilde{T}$, $A(S, T)$ denotes *the number of E_S-reduced representations of T by S*. Since we know that the series (36) converges absolutely, uniformly over compact sets in $\mathfrak{H}_n$, the function $f(S, H, Z, \tilde{Z}, 0)$ which we shall be justified in denoting as $f(S, Z)$, is regular in $\mathfrak{H}_n$, involving as it does no $\tilde{Z}$.

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{M}_n$, $r(C) = n$. From (29), replacing P^*, Q^* by $E^{(n)}$ and $\begin{pmatrix} \tilde{D}_1^* & -\tilde{B}_1^* \\ -\tilde{C}_1^* & \tilde{A}_1^* \end{pmatrix}$ by $\begin{pmatrix} \tilde{D} & -\tilde{B} \\ -\tilde{C} & \tilde{A} \end{pmatrix}$, we obtain

$$f(S, M^{-1}\langle Z\rangle)|-\tilde{C}Z + \tilde{A}|^{-r} = e^{\pi i r n/2}(\delta(S))^{-n}|d|^{-rn/2}|\tilde{C}|^{-r} \times$$
$$\times \sum_{\substack{E_S G = G \in \{\mathfrak{O}\}_{m,n} \\ E_S Q_1 = Q_1}} \eta(S[G+Q_1]Z) \sum_{\substack{P_1 \bmod C \\ E_S P_1 = P_1}} \eta\big(-S[P_1]C^{-1}D + (\widetilde{G+Q_1})SP_1C^{-1} +$$
$$+ \tilde{P}_1 S(G+Q_1)\tilde{C}^{-1} - S[G+Q_1]AC^{-1}\big)\,.$$

Let now $C \equiv 0 \pmod{(|d|\delta(S))}$. Then in the inner sum over P_1, we can replace P_1 by $P_1 + (\sqrt{d})^{-1}S^{-1}G_1C$ with arbitrary $G_1 \in \{\mathfrak{O}\}_{m,n}$ since

$E_S(\sqrt{d})^{-1}S^{-1}G_1C = (\sqrt{d})^{-1}S^{-1}G_1C \in \{\mathfrak{O}\}_{m,n}$. But then this replacement brings in an extra factor $\eta\left(-\frac{1}{\sqrt{d}}\tilde{Q}_1G_1 + \frac{1}{\sqrt{d}}\tilde{G}_1Q_1\right)$ with which the inner sum gets multiplied. If Q_1 is not integral, we can always find $G_1 \in \{\mathfrak{O}\}_{m,n}$ such that $\tilde{Q}_1G_1 \notin \{\mathfrak{O}\}_{m,n}$ and therefore this extra factor is different from 1. Thus the inner sum over P_1 is zero, unless $Q_1 = 0$, provided that $C \equiv 0 \pmod{(|d|\delta(S))}$. When $Q_1 = 0$, the inner sum may be easily identified with $G(S, -C^{-1}D) = \sum_{E_S P_1 = P_1 \bmod C} \eta(-S[P_1]C^{-1}D)$.

Thus in particular, for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{M}_n(|d|\delta(S))$ with $r(C) = n$, we have

$$f(S, M^{-1}\langle Z\rangle) = e^{\pi i r n/2}(\delta(S))^{-n}|d|^{-rn/2}|\tilde{C}|^{-r}|-\tilde{C}Z + \tilde{A}|^r G(S, -C^{-1}D)f(S, Z)\,.$$

For such M, define

$$v(M^{-1}) = e^{\pi i r n/2}\delta(S)^{-n}|d|^{-rn/2}|\tilde{C}|^{-r}G(S, -C^{-1}D)\,.$$

Then we have

$$(37)\qquad f(S, Z)/M^{-1} = v(M^{-1})f(S, Z)\,.$$

From [2, II] (see p. 94), we know that $|v(M^{-1})| \leqslant 1$. Taking M instead of M^{-1}, the same arguments give $|v(M)| \leqslant 1$. But from (37), we have $v(M^{-1})v(M) = 1$ since $f(S, Z) \not\equiv 0$. Thus for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{M}_n(|d|\delta(S))$ with $|C| \neq 0$, we have

$$(38)\qquad |v(M)| = 1\,.$$

Let us now take $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{M}_n(|d|\delta(S))$ with $t = r(C) < n$. Then clearly $D \neq 0^{(n)}$, since $r(CD) = n$. Hence $|C + xD|$ as a polynomial in x does not vanish identically. We can thus find a rational integer ν with $|d|\delta(S)/\nu$ such that $|C + \nu D| \neq 0$. Let now

$$M_1 = \begin{pmatrix} E^{(n)} & 0 \\ \nu E^{(n)} & E^{(n)} \end{pmatrix}, \qquad M_2 = MM_1 = \begin{pmatrix} A & B \\ C_1 & D_1 \end{pmatrix}.$$

Then $|C_1| \neq 0$. Thus $M = M_2M_1^{-1}$ and both M_2 and M_1^{-1} are of the type mentioned above so that

$$f(S, Z)/M = (f(S, Z)/M_2)/M_1^{-1} = v(M_2)f(S, Z)/M_1^{-1} = v(M_2)v(M_1^{-1})f(S, Z)\,.$$

Let us now define

$$(39)\qquad v(M) = v(M_2)v(M_1^{-1})\,.$$

Then from (38), $|v(M)| = 1$ and further

$$(40) \qquad f(S, Z)/M = v(M) f(S, Z)$$

for all $M \in \mathfrak{M}_n(|d|\delta(S))$. That $v(M)$ is uniquely defined by (39) is a consequence of (40) and the fact that $f(S, Z) \not\equiv 0$.

However, it does not necessarily follow from the foregoing that $v(M^{-1}AM) = 1$ for all $M \in \mathfrak{M}_n$ and $A = \begin{pmatrix} E^{(n)} & * \\ 0 & E^{(n)} \end{pmatrix} \in \mathfrak{M}_n(|d|\delta(S))$. From the form (29) of $f(S, Z)/M^{-1}$, we notice that

$$f(S, Z+V)/M^{-1} = f(S, Z)/M^{-1}$$

for $V = \tilde{V} \equiv 0 \pmod{\gamma^2}$. In other words, for $A_0 = \begin{pmatrix} E & * \\ 0 & E^{(n)} \end{pmatrix} \in \mathfrak{M}_n(\gamma^2)$ we have

$$(41) \qquad v(M^{-1}A_0 M) = 1\,, \quad \text{for all} \quad M \in \mathfrak{M}_n\,.$$

From (40), (38) and (41) we obtain

THEOREM 2. *For $S \geqslant 0$, the generalized theta-series $f(S, Z)$ is a hermitian modular form of degree n, dimension $-r(S)$, stufe γ^2 and belonging to the multiplier-system $\{v(M)\}$ defined by* (39).

We may now relax the condition that the signature of S be $(r, 0)$ and go back to consider the function $f(S, H, Z_1, Z_2, 0)$. Let $Z_1 = Z \in \mathfrak{H}_n$ and $Z_2 = \tilde{Z}$. Further let $N_j\langle Z\rangle \in \mathfrak{G}_n$ where $N_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \in \mathfrak{M}_n$. Then, from Theorem 1,

$$|f(S, H, Z, \tilde{Z}, 0)|$$
$$= \|C_j Z + D_j\|^{-p} \|C_j \tilde{Z} + D_j\|^{-q} \sum_{E_S G^* = G^* = \gamma(G_1+Q_1\ G_2+R_2)\tilde{P}^{*-1}} \lambda(G^*, N_j^{-1}) \times$$
$$\times \eta\left(\frac{1}{2\gamma^2}\left((S+H)[G^*]N_j\langle Z\rangle + (S-H)[G^*]N_j\langle\tilde{Z}\rangle\right)\right).$$

If $Y_j = I(N_j\langle Z\rangle)$, then by Lemma 4,

$$(42) \qquad |f(S, H, Z, \tilde{Z}, 0)|$$
$$\leqslant c_9 \|C_j Z + D_j\|^{-r} \sum_{E_S G^* = G^* \in \{\mathfrak{O}\}_{m,n}} \exp\left(-(2\pi/\gamma^2)\,\sigma(H[G^*]\,Y_j)\right).$$

The constant c_9 and the constants $c_{10}, c_{11}, \ldots$, to follow depend only on k, r, and n, unless otherwise stated. Now we have

$$(43) \qquad \sum_{E_S G^* = G^* \in \{\mathfrak{O}\}_{m,n}} \exp\left(-(2\pi/\gamma^2)\,\sigma(H[G^*]\,Y_j)\right)$$
$$= \sum_{AG^* = G_1 \in \mathfrak{N}} \exp\left(-\frac{2\pi}{\gamma^2}\sigma(H_1[G_1]\,Y_j)\right) < \sum_{G \in \{\mathfrak{O}\}_{r,n}} \exp\left(-\frac{2\pi}{\gamma^2}\sigma(H_1[G]\,Y_j)\right).$$

In (43), we may replace G by UGV with suitable $U \in \Omega_r$, $V \in \Omega_n$ so that we could suppose H_1 and Y_j to be reduced in the sense of Humbert. Now there exist Humbert matrices A_*, B_* such that $Y_j[A_*] \in \mathfrak{T}_0^{(n)}$ and $H_1[B_*] \in \mathfrak{T}_0^{(n)}$. Since $Y_j[A_*] \in \mathfrak{T}_0^{(n)}$, we have for a constant $c_{10} > 0$, $Y_j[A_*] \geqslant c_{10}(\min Y_j) E^{(n)}$. This is easily deduced by proving an analogue of a lemma of Siegel ([11], Lemma 12). Thus $Y_j > c_{11} y_j E^{(n)}$, where $y_j = \min Y_j$, in view of the fact that A_* belongs to the finite set of Humbert matrices in $\{\mathfrak{O}\}_{n,n}$. Thus

$$(44) \qquad \sum_{E_S G^* = G^* \in \{\mathfrak{O}\}_{m,n}} \exp\left(-\frac{2\pi}{\gamma^2}\sigma(H[G^*]\,Y_j)\right) < \sum_{G \in \{\mathfrak{O}\}_{r,n}} \exp\left(-c_{12} y_j \sigma(H_1[G])\right).$$

Let $h_1, \ldots, h_r$ be the diagonal elements of $H_1[B_*]$ in $\mathfrak{T}_0^{(r)}$. Now there exists a constant $c_{13} > 0$ depending only on k and r such that $H_1[B_*] \geqslant c_{13} H_1^*$ where H_1^* is a diagonal matrix with the diagonal elements $h_1, \ldots, h_r$. The proof of this fact again is exactly similar to the proof of Lemma 12 in [11]. Thus we have

$$(45) \qquad \sum_{G \in \{\mathfrak{O}\}_{r,n}} \exp\left(-c_{12} y_j \sigma(H_1[G])\right)$$
$$\leqslant \left(\prod_{k=1}^{r}\left(\sum_{\alpha \in \mathfrak{O}} \exp(-c_{14} h_k y_j |\alpha|^2)\right)\right)^n \leqslant \left(\sum_{\alpha \in \mathfrak{O}} \exp(-c_{14} h_k y_j |\alpha|^2)\right)^{rn}$$

using, again, the fact that B_* belongs to the fixed finite set of Humbert matrices in $\{\mathfrak{O}\}_{r,r}$. It is easy to verify that the last series in (45) is majorized by $c_{15} \prod_{k=1}^{r} \left(1 + c_{16}(h_k y_j)^{-n}\right)$.

Let us now consider $f(S, H, Z, \tilde{Z}, 0) - \lambda(0, N_j^{-1})\,|C_j Z + D_j|^{-p} \times \times |C_j \tilde{Z} + D_j|^{-q}$. We have to estimate the series $\sum_{G \in \mathfrak{N},\, G \neq 0} \exp\left(-\frac{2\pi}{\gamma^2}\sigma(H_1[G]\,Y_j)\right)$. We can easily prove that, for $G \in \{\mathfrak{O}\}_{r,n}$, $G \neq 0$,

$$\sigma(H_1[G]\,Y_j) = \sigma(H_1[B_*][B_*^{-1} G \tilde{A}_*^{-1}]\,Y_j[A_*])$$
$$\geqslant c_{17} y_j \sigma(H_1^*[B_*^{-1} G \tilde{A}_*^{-1}]) \geqslant c_{18} h_1 y_j\,.$$

Thus we have

$$\left|f(S, H, Z, \tilde{Z}, 0) - \lambda(0, N_j^{-1})\,|C_j Z + D_j|^{-p}\,|C_j \tilde{Z} + D_j|^{-q}\right|$$
$$\leqslant c_{19} \exp(-c_{18} h_1 y_j)\,\|C_j Z + D_j\|^{-r} \sum_{0 \neq G \in \{\mathfrak{O}\}_{r,n}} \exp\left(-\frac{\pi}{\gamma^2}\sigma(H_1[G]\,Y_j)\right)$$
$$\leqslant c_{20} \exp(-c_{18} h_1 y_j)\,\|C_j Z + D_j\|^{-r} \prod_{k=1}^{r} \left(1 + (h_k y_j)^{-n}\right)$$
$$\leqslant c_{21} \exp(-c_{18} h_1 y_j)\,\|C_j Z + D_j\|^{-r} \left(1 + (h_1 y_j)^{-rn}\right).$$

We have therefore proved

THEOREM 3. *For* $Z \in N_j^{-1}\langle\mathfrak{G}_n\rangle$, $N_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \in \mathfrak{M}_n$ *and for* $y_j = \min I(N_j\langle Z\rangle)$,

$$|f(S, H, Z, \tilde{Z}, 0)| \leqslant c_{22}\|C_jZ+D_j\|^{-r}\prod_{k=1}^{r}\left(1+c_{16}(h_k y_j)^{-n}\right),$$

$$(46)\qquad \left|f(S, H, Z, \tilde{Z}, 0)-\lambda(0, N_j^{-1})\,|C_jZ+D_j|^{-p}\,|C_j\tilde{Z}+D_j|^{-q}\right| \leqslant c_{23}\|C_jZ+D_j\|^{-r}\left(1+(h_1y_j)^{-rn}\right)\exp(-c_{18}h_1y_j).$$

We shall use the estimates (46) in § 5 in connection with the 'generalized Farey dissection'.

§ 4. Eisenstein series. With $f(S, H, Z, \tilde{Z}, 0)$, we shall associate for $r > 2n$, a function $\varphi(Z) = \varphi(S, Z)$ which 'behaves similarly' under the transformations $Z \to M\langle Z\rangle$ for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{M}_n$ with $r(C) = n$; more precisely, $\varphi(Z)$ has the property that

$$\left(f(S, H, i\lambda E, -i\lambda E, 0)-\varphi(i\lambda E)\right)/M$$

tends to zero as $\lambda\to\infty$, for all such $M \in \mathfrak{M}_n$. The function $\varphi(Z)$ is defined, for $r > 2n$, by

$$(47)\qquad \varphi(Z) = e^{\pi in(2p-r)/2}|\delta(S)|^{-n}\,|d|^{-nr/2}\times \sum_{(C_iD_i),\, r(C_i)=n}|\tilde{C}_i|^{-r}G(S, -C_i^{-1}D_i)\,|C_iZ+D_i|^{-p}\,|C_i\tilde{Z}+D_i|^{-q}$$

where (C_iD_i) runs over a complete set of representatives of classes of n-rowed coprime pairs (CD) with $|C| \neq 0$. In the first place, $\varphi(Z)$ is well-defined by (47); for, if we take $U(C_iD_i)$ with arbitrary $U \in \Omega_n$, instead of (C_iD_i), then $|\tilde{C}_i|^{-r}G(S, -C_i^{-1}D_i)\,|C_iZ+D_i|^{-p}\,|C_i\tilde{Z}+D_i|^{-q}$ remains unchanged. In view of the fact that $|\lambda(0, N_i^{-1})| \leqslant 1$ for $N_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in \mathfrak{M}_n$ with $|C_i| \neq 0$, we see that (47) is majorized by $\sum_{(C_iD_i)}\|C_iZ+D_i\|^{-r}$ which converges by [2, I], for $r > 2n$. Thus the series (47) which is a non-analytic Eisenstein series for $pq \neq 0$ converges absolutely, uniformly when Z lies in a compact subset of $\mathfrak{H}_n$. If $q = 0$, then indeed $\varphi(Z)$ is regular in $\mathfrak{H}_n$.

Using the one-one correspondence between the classes of n-rowed coprime pairs (CD) with $|C| \neq 0$ and hermitian matrices in $\{k\}_{n,n}$ we may rewrite (47) as

$$(48)\qquad \varphi(Z) = e^{\pi in(2p-r)/2}|\delta(S)|^{-n}\,|d|^{-rn/2}\times \sum_{R=\tilde{R}\in\{k\}_{n,n}} d(R)^{-2r}G(S, -R)|Z+R|^{-p}|\tilde{Z}+R|^{-q}.$$

The summation in (48) is over all hermitian matrices R in $\{k\}_{n,n}$ and further if $R = C^{-1}D$ with (CD) being a coprime pair, then $G(S, -R) = \sum_{G_1}\eta(-S[G_1]R)$ where now G_1 runs over a full system of modulo C incongruent elements of $\{\mathfrak{O}\}_{m,n}$, satisfying $E_SG_1 = G_1$. If $R = C^{-1}D$, $R_1 = C_1^{-1}D_1$ with (CD), (C_1D_1) being coprime pairs, and if $R-R_1$ is semi-integral, then we have

$$(49)\qquad d(R)^{-2r}G(S, -R) = d(R_1)^{-2r}G(S, -R_1)$$

as may be verified from (34), taking $a = c_6^5|d|^n\,|\delta(S)|^2\,\|C\|^2\,\|C_1\|^2$, for example. Thus we obtain

$$(50)\qquad \varphi(Z) = e^{\pi in(2p-r)/2}|\delta(S)|^{-n}\,|d|^{-rn/2}\times \sum_{R_1=\tilde{R}_1 \bmod \mathfrak{O}} d(R_1)^{-2r}G(S, -R_1)\sum_{R=\tilde{R}\equiv R_1 (\bmod \mathfrak{O})}|Z+R|^{-p}\,|\tilde{Z}+R|^{-q}$$

where the inner sum over all $R = \tilde{R} \in \{k\}_{n,n}$ with $R \equiv R_1 (\bmod \mathfrak{O})$ for a fixed $R_1 = \tilde{R}_1 \in \{k\}_{n,n}$ and the outer sum is over a complete set of hermitian matrices R_1 in $\{k\}_{n,n}$ incongruent modulo $\mathfrak{O}$.

In view of (49), $\varphi(Z+T) = \varphi(Z)$ for every semi-integral $T \in \{k\}_{n,n}$. By the usual arguments (cf. [2, I], pp. 845-849), it could be shown that $\varphi(Z)$ has the Fourier expansion

$$(51)\qquad \varphi(Z) = e^{\pi in(2p-r)/2}|\delta(S)|^{-n}\,|d|^{-rn/2}\times \sum_{T}\eta(TX)\int_{\mathfrak{E}}\Big(\sum_{R=\tilde{R}\in\{k\}_{n,n}} d(R)^{-2r}G(S, -R)|X^*+iY+R|^{-p}\,|X^*-iY+R|^{-q}\Big)\times \eta(-TX^*)\{dX^*\}$$

where $X = R(Z)$, $Y = I(Z)$, T runs over all semi-integral matrices in $\{\mathfrak{O}\}_{n,n}$, and the volume element $\{dX^*\}$ in $\mathcal{H}_n$ and the domain of integration $\mathfrak{E}$ are defined as follows. Namely, writing $X^* \in \mathcal{H}_n$ as $X_1^*+\omega X_2^*$ with real $X_1^* = (x_{ij}^{(1)})$ and $X_2^* = (x_{kl}^{(2)})$, then $\{dX^*\} = \prod_{1\leqslant i\leqslant j\leqslant n} dx_{ij}^{(1)}\prod_{1\leqslant k<l\leqslant n} dx_{kl}^{(2)}$ and $\mathfrak{E}$ is the 'cube' consisting of $X^* \in \mathcal{H}_n$ for which $0 \leqslant x_{ij}, x_{kl} \leqslant 1$ for $1 \leqslant i \leqslant j \leqslant n$ and $1 \leqslant k < l \leqslant n$. It is clear that if dX^* is the volume element in $\mathcal{H}_n$ defined in § 2, then

$$\{dX^*\} = \left(\frac{2}{\sqrt{|d|}}\right)^{n(n-1)/2} dX^*.$$

In view of the uniform convergence over $\mathfrak{E}$ of the series inside the sign of integration in (51), we have

$$\varphi(Z) = e^{(\pi in/2)(2p-r)}|\delta(S)|^{-n}\,|d|^{-rn/2}\sum_{T=\tilde{T}\text{ semi-integral}}\eta(TX)\times \sum_{R_1=\tilde{R}_1 \bmod \mathfrak{O}} d(R_1)^{-2r}G(S, -R_1)\int_{\mathcal{H}_n}|X^*+iY+R_1|^{-p}\,|X^*-iY+R_1|^{-q}\times \eta(-TX^*)\{dX^*\}$$

where now R_1 runs through the same set of matrices as R_1 in (50). It follows that

$$(52)\quad \varphi(Z) = e^{(\pi i n/2)(2p-r)}|\delta(S)|^{-n}|d|^{-rn/2}\sum_{T=\widetilde{T}}\eta(TX)\times$$
$$\times\sum_{R_1=\widetilde{R}_1 \bmod \mathfrak{D}}\eta(TR_1)d(R_1)^{-2r}G(S,-R_1)\int_{\mathcal{H}_n}|X^*+iY|^{-p}|X^*-iY|^{-q}\times$$
$$\times\,\eta(-TX^*)\{dX^*\}.$$

Again, in view of (49),

$$(53)\quad \sum_{R_1=\widetilde{R}_1 \bmod \mathfrak{D}} d(R_1)^{-2r}G(S,-R_1)\eta(TR_1) = |d|^{n(n-1)/2}\sum_{R=\widetilde{R} \bmod \mathfrak{D}} d(R)^{-2r}G(S,-R)\eta(TR)$$

where, on the right-hand side, R runs over a complete set of hermitian matrices in $\{k\}_{n,n}$ such that no two elements of the set differ by a semi-integral matrix.

For every rational prime p, let us define after Siegel [11], $a_p(S, T)$, the *p-adic density of representation of T by S*, by

$$(54)\quad a_p(S, T) = \lim_{a\to\infty} p^{an(n-2r)}A_{p^a}(S, T)$$

where $A_{p^a}(S, T)$ is the number of modulo (p^a) incongruent $G \in \{\mathfrak{D}\}_{m,n}$ for which $S[G] \equiv T \bmod (p^a)$ and $E_S G \equiv G \bmod (p^a)$. The definition (54) may be verified to be independent of the choice of E_S. Using the absolute convergence of the series (53) (since $r > 2n$, [2, I]) and proceeding as in Lemma 25 of [11] (cf. [2, II], p. 97, Lemma 1; [1], p. 140, Hilfssatz 51) it could be seen without difficulty that the densities $a_p(S, T)$ exist and further uniformly in T, the infinite series on the right-hand side of (53) is nothing but the infinite product $\prod_p a_p(S, T)$ of the p-adic densities (54) extended over all rational primes p. As a consequence

$$(55)\quad \varphi(Z) = e^{(\pi i n/2)(2p-r)}|\delta(S)|^{-n}|d|^{n(n-1)/2-rn/2}\times$$
$$\times\sum_{T=\widetilde{T}}\eta(TX)\prod_p a_p(S, T)\int_{\mathcal{H}_n}|X^*+iY|^{-p}|X^*-iY|^{-q}\eta(-TX^*)\{dX^*\}$$

where T runs over all n-rowed semi-integral hermitian matrices. The infinite series on the right-hand side of (53) is the "singular series of Siegel" associated with S and T.

If $S \geqslant 0$, then using the summation-formula (see [2, I], p. 847, Lemma 8),

$$\sum_{R=\widetilde{R}\equiv R_1 (\bmod \mathfrak{D})}|Z+R|^{-r} = \frac{|d|^{-n(n-1)/4}e^{-\pi i r n/2}(2\pi)^{rn-n(n-1)/2}}{\Gamma(r)\Gamma(r-1)\ldots\Gamma(r-n+1)}\times$$
$$\times\sum_{T=\widetilde{T}>0,\, T \text{ semi-integral}}|T|^{r-n}\eta\big(T(Z+R_1)\big)$$

one could, in this case, directly obtain the Fourier expansion of $\varphi(Z)$ viz.

$$(56)\quad \varphi(Z) = \prod_{j=r-n+1}^{r}\frac{(2\pi)^j}{\Gamma(j)\,|d|^{j/2}}\delta(S)^{-n}\sum_{T=\widetilde{T}>0}|T|^{r-n}\prod_p a_p(S, T)\eta(TZ).$$

In (56), T runs over all positive semi-integral matrices; the coefficients corresponding to $T \not> 0$ drop out thus when $q = 0$.

We now proceed to obtain estimates for $\varphi(Z)$ analogous to (46). These estimates will be needed in § 5 in connection with the 'generalized Farey dissection'. Moreover, these in particular, will enable us to verify the property of $\varphi(Z)$ in relation to $f(S, H, Z, \widetilde{Z}, 0)$ asserted at the beginning of this section.

THEOREM 4. *For* $Z^* = N_j\langle Z\rangle \in \mathfrak{G}_n$ *with* $N_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j\end{pmatrix} \in \mathfrak{M}_n$ *and* $y_j = \min I(Z^*)$, *we have*

$$(57)\quad \begin{aligned} &|\varphi(Z)| \leqslant c_{24}\|C_jZ+D_j\|^{-r},\\ &\big|\varphi(Z)-\lambda(0, N_j^{-1})|C_jZ+D_j|^{-p}|C_j\widetilde{Z}+D_j|^{-q}\big| \leqslant c_{25}y_j^{1-r}\|C_jZ+D_j\|^{-r}\\ &\hspace{20em}(|C_j| \neq 0).\end{aligned}$$

Proof. Since, by (31), $|\lambda(0, N^{-1})| \leqslant c_8$, we have clearly $|\varphi(Z)| \leqslant c_8\sum_{\{C,D\}}\|CZ+D\|^{-r}$, where the summation is over all classes of n-rowed coprime pairs. This gives

$$(58)\quad |\varphi(Z)| \leqslant c_8\sum_{\{C,D\}}\|C(\widetilde{D}_jZ^*-\widetilde{B}_j)+D(-\widetilde{C}_jZ^*+\widetilde{A}_j)\|^{-r}\|-\widetilde{C}_jZ^*+\widetilde{A}_j\|^{r}$$
$$\leqslant c_8\|C_jZ+D_j\|^{-r}\sum_{\{C,D\}}\|CZ^*+D\|^{-r}$$

since $(C\widetilde{D}_j-D\widetilde{C}_j\ \ -C\widetilde{B}_j+D\widetilde{A}_j)$ again runs over a complete system of representatives of classes of n-rowed coprime pairs when (CD) does so. Similarly, for $|C_j| \neq 0$, we have

$$(59)\quad \big|\varphi(Z)-\lambda(0, N_j^{-1})|C_jZ+D_j|^{-p}|C_j\widetilde{Z}+D_j|^{-q}\big| \leqslant c_8\|C_jZ+D_j\|^{-r}\sum_{\{C,D\}\neq\{0,E\}}\|CZ^*+D\|^{-r}$$

where $\{C, D\}$ runs over all classes of n-rowed coprime pairs except $\{0, E\}$. Let us define now, for $1 \leqslant t \leqslant n$,

$$(60)\quad \sigma_t = \sum_{\{C,D\},\, r(C)=t}\|CZ^*+D\|^{-r}$$

where the summation is over all classes $\{C, D\}$ of n-rowed coprime pairs with $r(C) = t$. From (9*) and Lemma 3, we have

(61) $$\sigma_t \leqslant \sum_{\substack{G_0 \in \mathfrak{L}(t,n) \\ Q=\binom{G_0}{G_0^*},\ L=\tilde{L}=Q^{-1}\begin{pmatrix}E^{(t)} & 0\\ 0 & 0\end{pmatrix} QL \in \{k\}_{n,n}}} d(L)^{-r} N\big(\delta(G_0)\big)^r \|G_0(Z^*+L)\tilde{G}_0\|^{-r}$$

$$= \sum_{\substack{G_0 \in \mathfrak{L}(t,n) \\ L=\tilde{L} \bmod \mathfrak{D},\ Q^{-1}\begin{pmatrix}E^{(t)} & 0\\ 0 & 0\end{pmatrix} QL=L}} d(L)^{-r} N\big(\delta(G_0)\big)^r \sum_{\substack{S=\tilde{S} \in \{\mathfrak{D}\}_{n,n} \\ Q^{-1}\begin{pmatrix}E^{(t)} & 0\\ 0 & 0\end{pmatrix} QS=S}} \|G_0(Z^*+L+S)\tilde{G}_0\|^{-r}.$$

Using an argument of H. Braun ([2, I], p. 850, (81)), we have

(62) $$\sum_{\substack{S=\tilde{S} \in \{\mathfrak{D}\}_{n,n} \\ Q^{-1}\begin{pmatrix}E^{(t)} & 0\\ 0 & 0\end{pmatrix} QS=S}} |G_0(Z^*+L+S)\tilde{G}_0|^{-r}$$

$$= N\big(\delta(G_0)\big)^{-t} \sum_{\substack{S=\tilde{S} \in \{\mathfrak{D}\}_{t,t} \\ R=\tilde{R} \bmod \mathfrak{D},\ \tilde{G}_0 R G_0 \text{ semi-integral}}} e^{-2\pi i \sigma(RS)} |G_0(Z^*+L)\tilde{G}_0 + S|^{-r}$$

where, on the right-hand side of (62), S runs over all t-rowed integral hermitian matrices and $R = (r_{ij})$ runs over a set of t-rowed hermitian matrices such that $\sqrt{d}\, r_{ij}$, r_{ll} $(1 \leqslant i < j \leqslant t,\ 1 \leqslant l \leqslant t)$ cover a full set of numbers in k incongruent modulo $\mathfrak{D}$, subject to the condition that $\tilde{G}_0 R G_0$ is semi-integral. On the other hand, it can be shown as in [11] (see p. 596, (60); cf. [2, I], p. 837, (29)) that

$$\sum_{S=\tilde{S} \in \{\mathfrak{D}\}_{t,t}} \|G_0(Z^*+L)\tilde{G}_0 + S\|^{-r}$$

$$\leqslant \exp\Big(c_{26}\sigma\big((I(Z^*[\tilde{G}_0]))^{-1}\big)\Big) \int_{\mathcal{H}_t} \|G_0(Z^*+L)\tilde{G}_0 + H\|^{-r}\, dH.$$

Since $G_0 \in \mathfrak{L}^{(t,n)}$, $N\big(\delta(G_0)\big) \leqslant c_2$ and it would then follow that

(63) $$\sum_{S=\tilde{S} \in \{\mathfrak{D}\}_{t,t}} \|G_0(Z^*+L)\tilde{G}_0 + S\|^{-r}$$

$$\leqslant \exp\Big(c_{26}\sigma\big((I(Z^*[\tilde{G}_0]))^{-1}\big)\Big) \|G_0 I(Z^*)\tilde{G}_0\|^{t-r} \int_{\mathcal{H}_t} \|E+H\|^{-r}\, dH$$

$$\leqslant c_{27} \exp\Big(c_{26}\sigma\big((I(Z^*[\tilde{G}_0]))^{-1}\big)\Big) \|I(Z^*)[\tilde{G}_0]\|^{t-r}$$

where $c_{27} = \int_{\mathcal{H}_t} \|E+H\|^{-r}\, dH < \infty$ since $r > 2t$ ([2, I]). Thus from (61), (62) and (63), we have

$$\sigma_t \leqslant c_{28} \sum_{G_0 \in \mathfrak{L}(t,n)} \sum_{\substack{L=\tilde{L} \bmod \mathfrak{D} \\ Q^{-1}\begin{pmatrix}E^{(t)} & 0\\ 0 & 0\end{pmatrix} QL=L}} d(L)^{-r} \exp\Big(c_{26}\sigma\big((I(Z^*[\tilde{G}_0]))^{-1}\big)\Big) \|I(Z^*)[\tilde{G}_0]\|^{t-r}.$$

Again, since $\sum_{L=\tilde{L} \bmod \mathfrak{D}} d(L)^{-r} < \infty$ for $r > 2n$ ([2, I], Lemma 4), we get

(64) $$\sigma_t \leqslant c_{29} \sum_{G_0 \in \mathfrak{L}(t,n)} \exp\Big(c_{26}\sigma\big((I(Z^*[\tilde{G}_0]))^{-1}\big)\Big) \|G_0 I(Z^*)\tilde{G}_0\|^{t-r}.$$

In the summation in (64), we can replace $\tilde{G}_0$ by $U\tilde{G}_0 V$ with suitable $U \in \Omega_n$, $V \in \Omega_t$ so that $I(Z^*)$ and $G_0 I(Z^*)\tilde{G}_0$ are both reduced in the sense of Humbert. Further, for a Humbert matrix $A_* \in \{\mathfrak{D}\}_{t,t}$, $I(Z^*)[\tilde{G}_0 A_*] \in \mathfrak{T}_0^{(t)}$ and if $\tilde{G}_0 A_* = (\underline{h}_1 \ldots \underline{h}_t)$, then $\|I(Z^*)[\tilde{G}_0 A_*]\| \geqslant c_{28} \prod_{l=1}^{t} I(Z^*)[\underline{h}_l]$ and as a consequence, $\|I(Z^*)[\tilde{G}_0]\| \geqslant c_{29} \prod_{l=1}^{t} I(Z^*)[\underline{h}_l]$. On the other hand, if we write $Y_1 = I(Z^*)[\tilde{G}_0]$, then $Y_1[A_*] \geqslant c_{30}(\min Y_1[A_*])E$ and since $\min Y_1[A_*] = \min Y_1 \geqslant \min I(Z^*)$ in view of G_0 and A_* being both integral, we have $Y_1[A_*] \geqslant c_{30}\big(\min I(Z^*)\big)E^{(t)}$. Thus, if $y_j = \min I(Z^*)$, then the characteristic roots of Y_1 are $\geqslant c_{31} y_j$ and hence

$$\sigma\big((I(Z^*[\tilde{G}_0]))^{-1}\big) = \sigma(Y_1^{-1}) \leqslant c_{32} y_j^{-1}.$$

By the above arguments, (64) now gives

(65) $$\sigma_t \leqslant c_{33} \sum_{\substack{G_0 \in \mathfrak{L}(t,n) \\ \tilde{G}_0 A_* = (\underline{h}_1 \ldots \underline{h}_t)}} \exp(c_{34} y_j^{-1}) \Big(\prod_{l=1}^{t} I(Z^*)[\underline{h}_l]\Big)^{t-r}$$

where G_0 runs over $\mathfrak{L}^{(t,n)}$ and we have assumed $I(Z^*[\tilde{G}_0])$ is reduced and further $I(Z^*[\tilde{G}_0 A_*]) \in \mathfrak{T}_0^{(t)}$ for a Humbert matrix $A_* \in \{\mathfrak{D}\}_{t,t}$. Again, there exists a Humbert matrix $B_* \in \{\mathfrak{D}\}_{n,n}$ such that $I(Z^*)[B_*] \in \mathfrak{T}_0^{(n)}$ and further

(66) $$I(Z^*)[\underline{h}_l] \geqslant I(Z^*[B_*])[B_*^{-1}\underline{h}_l] \geqslant c_{35} y_j E[B_*^{-1}\underline{h}_l] \geqslant c_{36} y_j E[\underline{h}_l].$$

From (65) and (66), we have

(67) $$\sigma_t \leqslant c_{37} \sum_{\substack{G_0 \in \mathfrak{L}(t,n) \\ \tilde{G}_0 A_* = (\underline{h}_1 \ldots \underline{h}_t)}} \exp(c_{34} y_j^{-1}) y_j^{t(t-r)} \Big(\prod_{l=1}^{t} E[\underline{h}_l]\Big)^{t-r}$$

$$\leqslant c_{37} \exp(c_{34} y_j^{-1}) y_j^{t(t-r)} \Big(\sum_{\underline{h} \neq 0} (E[\underline{h}])^{t-r}\Big)^t$$

where $\underline{h}$ runs over all non-zero integral n-rowed columns. Using the inequality $E[\underline{h}] \geqslant n\big(\prod_{i=1}^{n} |h^{(i)}|^2\big)^{1/n}$, for $\underline{h}' = (h^{(1)} \ldots h^{(n)})$ and the fact that $\sum_{0 \neq a \in \mathfrak{D}} N(a)^{-\lambda} < \infty$ for $\lambda > 1$, we see that $\sum_{0 \neq \underline{h} \in \{\mathfrak{D}\}_{n,1}} (E[\underline{h}])^{t-r} < \infty$ since $r > 2n$. Thus, from (67), we obtain, for $1 \leqslant t \leqslant n$,

(68) $$\sigma_t \leqslant c_{38} \exp(c_{34} y_j^{-1}) y_j^{t(t-r)}, \quad \text{i.e.} \quad \sigma_t \leqslant c_{38} \exp(c_{34} y_j^{-1}) y_j^{1-r}.$$

Now, for $Z^* \in \mathfrak{G}_n$, $y_j = \min(I(Z^*)) > \gamma'_n$ and therefore (58), (59) and (68) give us the required inequalities (57). Theorem 4 is thus proved.

From Theorem 3, taking $Z = i\mu E^{(n)}$, we have

$$f(S, H, i\mu E^{(n)}, -i\mu E^{(n)}, 0)/N_j^{-1} - \lambda(0, N_j^{-1}) \to 0$$

as $\mu \to \infty$ since for μ large, $i\mu E^{(n)} \in \mathfrak{G}_n$. From (57), we obtain for $N_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \in \mathfrak{M}_n$ with $|C_j| \neq 0$ that, for $\mu \to \infty$,

$$\varphi(i\mu E)/N_j^{-1} - \lambda(0, N_j^{-1}) \to 0 .$$

Thus, for such N_j, we have $\big(f(S, H, i\mu E^{(n)}, -i\mu E^{(n)}, 0) - \varphi(i\mu E)\big)/N_j^{-1} \to 0$ as $\mu \to \infty$ and this was what was asserted at the beginning of this section.

§ 5. Generalized Farey dissection. The theta-series $f(S, H, Z, \widetilde{Z}, 0)$ has the Fourier expansion

$$f(S, H, Z, \widetilde{Z}, 0) = \sum_T A(S, T; H, Y)\eta(TX)$$

where $X = R(Z)$, $Y = I(Z)$, T runs over all hermitian matrices in $\{\mathfrak{O}\}_{n,n}$ and further

$$A(S, T; H, Y) = \frac{1}{E_S} \sum_{\substack{G = \bar{G}'S[G] = T \\ G \in \{\mathfrak{O}\}_{m,n}}} \eta(iH[G]Y) .$$

If $S \geqslant 0$, then $A(S, T; H, Y) = \eta(iTY)A(S, T)$ (see (36)).

The associated $\varphi(S, Z)$ has the Fourier expansion (55) and using $\varphi(S, Z)$ as an approximation to $f(S, H, Z, \widetilde{Z}, 0)$, we estimate the Fourier coefficients of $f(S, H, Z, \widetilde{Z}, 0) - \varphi(S, Z)$. In other words, writing

$$f(S, H, Z, \widetilde{Z}, 0) = \varphi(S, Z) + f(S, H, Z, \widetilde{Z}, 0) - \varphi(S, Z)$$

we have

$$(69) \quad \int_{X \in \mathfrak{E}} f(S, H, X+iY, X-iY, 0)\eta(-TX)\{dX\} = \int_{X \in \mathfrak{E}} \varphi(S, X+iY)\eta(-TX)\{dX\} + \int_{X \in \mathfrak{E}} \big(f(S, H, X+iY, X-iY, 0) - \varphi(S, X+iY)\big)\eta(-TX)\{dX\}$$

where $\mathfrak{E}$ is the generalized 'unit cube' in $\mathcal{H}_n$ and $\{dX\}$ the volume element in $\mathcal{H}_n$ as defined on p. 55. Our object is to estimate the second integral on the right hand side of (69). To this end, we first carry over to $\mathfrak{H}_n$, the technique of Siegel's "generalized Farey dissection".

For fixed $Y = \widetilde{Y} > 0$, let $\mathfrak{E}^*(Y)$ be the set of $Z \in \mathfrak{H}_n$ with $I(Z) = Y$ and $R(Z) \in \mathfrak{E}$. Now, it is clear that the sets $N_i^{-1}\langle\mathfrak{G}_n\rangle = \{N_i^{-1}\langle Z\rangle \,|\, Z \in \mathfrak{G}_n\}$ cover $\mathfrak{H}_n$ without gaps and overlaps, when N_i runs over a complete set $\mathfrak{R}$ of representatives of the right cosets of $\mathfrak{M}_n$ modulo $\mathfrak{A}_n$, in view of the fact that $\mathfrak{F}_n$ is a fundamental region for $\mathfrak{M}_n$ in $\mathfrak{H}_n$. Therefore $\mathfrak{E}^*(Y) = \mathfrak{E}^*(Y) \cap \big(\bigcup_{N_i \in \mathfrak{R}} N_i^{-1}\langle\mathfrak{G}_n\rangle\big) = \bigcup_{N_i \in \mathfrak{R}} \big(\mathfrak{E}^*(Y) \cap N_i^{-1}\langle\mathfrak{G}_n\rangle\big)$. Denoting $\mathfrak{E}^*(Y) \cap N_i^{-1}\langle\mathfrak{G}_n\rangle$ by ϑ_i and setting $\mathfrak{E}_j^* = \mathfrak{D}_j - \bigcup_{i=1}^{j-1}\mathfrak{D}_i$, we see that the sets $\mathfrak{E}_i^*$, $i = 1, 2, \ldots$ give a non-overlapping covering of $\mathfrak{E}^*(Y)$. By property b) of $\mathfrak{F}_n$ (p. 37), all but finitely many of the sets $\mathfrak{E}_i^*$ are empty. We now choose $N_1 = E^{(2n)}$.

Let $\mathfrak{E}_i = \{R(Z) \,|\, Z \in \mathfrak{E}_i^*\}$. If $Z \in \mathfrak{E}_i^*$, then $Z^* = N_i\langle Z\rangle \in \mathfrak{G}_n$. Further, for a suitable $U \in \Omega_n$, $I(Z^*)[U]$ is reduced in the sense of Humbert.

Our subsequent discussion is split into two parts according as S is definite or indefinite.

We proceed with the case *when S is non-negative-definite* first i.e. $p = r$. Further let $T = T^{(n)} > 0$. In this case, we consider the 'dissection' of the cube $\mathfrak{E}^*(T^{-1})$ as defined above. From (69) and (56), we obtain

$$(70) \quad A(S, T) = \delta(S)^{-n} \prod_{j=r-n+1}^{r} \left(\frac{(2\pi)^j}{\Gamma(j)\,|d|^{j/2}}\right) \prod_p \alpha_p(S, T)|T|^{r-n} + e^{2\pi n}\int_{\mathfrak{E}} \big(f(S, X+iT^{-1}) - \varphi(X+iT^{-1})\big)\eta(-TX)\{dX\} .$$

Denoting by $c(T)$ the second-term on the right-hand side of (70), we have

$$c(T) = e^{2\pi n}\sum_j \int_{\mathfrak{E}_j} \big(f(S, X+iT^{-1}) - \varphi(X+iT^{-1})\big)\eta(-TX)\{dX\} .$$

Now, since we know that $\min T \leqslant \mu_n|T|^{1/n}$, we choose T such that $|T| > (\gamma'_n\mu_n^{-1})^n = c_{39}$ so that $\min T^{-1} < \gamma'_n$. Thus $\mathfrak{E}^*(T^{-1}) \cap \mathfrak{G}_n = \emptyset$ i.e. $\mathfrak{E}_1$ is empty. To estimate $c(T)$, we use the inequalities (46) and (57). We obtain

$$(71) \quad |c(T)| \leqslant c_{40} \sum_{N_j \in \mathfrak{R},\, j \neq 1} \int_{\mathfrak{E}_j} |f(S, X+iT^{-1}) - \varphi(X+iT^{-1})|dX = c_{40}\sum_{t=1}^{n} I_t$$

where, for $1 \leqslant t \leqslant n$,

$$I_t = \sum_{\substack{N_j = \begin{pmatrix} * & * \\ C_j & D_j \end{pmatrix} \in \mathfrak{R} \\ r(C_j) = t}} \int_{\mathfrak{E}_j} |f(S, X+iT^{-1}) - \varphi(X+iT^{-1})|dx .$$

If now $\{C_jD_j\} \leftrightarrow [G_0, L]$ then we denote $\mathfrak{E}_j$ by $\mathfrak{E}(G_0, L)$. With this notation, for $1 \leqslant t \leqslant n$, we have from (9*), (46) and (57),

$$I_t \leqslant c_{41} \sum_{\substack{G_0 \in \mathfrak{L}(t,n) \\ L=\widetilde{L} \in \{k\}_{n,n},\ Q^{-1}\begin{pmatrix} E^{(t)} & 0 \\ 0 & 0 \end{pmatrix} QL=L}} d(L)^{-r} \int_{\mathfrak{E}(G_0,L)} \|G_0(X+iT^{-1}+L)\widetilde{G}_0\|^{-r} dX .$$

For any $X \in \mathfrak{E}(G_0, L)$ (for fixed L_0 and $L \equiv L_0 \pmod{c_6^2}$) we can determine uniquely $S_0 = \widetilde{S}_0 = \begin{pmatrix} G_0(L-L_0)\widetilde{G}_0 & * \\ * & * \end{pmatrix} \in \{\mathfrak{D}\}_{n,n}$ such that $X[\widetilde{Q}] + \begin{pmatrix} G_0L_0\widetilde{G}_0 & 0 \\ 0 & 0 \end{pmatrix} + S_0$ has elements in the last $n-t$ rows of the form $\gamma + \delta\omega$ with $0 \leqslant \gamma,\ \delta < c_6^2$. Of course, S_0 depends on X, Q and L. But since the number of non-empty $\mathfrak{E}(G_0, L)$ is finite and the sets $\mathfrak{E}(G_0, L)$ are bounded, S_0 belongs to a finite set. The images of $\mathfrak{E}(G_0, L)$ (for fixed L_0 and $L \equiv L_0 \pmod{c_6^2}$) under the transformation

$$X \to X[\widetilde{Q}] + \begin{pmatrix} G_0L_0\widetilde{G}_0 & 0 \\ 0 & 0 \end{pmatrix} + S_0 \tag{72}$$

are all disjoint and lie in the subspace $\mathcal{H}_t^n$ of the space of n-rowed complex hermitian matrices with elements in the last $n-t$ rows of the form $\gamma+\delta\omega$ with $0 \leqslant \gamma,\ \delta < c_6^2$. Applying the transformation (72) to $\mathfrak{E}(G_0, L)$ and denoting the top t-rowed principal minor of X by X_t, we obtain

$$I_t \leqslant c_{42} \sum_{L=\widetilde{L} \bmod (c_6^2)} d(L)^{-r} \sum_{G_0 \in \mathfrak{L}(t,n)} \int_{\mathcal{H}_t^n} \|iT^{-1}[\widetilde{G}_0] + X_t\|^{-r} dX$$

where now L runs over a complete set of modulo (c_6^2) incongruent matrices in $\{k\}_{n,n}$. Since $\sum_{L=\widetilde{L} \bmod (c_6^2)} d(L)^{-r} < \infty$ for $r > 2n$, we have

$$\begin{aligned} I_t &\leqslant c_{43} \sum_{G_0 \in \mathfrak{L}(t,n)} \int_{\mathcal{H}_t} \|iT^{-1}[\widetilde{G}_0] + H\|^{-r} dH \\ &\leqslant c_{44} \sum_{G_0 \in \mathfrak{L}(t,n)} |T^{-1}[\widetilde{G}_0]|^{-(r-t)}, \end{aligned}$$

since $\int_{\mathcal{H}_t} \|E+iH\|^{-r} dH < \infty$, [2, I]. By applying the same arguments as on p. 59, we may deduce that

$$\begin{aligned} I_t &\leqslant c_{45} (\min T^{-1})^{-t(r-t)} \Big(\sum_{0 \neq \underline{h} \in \{\mathfrak{D}\}_{n,1}} (E^{(n)}[\underline{h}])^{t-r} \Big)^t \\ &\leqslant c_{46} (\min T^{-1})^{-t(r-t)} \end{aligned}$$

i.e.

$$I_t \leqslant c_{46} (\min T^{-1})^{-(r-n+1)(n-1)} \quad (1 \leqslant t \leqslant n-1) . \tag{73}$$

We now proceed to estimate I_n. For $Z \in \mathfrak{E}_j^*$ (with $N_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \in \mathfrak{R}$, $r(C_j) = n$ and $Z^* = N_j\langle Z\rangle$), we may derive, from (46) and (57), the inequality

$$\begin{aligned} |f(S, Z) - \varphi(Z)| &\leqslant \big| f(S, Z) - \lambda(0, N_j^{-1}) |C_jZ+D_j|^{-r} \big| + \\ &\quad + \big| \varphi(Z) - \lambda(0, N_j^{-1}) |C_jZ+D_j|^{-r} \big| \leqslant c_{47} \big(\min I(Z^*)\big)^{1-r} \|C_jZ+D_j\|^{-r} . \end{aligned}$$

Now

$$\begin{aligned} I_n &\leqslant c_{48} \sum_{L=\widetilde{L} \in \{k\}_{n,n}} d(L)^{-r} \int_{\mathfrak{E}(E^{(n)},L)} \|X+iT^{-1}+L\|^{-r} y^{1-r} dX \\ &\leqslant c_{48} \sum_{L_0=\widetilde{L}_0 \bmod \mathfrak{D}} d(L_0)^{-r} \sum_{L=\widetilde{L}\equiv L_0 (\bmod \mathfrak{D})} \int_{\mathfrak{E}(E^{(n)},L)} \|X+iT^{-1}+L\|^{-r} y^{1-r} dX \end{aligned}$$

where $y = \min I(Z^*) = \min\big((T[X+L]+T^{-1})^{-1}[C_j^{-1}]\big)$, $L = C_j^{-1}D_j$ and $L_0 = C_0^{-1}D_0$. Since $L \equiv L_0 (\bmod \mathfrak{D})$ it may be shown that $C_j = UC_0$ for $U \in \Omega_n$ and hence $y = \min\big((T[X+L]+T^{-1})^{-1}[C_0^{-1}]\big)$. We now apply to each $\mathfrak{E}(E, L)$, the transformation $X \to X+L$ and then the sets $\mathfrak{E}(E, L)$ corresponding to different $L \equiv L_0 (\bmod \mathfrak{D})$ for a fixed L_0 go over into non-overlapping sets of finite euclidean volume (in the space $\mathcal{H}_n$) and so we have

$$I_n \leqslant c_{48} \sum_{L_0=\widetilde{L}_0 \bmod \mathfrak{D}} d(L_0)^{-r} \int_{\mathcal{H}_n} \|X+iT^{-1}\|^{-r} y^{1-r} dX . \tag{74}$$

Applying the transformation $X \to X[C^{-1}]$ where $T^{-1} = \widetilde{C}C$, we get from (74)

$$I_n \leqslant c_{48} |T|^{r-n} \sum_{L_0=\widetilde{L}_0 \bmod \mathfrak{D}} d(L_0)^{-r} \int_{\mathcal{H}_n} \|E+X^2\|^{-r/2} y^{1-r} dX \tag{75}$$

where now, $y = \min\big((E+X^2)^{-1}[\widetilde{C}^{-1}C_0^{-1}]\big)$.

We may write $X \in \mathcal{H}_n$ as $W[U^{-1}]$ with W being a diagonal matrix with diagonal elements $w_1, \ldots, w_n$ satisfying $w_1 \geqslant w_2 \geqslant \ldots \geqslant w_n$ and U being unitary. Then from $XU = UW$, we get $XdU + dXU = dUW + UdW$ i.e. $U^{-1}dXU = \delta UW - W\delta U + dW$, where $\delta U = U^{-1}dU$. But now $\delta U = (\delta u_{ij})$ is skew-hermitian and $\prod_{i \neq j} \delta u_{ij}$ introduces a volume element $d\mu$ in the space of left cosets of the n-rowed unitary group modulo the subgroup of diagonal unitary matrices. It may then be seen that $dX = \prod_{i \neq j} (w_i - w_j)^2 \prod_{i=1}^{n} dw_i d\mu$. From the theory of classical groups, we know that $\int d\mu$ extended over the coset space is finite. On the other hand, we have

$$\begin{aligned} y = \min\big((E+W^2)^{-1}[\widetilde{U}\widetilde{C}^{-1}C_0^{-1}]\big) &\geqslant \min\big((1+w_1^2)^{-1}T[C_0^{-1}]\big) \\ &\geqslant (1+w_1^2)^{-1} d(L_0)^{-2} y_1 \end{aligned} \tag{76}$$

where $y_1 = \min T$ and $d(L_0) = \|C_0\|$. Moreover,

$$(77)\qquad \prod_{k\neq l}(w_k-w_l)^2 \leqslant \prod_{k\neq l}(1+w_k^2)^{1/2}(1+w_l^2)^{1/2} = \prod_{k=1}^{n}(1+w_k^2)^{n-1}.$$

From (75), (76), (77), we deduce that

(78)

$$I_n \leqslant c_{49}|T|^{r-n}\sum_{L_0=\widetilde{L}_0 \bmod \mathfrak{D}} d(L_0)^{-r}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\prod_{k=1}^{n}(1+w_k^2)^{n-1-r/2}y^{1-r}\,dw_1\dots dw_n\int d\mu$$
$$\leqslant c_{50}|T|^{r-n}\sum_{L_0=\widetilde{L}_0 \bmod \mathfrak{D}} d(L_0)^{-r}\int_{-\infty}^{\infty}(1+w_1^2)^{n-1-r/2}y^{1-r}\,dw_1.$$

Setting $v = (y_1/\gamma_n')^{1/2}d(L_0)^{-1}$ and $y_0 = \gamma_n'\max\big(1, v^2(1+w_1^2)^{-1}\big)$, we have $y \geqslant y_0$ since $y \geqslant \gamma_n'$ already. Further it can be shown without difficulty that

$$(79)\qquad \int_{-\infty}^{\infty}(1+w_1^2)^{n-1-r/2}y_0^{1-r}\,dw_1 \leqslant c_{51}\min\big(1, y_1^{(2n-1-r)/2}d(L_0)^{r-2n+1}\big).$$

Thus from (78) and (79), we have

$$(80)\qquad I_n \leqslant c_{52}|T|^{r-n}\Big(y_1^{(2n-1-r)/2}\sum_{\substack{L_0=\widetilde{L}_0 \bmod \mathfrak{D}\\ d(L_0)<y_1^{1/2}/\gamma_n'}} d(L_0)^{-2n+1} + \sum_{\substack{L_0=\widetilde{L}_0 \bmod \mathfrak{D}\\ d(L_0)\geqslant y_1^{1/2}/\gamma_n'}} d(L_0)^{-r}\Big).$$

We now use the following

LEMMA 6.

$$(81)\qquad u^{-s}\sum_{\substack{L=\widetilde{L} \bmod \mathfrak{D}\\ d(L)<u}} d(L)^{-2n+1} + \sum_{\substack{L=\widetilde{L} \bmod \mathfrak{D}\\ d(L)\geqslant u}} d(L)^{-s-2n+1} < c_{53}\Big(2+\frac{1}{s-1}\Big)u^{1-s}$$

where $u = y_1^{1/2}/\gamma_n' > 0$ *and* $s = r-2n+1 > 1$.

Proof. The proof is on the same lines as Lemma 11 of [11]. One has now to consider the Dirichlet series $\varphi(s) = \sum_{L=\widetilde{L} \bmod \mathfrak{D}} d(L)^{-s-2n+1} = \sum_{l=1}^{\infty} a_l l^{-s}$ which converges absolutely for real $s > 1$. Now let $\zeta(s)$ be the Riemann zeta function and let us write $\big(\zeta(s+1)\big)^{2n-4+2^{2n-1}}\zeta(s)$ as a Dirichlet series $\sum_{l=1}^{\infty} b_l l^{-s}$ (real $s > 1$). Then using some results of H. Braun ([2, I], p. 840) it can be shown that $a_l \leqslant c_{54}b_l$ for $1 \leqslant l < \infty$ and further that $\sum_{l<u} a_l < c_{55}u$ for a constant c_{55} independent of u. The proof of the lemma is then completed as in Lemma 11 of [11].

From (80) and (81) we obtain

$$(82)\qquad I_n \leqslant c_{56}|T|^{r-n}(\min T)^{(2n-r)/2}.$$

Finally then, for $T = \widetilde{T} > 0$ and $|T| > c_{39}$, we have from (71), (73) and (82),

$$(83)\qquad |c(T)| \leqslant c_{57}\big((\min T^{-1})^{(n-1)(n-r-1)} + |T|^{r-n}(\min T)^{(2n-r)/2}\big).$$

This estimate is of no use, unless we relate the growth of $\min T$ to that of $|T|$ as $|T|$ tends to infinity. We know, of course, that $\min T \leqslant \mu_n|T|^{1/n}$ but $\min T$ may remain fixed, even as $|T| \to \infty$. But as $|T| \to \infty$, $\min T^{-1} \to 0$. If therefore we require that as $|T| \to \infty$, $\min T \min T^{-1}$ should be bounded away from zero, the situation considered above cannot happen.

We may now impose the restriction on T that

$$(84)\qquad \min T \geqslant c|T|^{1/n} \quad \text{for a fixed } c > 0, \quad \text{as} \quad |T| \to \infty.$$

This condition implies that $\min T^{-1} \geqslant c' T^{-1/n}$ for a constant c' depending only on k, n and c. The proof of this is similar to that of Lemma 2 of [8]. This condition (84) for T reduced in the sense of Humbert, merely means that $|T|^{-1/n}T$ lies in a compact set in the space of n-rowed positive hermitian matrices of determinant 1.

Thus for $|T| > c_{39}$ and under the condition (84), we have for $r > 2n$,

$$(85)\qquad |c(T)| \leqslant c_{58}\big(|T|^{(n-1)(r-n+1)/n} + |T|^{r-n+(2n-r)/2n}\big)$$
$$\leqslant c_{59}|T|^{r-n+(2n-r)/2n}$$

the constant c_{59} depending only on S, k, n *and* c but *not* on T. We have thus proved

THEOREM 5. *If $S^{(m)}$, $T^{(n)}$ are non-negative integral hermitian matrices of rank $r \leqslant m$ and rank n respectively, then for $r > 2n$, we have*

(86) $A(S, T)$

$$= |T|^{r-n}\big(\delta(S)\big)^{-n}\prod_{j=r-n+1}^{r}\frac{(2\pi)^j|d|^{-j/2}}{\Gamma(j)}\prod_{p}\alpha_p(S, T) + O\big(|T|^{r-n-(r-2n)/2n}\big)$$

provided that $|T| > c_{39}$ *and* $\min T \geqslant c|T|^{1/n}$. (The constants in the O-term in (86) involve c.)

Remark. From the working above, it is clear that if we replace $\varphi(Z)$ by another function $\varphi_1(Z)$ which is, say, a hermitian modular form of degree n and stufe γ^2 and if further, for all $N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{M}_n$ with $r(C) = n$, the constant term in the Fourier expansion of $\varphi_1(Z)/N^{-1}$ is the same as $\lambda(0, N^{-1})$, then, for the Fourier coefficients $c_1(T)$ of $f(S, Z)-\varphi_1(Z)$ with $T>0$ again, we have the estimate (85). The constants in the estimate now depend also on $\varphi_1(Z)$ in general. Thus obtaining an estimate of the type (85) depends only on the fact that $\varphi_1(Z)$ "mimics" $f(S, Z)$ under such hermitian modular transformations. However, for arithmetical applications, it is necessary to know the precise Fourier expansion of $\varphi_1(Z)$.

We now take up the case when S is indefinite i.e. $pq>0$. Carrying through the Farey dissection of the cube $\mathfrak{E}^*(\varepsilon E^{(n)})$ in $\mathfrak{H}_n$ $(0<\varepsilon<\gamma_n')$ exactly as above, with $f(S, H, Z, \widetilde{Z}, 0)$ and $\varphi(S, Z)$ we obtain for fixed $T=\widetilde{T}\in\{\mathfrak{O}\}_{n,n}$, the formula

$$(87)\qquad A(S, T; H, \varepsilon E) = e^{\pi i n(2p-r)/2}|\delta(S)|^{-n}|d|^{n(n-1)/2-rn/2}\times \\ \times\prod_p \alpha_p(S, T)\int_{\mathcal{H}_n}|X+i\varepsilon E|^{-p}|X-i\varepsilon E|^{-q}\eta(-TX)\{dX\}+o(\varepsilon^{-n(r-n)}).$$

The o-sign refers to the passage to the limit $\varepsilon\to 0$ and holds uniformly when $\min H_1=\min H[A^{-1}]=h_1>h_0$ (any given positive number) (for definition of H_1, see (19)). Now

$$(88)\qquad \int_{\mathcal{H}_n}|X+i\varepsilon E|^{-p}|X-i\varepsilon E|^{-q}\eta(-TX)\{dX\} \\ = e^{\pi i n(r-2p)/2}\varepsilon^{-n(r-n)}\int_{\mathcal{H}_n}|E-iX|^{-p}|E+iX|^{-q}\eta(-\varepsilon TX)\{dX\}.$$

The integral on the right-hand side of (88) converges absolutely since $r>2n$ ([2, I]) and the value of the integral with the integrand replaced by its absolute value is independent of ε. Thus this integral converges uniformly with regard to ε as $\varepsilon\to 0$ and its limit is equal to $\int_{\mathcal{H}_n}|E-iX|^{-p}\times$ $\times|E+iX|^{-q}\{dX\}$. The value of this last intergal may be computed as in [11] (Lemma 24) to be

$$\left(\frac{2}{\sqrt{|d|}}\right)^{n(n-1)/2}2^{n(n+1-r)}\frac{\varrho_p\varrho_q\varrho_{r-2n}}{\varrho_{p-n}\varrho_{q-n}\varrho_{r-n}}$$

where, for (rational integral) $\mu>0$, we have set

$$(89)\qquad \varrho_\mu=\prod_{k=1}^{\mu}\frac{\pi^k}{\Gamma(k)},\qquad \varrho_0=1.$$

Multiplying both sides of (87) by $\varepsilon^{n(r-n)}$ and letting ε tend to zero, we obtain

THEOREM 6. *If $S=\widetilde{S}$ is an m-rowed integral matrix of signature (p, q) and rank r $(=p+q)$ and if $T=\widetilde{T}$ is an n-rowed integral matrix, then for $p, q\geqslant n$ and $r>2n$, we have*

$$(90)\qquad \lim_{\varepsilon\to 0}\varepsilon^{n(r-n)}\sum_{\substack{S[G]=T\\ E_SG=G\in\{\mathfrak{O}\}_{m,n}}}\eta(i\varepsilon H[G]) \\ =|\delta(S)|^{-n}2^{n(3n+1-2r)/2}|d|^{n(n-1-2r)/4}\frac{\varrho_p\varrho_q\varrho_{r-2n}}{\varrho_{p-n}\varrho_{q-n}\varrho_{r-n}}\prod_p\alpha_p(S, T)$$

uniformly for $\min H_1\geqslant h_0>0$, where E_S is a fixed r-unit of S and H, H_1 satisfy (17), (19).

Remark. If $|T|\neq 0$, then we know from [1] (Hilfssatz 51, p. 140) that the absolutely convergent infinite product $\prod_p\alpha_p(S, T)$ $(r>2n!)$ vanishes if and only if at least one $\alpha_p(S, T)=0$. Again, by [1] (Hilfssatz 11, p. 83), we know that $\alpha_p(S, T)\neq 0$ if and only if there exists $G^{(m,n)}$ with elements in the ring R_p of p-adic integers over k such that $S[G]=T$, $E_SG=G$. On the other hand, the left-hand side of (90) vanishes if there is no E_S-reduced $G\in\{\mathfrak{O}\}_{m,n}$ with $S[G]=T$ and then $\prod_p\alpha_p(S, T)=0$, implying that $\alpha_p(S, T)=0$ for at least one p. We have thus the following

COROLLARY. *Under the conditions of Theorem 6 together with the condition $|T|\neq 0$, there exists an E_S-reduced $G\in\{\mathfrak{O}\}_{m,n}$ for which $S[G]=T$ if and only if for every rational prime p, $S[G]=T$ for an E_s-reduced G with elements in R_p.*

§ 6. Measure of representation by indefinite forms. Throughout this section, we shall assume that S is an m-rowed indefinite integral hermitian matrix of signature (p, q) and T an n-rowed integral hermitian matrix of signature (p', q') with $p, q\geqslant n$. If $S[G]=T$ for $G\in\{\mathfrak{O}\}_{m,n}$, then we shall define, after Siegel, the "measure of the representation G". First we need a few preliminaries.

By the (reduced) *orthogonal group* $\Omega(S)$ of S, we mean the group of all m-rowed complex square matrices U satisfying

$$(91)\qquad S[U]=S,\qquad E_SUE_S=U,$$

where E_S is a fixed r-unit of S. The inverse U^{-1} of U is uniquely determined by the conditions

$$UU^{-1}=E_S,\qquad U^{-1}U=\widetilde{E}_S.$$

The set of $U\in\Omega(S)$ with $U\in\{\mathfrak{O}\}_{m m}$ is a subgroup $\Gamma(S)$, called the (reduced) *unit group of S*. The group $\Omega(S)$ is a locally compact topological group and $\Gamma(S)$ is a discrete subgroup of $\Omega(S)$. The group $\Omega(S)$ as defined by (91) may be seen to be independent of the particular choice of E_S, for, if $\Omega^*(S)$ is the orthogonal group defined with respect to the r-unit

E_S^*, then under the mapping $U \in \Omega(S) \to E_S^* U \in \Omega^*(S)$, the groups $\Omega(S)$ and $\Omega^*(S)$ may be seen to be isomorphic and in a similar sense, $\Gamma(S)$ is independent of the choice of E_S.

The space $\mathfrak{P}(S)$ (see § 3) associated with S admits of a parametrization as follows. In fact, let C be a complex non-singular matrix such that $S_1[C] = \begin{pmatrix} E^{(p)} & 0 \\ 0 & -E^{(q)} \end{pmatrix}$. If $Z^{(p,q)}$ is an arbitrary complex matrix satisfying $E - Z\tilde{Z} > 0$ and if

$$K_1 = \begin{pmatrix} (E-Z\tilde{Z})^{-1} & -(E-Z\tilde{Z})^{-1}Z \\ -\tilde{Z}(E-Z\tilde{Z})^{-1} & \tilde{Z}(E-Z\tilde{Z})^{-1}Z \end{pmatrix},$$

then $H_1 = 2K_1[C^{-1}] - S_1 \in \mathfrak{P}(S_1)$ and $H = 2K_1[C^{-1}A] - S \in \mathfrak{P}(S)$ (see [12], for the case of quadratic forms over Γ).

In the space $\mathfrak{P}(S_1)$, there is a metric $ds^2 = \sigma(H_1^{-1}dH_1H_1^{-1}dH_1)$ induced from the space of r-rowed positive hermitian matrices and this is invariant under the transformations $H_1 \to H_1[L]$ for arbitrary r-rowed complex non-singular L. The corresponding volume element in $\mathfrak{P}(S_1)$ in terms of the parameter Z is

$$dv = \|E - Z\tilde{Z}\|^{-r} \prod_{\substack{1 \leqslant k \leqslant p \\ 1 \leqslant l \leqslant q}} dx_{kl}\, dy_{kl}$$

where $Z = (z_{kl})$, $z_{kl} = x_{kl} + iy_{kl}$. This volume element taken in $\mathfrak{P}(S)$ is invariant under the transformations $H \to H[M]$ with $E_S M E_S = M$, for, this is equivalent to the transformation $H_1 \to H_1[L]$ for suitable non-singular L. Further it may be seen to be independent of the special choice of A or E_S.

For $U \in \Gamma(S)$ the mapping $H \to H[U]$ for $H \in \mathfrak{P}(S)$ is a homeomorphism of $\mathfrak{P}(S)$ onto itself and in this way, we have a representation of $\Gamma(S)$ as a discontinuous group of mappings of $\mathfrak{P}(S)$ onto itself. This representation is indeed faithful, if we agree to identify U and ϱU in $\Gamma(S)$ for every root of unity ϱ in k. We shall construct a fundamental region for $\Gamma(S)$ in $\mathfrak{P}(S)$ and define, after Siegel ([11], [9]) a 'measure of the unit group $\Gamma(S)$'.

Let W be an m-rowed complex hermitian matrix of signature (p, q) again and having E_S as a r-unit. We define $\Omega(S, W)$ to be the space of complex (m, m) matrices X for which

$$S[X] = W, \quad E_S X E_S = X.$$

For $S = W$, $\Omega(S, W)$ is the same as $\Omega(S)$. The unit group $\Gamma(S)$ acts discontinuously on $\Omega(S, W)$ as the group of mappings $X \to UX$ of $\Omega(S, W)$ onto itself (for $U \in \Gamma(S)$).

Let now $S_1 = S[A^{-1}]$ (see (16)) and $W_1 = W[A^{-1}]$. Then $\Omega(S, W)$ and $\Omega(S_1, W_1)$ are homeomorphic under the mapping $X \in \Omega(S, W) \leftrightarrow X_1 = AXA^{-1} \in \Omega(S_1, W_1)$.

Writing $X_1 = X^{(1)} + iX^{(2)}$ with real $X^{(1)}, X^{(2)}$ and similarly $W_1 = W^{(1)} + iW^{(2)}$, we have in the space $\Omega(S_1, W_1)$ the volume element $\dfrac{\{dX_1\}}{\{dW_1\}} = \dfrac{\{dX^{(1)}\}\{dX^{(2)}\}}{\{dW^{(1)}\}\{dW^{(2)}\}}$ after Siegel ([12]; § 6, Chapter IV). We carry over this volume element to $\Omega(S, W)$, with an extra factor $N(\delta(A^{-1}))^r$. The reason for putting in this extra factor is clear since $N(\delta(A^{-1}))^r \times \dfrac{\{dX_1\}}{\{dW_1\}}$ is independent of the special choice of A. In fact, A^{-1} is unique upto multiplication on the right by a non-singular $L \in \{k\}_{r,r}$ and therefore, if $A^{-1} \to A^{-1}L$, then $S_1 \to S_1[L]$, $W_1 \to W_1[L]$, $X_1 \to L^{-1}X_1L$ and then $N(\delta(A^{-1}))^r \dfrac{\{dX_1\}}{\{dW_1\}} \to N(\delta(A^{-1}))^r N(\delta(L))^r N(\delta(L))^{-r} \dfrac{\{dX_1\}}{\{dW_1\}}$.

The group $A\Gamma(S)A^{-1}$ acts discontinuously on $\Omega(S_1, W_1)$ as the group of mappings $X_1 \to UX_1$ for $X_1 \in \Omega(S_1, W_1)$ and $U \in A\Gamma(S)A^{-1}$. It contains a subgroup G_A of finite index, which is also of finite index in the unit group $\Gamma(S_1)$ of S_1. One can construct a fundamental region $\tilde{F}$ for $\Gamma(S_1)$ and hence for $A\Gamma(S)A^{-1}$ in $\Omega(S_1, W_1)$ and using the homeomorphism between $\Omega(S_1, W_1)$ and $\Omega(S, W)$, one can get a fundamental set for $\Gamma(S)$ in $\Omega(S, W)$.

We now define $\mu(S)$, *the measure of the unit group* $\Gamma(S)$ (cf. [9]) by

$$(92) \qquad \mu(S) = \left(\frac{2}{\sqrt{|d|}}\right)^{r(r+1)/2} \frac{(\Gamma(S_1): G_A)}{(A\Gamma(S)A^{-1}: G_A)} N(\delta(A^{-1}))^r \int_{\tilde{F}} \frac{\{dX_1\}}{\{dW_1\}},$$

where $\tilde{F}$ is a fundamental set for $\Gamma(S_1)$ in $\Omega(S_1, W_1)$. The right-hand side of (92) is seen to be independent of the special choice of A. Further, let us observe that $\dfrac{(\Gamma(S_1): G_A)}{(A\Gamma(S)A^{-1}: G_A)} \displaystyle\int_{\tilde{F}} \dfrac{\{dX_1\}}{\{dW_1\}}$ is exactly the volume of a fundamental set for $A\Gamma(S)A^{-1}$ in $\Omega(S_1, W_1)$.

It is known from [9] that there exists a fundamental region F in $\mathfrak{P}(S_1)$ for $\Gamma(S_1)$ modulo the subgroup of units of S_1 of the form $\varrho E^{(r)}$ with ϱ, a root of unity in k and further for $r \geqslant 1$, $\int_F dv < \infty$ (Lemma 7, [9]). Proceeding exactly as in ([12]; § 6, Chapter IV) one can connect F with the fundamental set $\tilde{F}$ by applying now an analogue of a lemma of Siegel's (viz. Lemma 9, [9]) and prove the formula

$$(93) \qquad w\mu(S) = \left(\frac{2}{\sqrt{|d|}}\right)^{r(r+1)/2} |\delta(S)|^{-r} \varrho_p \varrho_q \frac{(\Gamma(S_1): G_A)}{(A\Gamma(S)A^{-1}: G_A)} \int_F dv.$$

In (93), w denotes the number of roots of unity in k and ϱ_p, ϱ_q are defined by (89). Thus, for $r \geqslant 1$, $\mu(S) < \infty$.

If S is definite, $\frac{1}{w} \cdot \frac{(\Gamma(S_1)\colon G_A)}{(A\Gamma(S)A^{-1}\colon G_A)} \int\limits_F dv$ is just $\varrho(S)^{-1}$ where $\varrho(S)$ is the order of the reduced unit group of S and therefore

$$\mu(S) = \left(\frac{2}{\sqrt{|d|}}\right)^{r(r+1)/2} \varrho_p \delta(S)^{-r} \varrho(S)^{-1}. \tag{94}$$

Let now $T = \tilde{T}$ be n-rowed, integral, of rank t and of signature (p', q') and let $r > 2n$ and $p, q \geqslant n$. Further let G be an E_S-reduced representation of T by S, of rank $h > 0$; clearly $t \leqslant h \leqslant n$. Before we define the "measure of the representation G", we need the following

LEMMA 7. *$G = C\begin{pmatrix} E^{(t)} & 0 \\ 0 & D \end{pmatrix} Q^*$ where $C = C^{(m,h)} = E_S C$, $r(C) = h$, $D = D^{(h-t,n-t)}$, $r(Q^*) = n$ and further, there exists a rational integer c_{60} depending only on k and n such that $c_{60}C$, $c_{60}D$, $c_{60}Q^{*-1}$ and Q^* are integral and $N(\delta(D)) \leqslant c_{60}$. If $h = n$, we take $D = E^{(n-t)}$ and if $t = n$, we take $Q^* = E^{(n)}$ and $G = C$.*

(Note. The constants $c_{60}, c_{61}, \ldots, c_{64}$ occurring in the course of the proof below are positive rational integers depending only on k and n.)

Proof. Let, first, $h < n$. Then G has a r-unit $\check{O}$ of rank h and using (5) and (6), we have

$$G = C_1 B, \quad r(C_1) = h, \quad B \in \{\mathfrak{O}\}_{h,n},$$
$$B\check{O} = B, \quad c_{61} C_1 \in \{\mathfrak{O}\}_{n,h}, \quad N(\delta(B)) \leqslant c_{62}$$

for constants c_{61}, c_{62}. By Hilfssatz 24 of [10], upto a left sided unimodular factor, there can be at most finitely many integral $B^{(h,n)}$ with r-unit $\check{O}$ and $N(\delta(B)) \leqslant c_{62}$. Let $s = r(S[C_1])$. Again using (5) and (6), there exists an integral non-singular Q_2 with $c_{63}Q^{-1} \in \{\mathfrak{O}\}_{h,h}$ (for a suitable constant c_{63}) such that $S[C_1Q_2^{-1}] = \begin{pmatrix} T_1^{(s)} & 0 \\ 0 & 0 \end{pmatrix}$. We may assume by replacing C_1, B by $C_1Q_2^{-1}$, Q_2B respectively that $S[C_1] = \begin{pmatrix} T_1^{(s)} & 0 \\ 0 & 0 \end{pmatrix}$ already. If $h = n$, we choose $C_1 = G$ and $B = E^{(n)}$.

Let now $t < n$. There exist then non-singular integral Q^* and a constant c_{64} with $c_{64}Q^{*-1}$ integral such that

$$T[Q^{*-1}] = \begin{pmatrix} T_2^{(t)} & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $G^* = GQ^{*-1}$ and $B^* = BQ^{*-1} = \begin{pmatrix} B_1^{(s,t)} & B_2 \\ B_3 & B_4 \end{pmatrix}$. From

$$S[G^*] = \begin{pmatrix} T_1^{(s)} & 0 \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}\right] = \begin{pmatrix} T_2^{(t)} & 0 \\ 0 & 0 \end{pmatrix},$$

we obtain $t \leqslant s$ and $T_1[(B_1 B_2)] = \begin{pmatrix} T_2 & 0 \\ 0 & 0 \end{pmatrix}$. But $(B_1 B_2)$ has n columns and rank s. Since $|T_1| \neq 0$, $|T_2| \neq 0$, this means $s \leqslant t$. Thus $s = t$ and $|B_1| \neq 0$. Further $B_2 = 0$ necessarily. Now replacing C_1 by $C_1\begin{pmatrix} B_1 & 0 \\ B_3 & E^{(h-t)} \end{pmatrix}$ and B^* by $\begin{pmatrix} B_1^{-1} & 0 \\ -B_3B_1^{-1} & E^{(h-t)} \end{pmatrix} B^*$, we may conclude that

$$G^* = GQ^{*-1} = C\begin{pmatrix} E^{(t)} & 0 \\ 0 & D \end{pmatrix}$$

where $D = D^{(h-t,n-t)}$ and $C = C^{(m,h)} = E_S C$ are of rank $h-t$ and h respectively and furthermore, there exists a rational integer c_{60} such that $c_{60}C$, $c_{60}D$ are integral and $N(\delta(D)) \leqslant c_{60}$.

If $h = n$, then we may choose $D = E^{(n-t)}$ and $C = GQ^{*-1}$. If $t = n$, we take $Q^* = E^{(n)}$ and since $h = n$ necessarily, we take $C = G$. Our lemma is proved.

We proceed to define after Siegel, the *measure of the E_S-reduced representation G of T by S*, as follows. Let $\Gamma(S, G)$ be the group of E_S-reduced units U of S for which $UG = G$. Invoking the form of G in Lemma 7, we see that $\Gamma(S, G)$ is the same as $\Gamma(S, C)$, the group of $U \in \Gamma(S)$ for which $UC = C$.

Let $W_1 = \begin{pmatrix} S[C] & Q \\ \tilde{Q} & R \end{pmatrix}$ be an r-rowed complex hermitian matrix of signature (p, q). Consider the space $\Omega(S, W_1; C)$ of complex matrices $Y^{(m,r-h)}$ for which

$$E_S Y = Y, \quad S[(C\ Y)] = W_1.$$

Writing $S[(C\ Y)] = W_1$ as $S_1[(AC\ AY)] = W_1$, we see that $\Omega(S, W_1; C)$ is homeomorphic with the space $\Omega(S_1, W_1; AC)$ of complex matrices $Y_1 = Y_1^{(r,r-h)}$ for which $S_1[(AC\ Y_1)] = W_1$. The group $A\Gamma(S, C)A^{-1}$ contains a subgroup $G_{A,C}$ of finite index, which is also of finite index in $\Gamma(S_1, AC)$. One can construct a fundamental set $\tilde{F}(AC)$ for $\Gamma(S_1, AC)$ in $\Omega(S_1, W_1; AC)$ and hence, for $A\Gamma(S, C)A^{-1}$. Pulling this back to $\Omega(S, W_1; C)$ we get a fundamental set for $\Gamma(S, C)$ in $\Omega(S, W_1; C)$. Let $a = \frac{(\Gamma(S_1, AC)\colon G_{A,C})}{(A\Gamma(S, C)A^{-1}\colon G_{A,C})}$. Now, on $\Omega(S_1, W_1; AC)$, we have after Siegel ([12]; § 6, Chapter IV), the volume element $\frac{\{dY_1^{(1)}\}\{dY_1^{(2)}\}}{\{dQ^{(1)}\}\{dQ^{(2)}\}\{dR^{(1)}\}\{dR^{(2)}\}}$, where we have written $Y_1 = Y_1^{(1)} + iY_1^{(2)}$, $Q = Q^{(1)} + iQ^{(2)}$, $R = R^{(1)} + iR^{(2)}$ with real $Y_1^{(1)}$, $Y_1^{(2)}$, $Q^{(1)}$, $Q^{(2)}$, $R^{(1)}$ and $R^{(2)}$. Denoting this briefly by $\frac{\{dY_1\}}{\{dQ\}\{dR\}}$ we see, in the first place, that $N(\delta(A^{-1}))^{r-h}\frac{\{dY_1\}}{\{dQ\}\{dR\}}$ is independent of the special choice of A. For, if $A^{-1}\to A^{-1}L$ with $|L|\neq 0$, then $\{dY_1\}\to N(|L|)^{-(r-h)}\{dY_1\}$ and $\{dQ\}\{dR\}$ is unchanged. Again, if W_1 is replaced by $W_1\left[\begin{pmatrix} E^{(h)} & L^* \\ 0 & P\end{pmatrix}\right]$ with arbitrary non-singular $P^{(r-h)}$ and $(CY_1)\to(CY_1)\begin{pmatrix} E & L^* \\ 0 & P\end{pmatrix}$, then again $\frac{\{dY_1\}}{\{dQ\}\{dR\}}$ is unchanged. Thus we may assume W_1 to be in the form

$$W_1 = \begin{pmatrix} S[C] & Q_1 & Q_2 \\ \tilde{Q}_1 & T_0 & Q_3 \\ \tilde{Q}_2 & \tilde{Q}_3 & Q_4\end{pmatrix} \quad\text{with}\quad W_2^{(p+h)} = \begin{pmatrix} S[C] & Q_1 \\ \tilde{Q}_1 & T_0\end{pmatrix},\quad T_0 > 0 \text{ and } |W_2|\neq 0\,.$$

Then $Q = (Q_1Q_2)$ and $R = \begin{pmatrix} T_0 & Q_3 \\ \tilde{Q}_3 & Q_4\end{pmatrix}$. Since $T_0 > 0$, W_2 has the signature (p, h).

The measure $\mu(S, G)$ of the representation G is defined by

$$a\mu(S, G) = \|Q^*\|^{2(r-h)}\left(\frac{2}{\sqrt{|d|}}\right)^{(r-h)(r-h+1)/2} N(\delta(A^{-1}))^{r-h}\int\limits_{\tilde{F}(AC)}\frac{\{dY_1\}}{\{dQ\}\{dR\}} \tag{95}$$

(referring to the notation of Lemma 7). Proceeding as in [12] (§ 6, Chap. IV) and applying Lemma 9 of [9] repeatedly, we can prove that

$$\begin{aligned} a\mu(S, G)&\int\limits_{L>0, L>S[C]} |L-S[C]|^{q-h}|L|^{p-h}g(L)\,dL \\ &= \varrho_{p-h}\varrho_{q-h}|\delta(S)|^{h-r}\|Q^*\|^{2(r-h)}\left(\frac{2}{\sqrt{|d|}}\right)^{(r-h)(r-h+1)/2}\int\limits_{F(AC)} g\left(\tfrac{1}{2}(S+H)[C]\right)dv\end{aligned} \tag{96}$$

where, on the left-hand side, the integration is over the space of all h-rowed positive L with $L > S[C] = \begin{pmatrix} T_2^{(l)} & 0 \\ 0 & 0\end{pmatrix}$ and further $F(AC)$ is a fundamental region for $\Gamma(S_1; AC)$ in $\mathfrak{P}(S_1)$. Moreover, in (96), the function $g(\cdot)$ is so chosen as to make all the concerned integrals converge.

Let $0 < r(G) = h < n$. With the notation of Lemma 7, we define for $L^{(h)} = \tilde{L}$ and $\varepsilon > 0$,

$$g(L) = \exp\left(-\frac{\pi\varepsilon}{2}\sigma(2L[PQ^*]-T)\right)$$

where $P = \begin{pmatrix} E^{(l)} & 0 \\ 0 & D\end{pmatrix}$. For $h = n$, we define for $L^{(n)} = \tilde{L}$,

$$g(L) = \exp\left(-\frac{\pi\varepsilon}{2}\sigma(2L[Q^*]-T)\right).$$

Then, for $0 < h < n$, we have

$$\begin{aligned} g(L) = \exp\left(-\frac{\pi\varepsilon}{2}\sigma\big((H+S)[CPQ^*]-T\big)\right) &= \exp\left(-\frac{\pi\varepsilon}{2}\sigma\big((H+S)[G]-T\big)\right) \\ &= \exp\left(-\frac{\pi\varepsilon}{2}\sigma(H[G])\right)\end{aligned}$$

and further

$$\begin{aligned} g(\varepsilon^{-1}L) &= \exp\left(-\frac{\pi}{2}\sigma\left(2L[PQ^*]-\varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0\end{pmatrix}[Q^*]\right)\right) \\ &= \exp\left(-\frac{\pi}{2}\sigma\left(\left(2L[P]-\varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0\end{pmatrix}\right)R^*\right)\right)\end{aligned} \tag{97}$$

where $R^* = E[\tilde{Q}^*]$. For $h = n$, we have

$$g(\varepsilon^{-1}L) = \exp\left(-\frac{\pi}{2}\sigma\left(\left(2L-\varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0\end{pmatrix}\right)R^*\right)\right). \tag{98}$$

With $g(L)$ chosen in this way, we replace L by $\varepsilon^{-1}L$ on the left-hand side of (96) and obtain, in view of (97), that

$$\begin{aligned} \varepsilon^{-h(r-h)}a\mu(S, G)&\int\limits_{L=L^{(h)}>0,\, L>\varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0\end{pmatrix}} \left|L-\varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0\end{pmatrix}\right|^{q-h}|L|^{p-h}\times \\ &\times\exp\left(-\frac{\pi}{2}\sigma\left(\left(2L[P]-\varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0\end{pmatrix}\right)R^*\right)\right)dL \\ = \varrho_{p-h}\varrho_{q-h}&\left(\frac{2}{\sqrt{|d|}}\right)^{(r-h)(r-h+1)/2}|\delta(S)|^{h-r}\|Q^*\|^{2(r-h)}\int\limits_{F(AC)}\exp\left(-\frac{\pi\varepsilon}{2}\sigma(H[G])\right)dv\end{aligned} \tag{99}$$

for $0 < h < n$ and further, if $r(G) = n$, then

$$\begin{aligned} \varepsilon^{-n(r-n)}a\mu(S, G)&\int\limits_{L=L^{(n)}>0,\, L>\varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0\end{pmatrix}} \left|L-\varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0\end{pmatrix}\right|^{q-n}|L|^{p-n}\times \\ &\times\exp\left(-\frac{\pi}{2}\sigma\left(\left(2L-\varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0\end{pmatrix}\right)R^*\right)\right)dL \\ = \varrho_{p-n}\varrho_{q-n}&\left(\frac{2}{\sqrt{|d|}}\right)^{(r-n)(r-n+1)/2}|\delta(S)|^{n-r}\|Q^*\|^{2(r-n)}\int\limits_{F(AG)}\exp\left(-\frac{\pi\varepsilon}{2}\sigma(H[G])\right)dv\,.\end{aligned} \tag{100}$$

We shall investigate the right-hand side of (99) and (100) more closely. Since, for $U \in \Gamma(S, C)$, we know that $\varrho U \notin \Gamma(S, C)$ for a root of unity $\varrho \neq 1$ in k, we can choose left coset representatives of $\Gamma(S)$ modulo $\Gamma(S, C)$ as $U_1, \varrho U_1, \ldots, U_2, \varrho U_2, \ldots$, where ϱ runs over the $w-1$ roots of unity in k different from 1. For $A\Gamma(S)A^{-1}$ modulo $A\Gamma(S, C)A^{-1}$, the left coset representatives may therefore be chosen as AU_1A^{-1}, $\varrho AU_1A^{-1}, \ldots, AU_2A^{-1}, \varrho AU_2A^{-1}, \ldots$ Now, since for U_1 in $A\Gamma(S, C)A^{-1}$ or $\Gamma(S_1, AC)$, we have $U_1AG = AG$, we can easily verify that

$$\frac{w}{a} \int\limits_{F(AC)} e^{-\pi\varepsilon\sigma(H[G])/2}\, dv = w \int\limits_{F(A\Gamma(S,C)A^{-1})} e^{-\pi\varepsilon\sigma(H[G])/2}\, dv \tag{101}$$

$$= \sum_{G^*} \int\limits_{F(A\Gamma(S)A^{-1})} e^{-\pi\varepsilon\sigma(H[G^*])/2}\, dv \tag{102}$$

where, in (102), G^* runs over all distinct matrices in $\{\hat{\mathfrak{D}}\}_{m,n}$ which are associated with G on the left with respect to $\Gamma(S)$, $F(A\Gamma(S)A^{-1})$ denotes a fundamental region for $A\Gamma(S)A^{-1}$ in $\mathfrak{P}(S_1)$ (identifying $U \in A\Gamma(S)A^{-1}$ and ϱU, for every root of unity ϱ in k) and further $F(A\Gamma(S, C)A^{-1})$ denotes a fundamental region for $A\Gamma(S, C)A^{-1}$ in $\mathfrak{P}(S_1)$. If $r(G) = n$, then again, formulae (101) and (102) are valid with $F(AC)$ in (101) and replaced by $F(AG)$.

We still do not know if the measures $\mu(S, G)$ defined above are finite or not. To this end, we proceed as follows. Summing over all E_S-reduced representations $G \neq 0$ of T by S, we have, from (99), (100) and (102),

$$\varepsilon^{n(r-n)} \sum_{\substack{S[G]=T \\ E_S G = G \neq 0,\, G \in \{\mathfrak{D}\}_{m,n}}} \int\limits_{F(A\Gamma(S)A^{-1})} e^{-\pi\varepsilon\sigma(H[G])/2}\, dv = J_1(\varepsilon) + J_2(\varepsilon), \tag{103}$$

where

$$J_1(\varepsilon) = w \frac{|\delta(S)|^{r-n} \|Q^*\|^{2(n-r)}}{\varrho_{p-n}\varrho_{q-n}} \left(\frac{\sqrt{|d|}}{2}\right)^{(r-n)(r-n+1)/2} \times \tag{104}$$
$$\times \sum_{\substack{S[G_1]=T \\ r(G_1)=n}}{}' \mu(S, G_1) \int\limits_{\substack{L=\bar{L}^{(n)}>0 \\ L>\varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0 \end{pmatrix}}} \left| L - \varepsilon \begin{pmatrix} T_2 & 0 \\ 0 & 0 \end{pmatrix} \right|^{q-n} |L|^{p-n} \times$$
$$\times \exp\left(-\frac{\pi}{2}\sigma\left(\left(2L - \varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0 \end{pmatrix}\right) R^*\right)\right) dL$$

and

$$J_2(\varepsilon) = \tag{105}$$
$$= \sum_{\substack{S[G_2]=T \\ 0<h=r(G_2)<n}}{}' \varepsilon^{n(r-n)-h(r-h)} w \frac{|\delta(S)|^{r-h} \|Q^*\|^{2(h-r)}}{\varrho_{p-h}\varrho_{q-h}} \left(\frac{\sqrt{|d|}}{2}\right)^{(r-h)(r-h+1)/2} \mu(S, G_2) \times$$
$$\times \int\limits_{\substack{L=\bar{L}^{(h)}>0 \\ L>\varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0 \end{pmatrix}}} \left| L - \varepsilon \begin{pmatrix} T_2 & 0 \\ 0 & 0 \end{pmatrix} \right|^{q-h} |L|^{p-h} \exp\left(-\frac{\pi}{2}\sigma\left(\left(2L[P] - \varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0 \end{pmatrix}\right) R^*\right)\right) dL.$$

In (104) and (105), G_1, G_2 run through a complete set of E_S-reduced representations of T by S not mutually associated on the left with respect to $\Gamma(S)$, with $r(G_1) = n$ and $r(G_2) = h$ $(0 < h < n)$, respectively.

We now obtain an estimate for

$$\varepsilon^{n(r-n)} A\left(S, T; H, \frac{\varepsilon}{4}E\right) = \varepsilon^{n(r-n)} \sum_{\substack{S[G]=T \\ E_S G = G \in \{\mathfrak{D}\}_{m,n}}} e^{-\pi\varepsilon\sigma(H[G])/2},$$

which is valid uniformly for H_1 in $\mathfrak{P}(S_1)$. We know that

$$A\left(S, T; H, \frac{\varepsilon}{4}E\right) = \int\limits_{X \in \mathfrak{E}} f\left(S, H, X + i\frac{\varepsilon}{4}E, X - i\frac{\varepsilon}{4}E, 0\right) \eta(-TX)\{dX\}.$$

Hence

$$A\left(S, T; H, \frac{\varepsilon}{4}E\right) \leqslant \int\limits_{\mathfrak{E}^*\left(\frac{\varepsilon}{4}E\right)} |f(S, H, Z, \tilde{Z}, 0)|\{dX\}.$$

We now proceed exactly as we did for the proof of (87) but use the full force of inequality (46), viz. for $N_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \in \mathfrak{M}_n$ and $Z \in N_j^{-1}\langle\mathfrak{G}_n\rangle$,

$$|f(S, H, Z, \tilde{Z}, 0)| \leqslant c_{22}\|C_j Z + D_j\|^{-r} \prod_{k=1}^{r} \left(1 + c_{16}(h_k y_j)^{-n}\right)$$

where $h_1, \ldots, h_r$ are the diagonal elements of the matrix in the Humbert domain $\mathfrak{T}_0^{(r)}$, corresponding to $H_1 = H[A^{-1}]$. We then obtain as in § 5 that

$$A\left(S, T; H, \frac{\varepsilon}{4}E\right)$$
$$\leqslant c_{65} \prod_{k=1}^{r} (1 + c_{66}h_k^{-n}) \sum_{\substack{t=1 \\ G_0 \in \mathfrak{L}^{(t,n)} \\ L=\tilde{L} \in \{k\}_{n,n} \\ Q^{-1}\begin{pmatrix} E^{(t)} & 0 \\ 0 & 0 \end{pmatrix} QL = L}}^{n} \int\limits_{\mathfrak{E}(G_0, L)} \left\| G_0\left(X + i\frac{\varepsilon}{4}E + L\right)\tilde{G}_0 \right\|^{-r} dX$$

and finally that

(106)
$$\varepsilon^{n(r-n)} \sum_{\substack{E_S G=G \\ S[G]=T}} e^{-\pi\varepsilon\sigma(H[G])/2} \leqslant c_{67} \prod_{k=1}^{r} (1+h_k^{-n})$$

uniformly for $h_1 = \min H[A^{-1}] \geqslant h_0 > 0$. The constant c_{67} depends only on r, n and $|\delta(S)|$. Now, analogous to a lemma of Siegel (Lemma 29, [11]; cf. Lemma 6, [9]), it can be proved that for $r > 2n$,

$$\int_F \prod_{k=1}^{r} (1+h_k^{-n})\, dv < \infty,$$

F being a fundamental region for $\Gamma(S_1)$ in $\mathfrak{P}(S_1)$. In view of the commensurability of $\Gamma(S_1)$ and $A\Gamma(S)A^{-1}$, it follows that

(107)
$$\int_{F(A\Gamma(S)A^{-1})} \prod_{k=1}^{r} (1+h_k^{-n})\, dv = c_{68} < \infty.$$

By (106) and (107), it turns out that the left-hand side of (103) and therefore $J_1(\varepsilon)+J_2(\varepsilon)$ is finite. Let us assume for the moment that all the integrals occurring in (104) and (105) are finite; as a matter of fact, we would presently see that these integrals exist and even converge uniformly with regard to ε, for $0<\varepsilon<1$. Since all terms involved in $J_1(\varepsilon)$ and $J_2(\varepsilon)$ are non-negative, it follows, in particular, that *the measures* $\mu(S, G)$ *are all finite.*

For the case of indefinite quadratic forms over an arbitrary algebraic number field the analogous notions of "measure of the (reduced) unit group" and "measure of (reduced) representation" etc. have been studied in [7].

In (90), we replace ε by $\varepsilon/4$, multiply both sides by the volume element dv in $\mathfrak{P}(S_1)$ and integrate over $F(A\Gamma(S)A^{-1})$. In view of (106) and (107), we are also justified in interchanging the integration over $F(A\Gamma(S)A^{-1})$ and passage of ε to the limit 0. We then obtain

(108)
$$\lim_{\varepsilon\to 0} \varepsilon^{n(r-n)} \sum_{\substack{S[G]=T \\ E_S G=G \in \{\mathfrak{O}\}_{m,n}}} \int_{F(A\Gamma(S)A^{-1})} e^{-\pi\varepsilon\sigma(H[G])/2}\, dv$$
$$= |\delta(S)|^{-n} 2^{n(2r-n+1)/2} |d|^{n(n-1-2r)/4} \frac{\varrho_p\, \varrho_q\, \varrho_{r-2n}}{\varrho_{p-n}\, \varrho_{q-n}\, \varrho_{r-n}} \prod_p \alpha_p(S, T) \int_{F(A\Gamma(S)A^{-1})} dv.$$

But the left-hand side of (108) is precisely $\lim_{\varepsilon\to 0}\big(J_1(\varepsilon)+J_2(\varepsilon)\big)$, since even if $G=0$ occurs on the left-hand side of (108), under the limit $\varepsilon\to 0$, it disappears, in view of $r>2n$ and the fact that $\int_{F(A\Gamma(S)A^{-1})} dv < \infty$. We shall now proceed to determine $\lim_{\varepsilon\to 0}\big(J_1(\varepsilon)+J_2(\varepsilon)\big)$.

Writing $L^{(h)} = \begin{pmatrix} L_1^{(t)} & 0 \\ 0 & L_2^{(h-t)} \end{pmatrix} \left[\begin{pmatrix} E^{(t)} & 0 \\ L_3 & E \end{pmatrix}\right]$, let us observe that the conditions "$L>0$, $L>\varepsilon\begin{pmatrix} T_2^{(t)} & 0 \\ 0 & 0\end{pmatrix}$" are equivalent to the conditions "$L_1>0$, $L_1>\varepsilon T_2$, $L_2>0$ and $L_3^{(h-t,t)}$ arbitrary complex". Further if dL_1, dL_2, dL_3 are products of differentials defined analogously to dL, we have $dL = |L_2|^{2t} dL_1\, dL_2\, dL_3$. Moreover, $|L| = |L_1||L_2|$, $\sigma(L[P]) = \sigma(L_1)+\sigma(L_2[L_3]) + \sigma(L_2[D])$. For $0<h<n$, we have,

(109)
$$\left| \int_{L>0,\, \varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0\end{pmatrix}} \left|L-\varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0\end{pmatrix}\right|^{q-h} |L|^{p-h} \exp\left(-\frac{\pi}{2}\sigma\left(\left(2L[P]-\varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0\end{pmatrix}\right)R^*\right)\right) dL \right|$$
$$\leqslant \int_{L>0,\, L>\varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0\end{pmatrix}} \left|L-\varepsilon\begin{pmatrix} T_2 & 0 \\ 0 & 0\end{pmatrix}\right|^{q-h} |L|^{p-h} \exp\left(-\pi\lambda\sigma\left(L[P]-\frac{\varepsilon}{2}\begin{pmatrix} T_2 & 0 \\ 0 & 0\end{pmatrix}\right)\right) dL$$

since, for a constant $\lambda>0$ depending only on k and h and hence, only on k and n, we have $R^* > \lambda E^{(h)}$. The integral (109) is precisely

(110)
$$\frac{\varrho_{r-2h+t}}{\varrho_{r-h}} \lambda^{(h-t)(h-r-t)} |D\widetilde{D}|^{h-r} \int_{L_1^{(t)}>0,\, L_1>\varepsilon T_2} e^{-\pi\lambda\sigma\left(L_1-\frac{\varepsilon}{2}T_2\right)} |L_1-\varepsilon T_2|^{q-h} |L_1|^{p-h}\, dL_1$$

since $\int_{L_3^{(h-t,t)}} e^{-\pi\lambda\sigma(L_2[L_3])} dL_3 = \lambda^{-t(h-t)} |L_2|^{-t}$ (the integration being over the space of all complex matrices L_3 of $h-t$ rows and t columns) and

$$\int_{L_2^{(h-t)}>0} e^{-\pi\lambda\sigma(L_2 D\widetilde{D})} |L_2|^{r-2h+t} dL_2 = \lambda^{(h-t)(h-r)} |D\widetilde{D}|^{h-r} \frac{\varrho_{r-2h+t}}{\varrho_{r-h}}.$$

We now claim that the integral (110) converges uniformly with respect to ε for $0<\varepsilon<1$. For, suppose first that $q \geqslant p$. Then $q-h>0$ and we have

(111)
$$|L_1-\varepsilon T_2|^{q-h} \leqslant c_{69} e^{\lambda\sigma(L_1-\varepsilon T_2)}$$

and the integral (110) is majorized by

$$c_{69} \int_{L_1>0,\, L_1>\varepsilon T_2} |L_1|^{p-h} \exp\left(-\pi\lambda\sigma\left(L_1-\frac{\varepsilon}{2}T_2\right)+\lambda\sigma(L_1-\varepsilon T_2)\right) dL_1.$$

If $p \geqslant q$, then in the integral (110), we effect the transformation $L_1 \to L_1 + \varepsilon T_2$ and use, instead of (111) the inequality

$$|L_1+\varepsilon T_2|^{p-h} \leqslant c_{70} e^{\lambda\sigma(L_1+\varepsilon T_2)}.$$

Thus, as $\varepsilon\to 0$, the limit of the integral (110) exists and is, in fact, equal to $\int\limits_{L_1=L_1^{(l)}>0} e^{-\pi\lambda\sigma(L_1)}|L_1|^{r-2h}dL_1$ which converges since $r>2n>2h$. Thus we may conclude that

(112) $$\lim_{\varepsilon\to 0} J_2(\varepsilon)=0\,.$$

Again

$$\int\limits_{\substack{L=L^{(n)}>0\\ L>\varepsilon\begin{pmatrix}T_2&0\\0&0\end{pmatrix}}} \left|L-\varepsilon\begin{pmatrix}T_2&0\\0&0\end{pmatrix}\right|^{q-n}|L|^{p-n}\exp\left(-\frac{\pi}{2}\sigma\left(\left(2L-\varepsilon\begin{pmatrix}T_2&0\\0&0\end{pmatrix}\right)R^*\right)\right)dL$$

$$=\|Q^*\|^{2(r-n)}\int\limits_{L>0,\,L>\varepsilon T}|L-\varepsilon T|^{q-n}|L|^{p-n}\exp\left(-\pi\sigma\left(L-\frac{\varepsilon}{2}T\right)\right)dL\,.$$

By the same arguments as before, the integral on the right-hand side converges uniformly with regard to ε for $0<\varepsilon<1$ and its limit, as $\varepsilon\to 0$, is $\int\limits_{L>0}|L|^{r-2n}e^{-\pi\sigma(L)}dL=\frac{\varrho_{r-2n}}{\varrho_{r-n}}$. Thus

(113) $$\lim_{\varepsilon\to 0} J_1(\varepsilon)=w\frac{|\delta(S)|^{r-n}\varrho_{r-2n}}{\varrho_{p-n}\varrho_{q-n}\varrho_{r-n}}\left(\frac{\sqrt{|d|}}{2}\right)^{(r-n)(r-n+1)/2}\sum_{G_1}{}'\mu(S,G_1)\,.$$

We define $M(S,T)$ the *measure of representation of* T *by* S by

$$M(S,T)=\sum\mu(S,G)$$

where the summation is over a complete set of E_S-reduced representations G of T by S with $r(G)=n$ and not mutually associated on the left with respect to $\Gamma(S)$. When $S\geqslant 0$, $M(S,T)$ just corresponds to the number $A(S,T)$ of E_S-reduced representation of T by S; in fact, for $S\geqslant 0$ and $T>0$,

$$M(S,T)=\left(\frac{2}{\sqrt{|d|}}\right)^{(r-n)(r-n+1)/2}\varrho_{r-n}\left(\frac{|\delta(S)|}{|T|}\right)^{n-r}\frac{A(S,T)}{\varrho(S)}$$

where $\varrho(S)$ is the order of the reduced unit group of S.

From (103), (108), (112) and (113) we have

$$\frac{w|\delta(S)|^{r-n}\varrho_{r-2n}}{\varrho_{p-n}\varrho_{q-n}\varrho_{r-n}}\left(\frac{\sqrt{|d|}}{2}\right)^{(r-n)(r-n+1)/2}M(S,T)$$

$$=|\delta(S)|^{-n}2^{n(2r-n+1)/4}|d|^{n(n-1-2r)/2}\frac{\varrho_p\varrho_q\varrho_{r-2n}}{\varrho_{p-n}\varrho_{q-n}\varrho_{r-n}}\prod_p\alpha_p(S,T)\int\limits_{F(A\Gamma(S)A^{-1})}dv\,.$$

But from (93), we know that

$$\int\limits_{F(A\Gamma(S)A^{-1})}dv=\frac{(\Gamma(S_1):G_A)}{(A\Gamma(S)A^{-1}:G_A)}\int\limits_F dv=\frac{w}{\varrho_p\varrho_q}|\delta(S)|^r\left(\frac{\sqrt{|d|}}{2}\right)^{r(r+1)/2}\mu(S)\,.$$

We are thus finally led to Siegel's main theorem for indefinite hermitian forms, viz.

THEOREM 7. *If S is an integral hermitian matrix of signature (p,q) and T an n-rowed integral hermitian matrix, then for $p+q>2n$, $p\geqslant n$, $q\geqslant n$, we have*

$$M(S,T)=\mu(S)\prod_l\alpha_l(S,T)$$

l running over all the rational primes.

§ 7. Analogue of a theorem of Tartakowsky. In this section, we shall give two applications of formula (86) for $A(S,T)$. The first application will concern an analogue of a well-known theorem of Tartakowsky ([14]) for the case of representation of 2-rowed positive definite integral matrices T by a given m-rowed integral positive definite matrix S with $m>4$. The second application will be to get a 'truly asymptotic' formula (in a sense to be made precise presently) for the number of representations of a positive hermitian form in 2 variables with coefficients in $\mathfrak{O}$ as sum of m (>4) squares of absolute values of linear forms with coefficients in $\mathfrak{O}$. This formula is an analogue of the 'truly asymptotic formula' of Hardy-Ramanujan for the number of representations of a rational integer as sum of squares of integers (see [5]). These two applications carry over to the hermitian case, results obtained earlier in [8] for integral quadratic forms over Γ.

We shall suppose, throughout this section, that $S=S^{(m)}>0$ and $T=T^{(2)}>0$ are integral hermitian matrices and $m>4$. In formula (86) for $A(S,T)$, although it is true that the power of $|T|$ occurring in the error term is strictly less than the power of $|T|$ explicitly occurring in the principal term, we can not say that the formula is "truly asymptotic", i.e. as $|T|$ tends to infinity the order of the error term in $|T|$ is strictly less than that of the principal term. For, it could happen, for example, that the infinite product $\prod_p\alpha_p(S,T)$ tends to zero like a negative power of $|T|$ as $|T|$ tends to infinity and diminishes the order of the principal term. But, if we know that $\prod_p\alpha_p(S,T)$ is bounded away from zero as $|T|$ tends to infinity, then the formula will be truly asymptotic.

With the help of the lemmas to follow, we will see that for integral $S=S^{(m)}>0$, $T=T^{(2)}>0$ and $m>4$, we have either $\prod_p\alpha_p(S,T)=0$

or $\prod_p \alpha_p(S, T) \geqslant \varrho > 0$ for a constant ϱ independent of T, when T tends to infinity under the condition "$\min T \geqslant c|T|^{1/2}$".

Let, for a rational prime p, a rational integer ϱ with $p \nmid \varrho$ and for $n \geqslant 1$,

$$G(\varrho; p^n) = \sum_{a \bmod (p^n)} e^{2\pi i \varrho a \bar{a}/p^n}$$

where a runs through a complete set of representatives of $\mathfrak{O}$ modulo (p^n). We then have

LEMMA 8.

$$|G(\varrho; p^n)| \leqslant \begin{cases} p_n, & (d/p) = \pm 1,\ p \text{ odd or even}, \\ p^{n+1/2}, & (d/p) = 0,\ p \text{ odd}, \\ p^{n+1}, & (d/p) = 0,\ \text{for } p = 2,\ 4\|d, \\ p^{n+3/2}, & (d/p) = 0,\ \text{for } p = 2,\ 8\|d. \end{cases}$$

(By (d/p), we mean the Legendre-Jacobi-Kronecker symbol.)

The proof is given as usual, by multiplying $G(\varrho; p^n)$ and $\overline{G(\varrho; p^n)}$, writing $a = \beta p^{n-1}+\gamma$ with β running modulo (p) and γ modulo (p^{n-1}) and successive reduction of n. One uses repeatedly the formula $\sum_{\mathfrak{O} \ni \beta \bmod \mathfrak{b}} e^{2\pi i Tr(\gamma a \beta)} = 0$ or $N(\mathfrak{b})$ according as $\mathfrak{b} \nmid \mathfrak{a}$ or $\mathfrak{b} | \mathfrak{a}$, where $(\gamma) = \mathfrak{a}\big(\mathfrak{b}(\sqrt{d})\big)^{-1}$ and $(\mathfrak{a}, \mathfrak{b}) = \mathfrak{O}$.

Remark. It is useful to note that $G(\varrho; 4) = 0$, $G(\varrho; 2) = 0$ for ϱ odd and $8\|d$ and $G(\varrho; 2) = 0$ for ϱ odd and $4\|d$.

LEMMA 9. *Let S be an m-rowed non-singular integral hermitian matrix with $m \geqslant 5$ and p a rational prime not dividing $|d||S|$. Then, for every 2-rowed integral hermitian T, the p-adic density $\alpha_p(S, T)$ satisfies*

$$\alpha_p(S, T) > \begin{cases} 1-p^{-9/8}, & p \text{ odd}, \\ \tfrac{5}{9}, & p = 2. \end{cases}$$

Proof. We shall sketch the proof for odd p with $(d/p) = +1$. The proof in the case $p = 2$, $(d/2) = +1$ is similar. For the case $(d/p) = -1$, the proof is much simpler and follows the same pattern as in [8], Lemma 3. We could suppose that S is diagonal with diagonal elements $a_1, \ldots, a_m$ $(p \nmid a_i)$ in view of Hilfssatz 9 of [1]. Now, for a large positive integer λ, we know from [1] that

$$p^{4\lambda(m-1)} \alpha_p(S, T) = A_{p^\lambda}(S, T)$$

i.e.

$$(114)\quad p^{4\lambda m} \alpha_p(S, T) = \sum_{\substack{C = (C_1 C_2) \bmod (p^\lambda) \\ \varrho, \varkappa, \tau \bmod (p^\lambda)}} \eta\left(p^{-\lambda}(\tilde{C} S C - T)\begin{pmatrix} \varrho & -\bar{\varkappa}/\sqrt{d} \\ \varkappa/\sqrt{d} & \tau \end{pmatrix}\right)$$

where $\underline{C}_1$, $\underline{C}_2$ run independently through a full system of m-rowed integral columns incongruent modulo (p^λ), ϱ, τ through a complete set of rational integers incongruent modulo p^λ and $\varkappa$ through a complete set of representatives of $\mathfrak{O}$ modulo (p^λ). For $B = \begin{pmatrix} \varrho & -\bar{\varkappa}/\sqrt{d} \\ \varkappa/\sqrt{d} & \tau \end{pmatrix}$, denoting the sum $\sum_{C \bmod (p^\nu)} \eta(p^{-\nu}\tilde{C} S C B)$ by $G^{(\nu)}(B)$, we observe that $G^{(\nu)}(p^g B) = p^{4mg} G^{(\nu-g)}(B)$. We now split up, in (114), the residue system $(\varrho, \varkappa, \tau) \bmod (p^\lambda)$ as $p^g(\varrho_1, \varkappa_1, \tau_1)$ with $\varrho_1, \varkappa_1, \tau_1$ running through a similar residue system modulo $(p^{\lambda-g})$ with the additional restriction that $p \nmid (\varrho_1, \varkappa_1, \tau_1)$, and further, with g running from 0 to λ. We may thus rewrite (114) as

$$\alpha_p(S, T) - 1 = \sum_{\beta=1}^{r} p^{-4m\beta} \sum_{\varrho, \tau \bmod p^\beta,\ \varkappa \bmod (p^\beta)} G^{(\beta)}\left(\begin{pmatrix} \varrho & -\bar{\varkappa}/\sqrt{d} \\ \varkappa/\sqrt{d} & \tau \end{pmatrix}\right) \times$$
$$\times\, \eta\left(-p^{-\beta} T \begin{pmatrix} \varrho & -\bar{\varkappa}/\sqrt{d} \\ \varkappa/\sqrt{d} & \tau \end{pmatrix}\right) = \sum_{\beta=1}^{\lambda} p^{-4m\beta} \sum_{i=1}^{6} C_\beta^{(i)}$$

where

$$C_\beta^{(i)} = \sum_{\varrho, \tau \bmod p^\beta,\ \varkappa \bmod (p^\beta)}{}' G^{(\beta)}\left(\begin{pmatrix} \varrho & -\bar{\varkappa}/\sqrt{d} \\ \varkappa/\sqrt{d} & \tau \end{pmatrix}\right) \eta\left(-p^{-\beta} T \begin{pmatrix} \varrho & -\bar{\varkappa}/\sqrt{d} \\ \varkappa/\sqrt{d} & \tau \end{pmatrix}\right),$$

the accent over $\sum$ indicating that ϱ, τ run over rational integers modulo p^β and $\varkappa$ over representatives of $\mathfrak{O}$ modulo (p^β), satisfying the additional conditions:

$i = 1$: $p \nmid \varrho$, $\varkappa$, τ arbitrary,
$i = 2$: $p | \varrho$, $p \nmid \tau$, $\varkappa$ arbitrary,
$i = 3$: $p | \varrho$, $p | \tau$, $(\varkappa, p) = \mathfrak{O}$,
$i = 4$: $p | \varrho$, $p | \tau$, $(\varkappa, p^\beta) = \mathfrak{p}$ or $\bar{\mathfrak{p}}$ (where $(p) = \mathfrak{p}\bar{\mathfrak{p}}$),
$i = 5$: $p | \varrho$, $p | \tau$, $(\varkappa, p^\beta) = \mathfrak{p}^a$ or $\bar{\mathfrak{p}}^b$, $2 \leqslant a$, $b \leqslant \beta-1$,
$i = 6$: $p | \varrho$, $p | \tau$, $(\varkappa, p^\beta) = \mathfrak{p}^\beta$ or $\bar{\mathfrak{p}}^\beta$, $\beta \geqslant 2$.

By definition, we have $C_1^{(5)} = 0$, $C_2^{(5)} = 0$, $C_1^{(6)} = 0$. Let

$$I_k = \sum_{\beta=1}^{\lambda} p^{-4m\beta} C_\beta^{(k)}, \quad 1 \leqslant k \leqslant 6.$$

We now use the estimates given by Lemma 8, as also the following easily obtained estimates (for p not dividing $|d|\,|S|$), viz.

$$\left|\sum_{C_1, C_2 \bmod (p^n)} \exp\left(\frac{2\pi i}{p^n}\left(\varrho \underline{\tilde{C}}_1 S \underline{C}_1 + Tr\left(\frac{\varkappa}{\sqrt{d}} \underline{\tilde{C}}_1 S \underline{C}_2\right) + \tau \underline{\tilde{C}}_2 S \underline{C}_2\right)\right)\right|$$
$$\leqslant \begin{cases} p^{2nm} & \text{for } (d/p) = +1,\ p \nmid \varkappa\bar{\varkappa},\ p|\varrho, p|\tau, \\ p^{(2n+1)m} & \text{for } (d/p) = +1,\ p|\varrho, p|\tau,\ \mathfrak{p} \text{ or } \bar{\mathfrak{p}} \| \varkappa. \end{cases}$$

Proceeding now as in Lemma 3 of [8], we can prove that

$$|I_1| \leqslant \sum_{\beta=1}^{\lambda} p^{-4m\beta}\left[p^{\beta(2m+4)}\frac{(p-1)^2}{p^2}\cdot\frac{p^{\beta(m-1)}-1}{p^{m-1}-1}+p^{\beta(3m+3)}\frac{(p-1)}{p}\right] \leqslant p^{-15/8},$$

$$|I_2| \leqslant \frac{2(p-1)^2}{p^m}+2\sum_{\beta=2}^{\lambda} p^{-4m\beta}p^{(2m+4)\beta-5}(p-1)^4 p^{m-1}\frac{p^{(\beta-1)(m-1)}-1}{p^{m-1}-1}+$$
$$+\frac{2(p-1)^3}{p^4}\sum_{\beta=2}^{\lambda} p^{-(m-3)\beta} < p^{-2},$$

$$|I_3| \leqslant (p-1)^2 p^{-4}\sum_{\beta=1}^{\lambda} p^{-4m\beta}p^{\beta(2m+4)} < \tfrac{729}{728}p^{-(2m-2)},$$

$$|I_4| \leqslant \frac{2(p-1)}{p^m}+2\sum_{\beta=2}^{\lambda} p^{-4m\beta}p^{\beta(2m+4)+m}(p-1)^2 p^{-5} < p^{-(m-2)},$$

$$|I_5| \leqslant 2p^{-(5m-2)}\big[p^2(p-1)^3\{p^{2m}+p^m(p-1)\}+p^{3m+9+2m-2}(p-1)^2 p^{-(4m+4)}\big]+$$
$$+2\sum_{\beta=4}^{\lambda} p^{-3m\beta+4m+2}\sum_{a=2}^{\beta-1}\big[p^{(2m+3)\beta-4m-5-a}(p-1)^3(a-1)(p^m-1)p^{-1}(p^{m-1}-1)^{-1}-$$
$$-p^{(m+4)\beta-2m-a-8}(p-1)^4(p^{(m-1)(a-1)}-1)(p^{m-1}-1)^{-2}+$$
$$+p^{(m+4)\beta}(p-1)^2 p^{(m-2)a}p^{-(4m+5)}\big] < 2p^5(p-1)^2 p^{-3m}+$$
$$+\tfrac{9}{4}p^{-3(m-3)-1/2}(p^{m-2}-1)^{-1}+\tfrac{27}{13}p^{-4(m-2)}(p^{m-1}-p)^{-1} < p^{-(3m-8)},$$

$$|I_6| \leqslant (2p-2)p^{-2(m-3)-3}+\sum_{\beta=3}^{\lambda} p^{-4m\beta}|C_\beta^{(6)}|$$
$$\leqslant (2p-2)p^{-2(m-3)-3}+2p^{-(5m-4)}(p-1)(p^{2m+1}+p^{2m}+p^{m+1}-p^m)+$$
$$+\sum_{\beta=4}^{\lambda} p^{-4m\beta}|C_\beta^{(6)}|$$
$$\leqslant (2p-2)p^{-2(m-3)-3}+2p^{-(5m-4)}(p-1)(p^{2m+1}+p^{2m}+p^{m+1}-p^m)+$$
$$+\sum_{\beta=4}^{\lambda} 2p^{-(3m-1)\beta+4m+1}(p-1)\Big[\sum_{\gamma=0}^{\beta-3} p^{(2m+1)\gamma}p^{\beta-\gamma-3}(p-1)\times$$
$$\times\Big\{p^{2m(\beta-2-\gamma)}\,p^{(m+1)(\beta-\gamma-3)}\sum_{k=0}^{\beta-\gamma-3} p^{m(k+1)-k}(p-1)\Big\}+p^{(2m+1)(\beta-2)}\Big]$$
$$\leqslant p^{-(2m-9/2)}+p^{-(3m-7)}+p^{-(4m-21/2)} < p^{-(2m-5)}.$$

It is now easy to conclude that for odd p with $(d/p) = +1$, $p \nmid |d|\,|S|$, and $m \geqslant 5$, we have

$$a_p(S, T) > 1-p^{-9/8}.$$

In a similar way, we can also prove

LEMMA 10. *For $p|d$ and $m \geqslant 5$, we have for any 2-rowed integral hermitian matrix T,*

$$a_p(E^{(m)}, T) > \begin{cases} 1/6, & \text{for } p \text{ odd},\\ 1/5, & \text{for } p = 2,\ 8\|d,\\ 17/26, & \text{for } p = 2,\ 4\|d. \end{cases}$$

LEMMA 11. *If S is an m-rowed non-singular integral hermitian matrix, T a 2-rowed integral hermitian matrix and if S represents T primitively modulo p^{2f+1} where $f = \lambda$ for odd p and $f = \lambda+1$ for $p = 2|d$ with p^λ being the highest power of p dividing $|S|$, then, for $m \geqslant 5$, we have*

$$a_p(S, T) \geqslant p^{-(2f+1)(4m-4)}. \tag{115}$$

Proof. The proof may be carried out in exactly the same way as in Hilfssatz 11 of [1] provided one notes the following facts implicitly contained in Lemma 37 of [11]. Let $\mu \geqslant 2f+1$ and $S[C_1] \equiv T\ (\mathrm{mod}\,(p^\mu))$ with C_1 primitive modulo (p^μ). One compares $A(C_1;\mu)$, the number of modulo (p^μ) incongruent integral solutions of the form $C_1+p^{\mu-f}X$ satisfying $S[C_1+p^{\mu-f}X] \equiv T\ (\mathrm{mod}\,(p^\mu))$ with $A(C_1;\mu+1)$ the number of primitive C incongruent $(\mathrm{modulo}\ (p^{\mu+1}))$ satisfying $S[C] \equiv T\ (\mathrm{mod}\,(p^{\mu+1}))$ and $C \equiv C_1\ (\mathrm{mod}\,(p^{\mu-f}))$. For this purpose, one finds unimodular matrices U_1, U_2 for which $U_1SC_1U_2 \equiv \begin{pmatrix} D^{(2)} \\ 0 \end{pmatrix}\ (\mathrm{mod}\,(p^{\mu+1}))$ with diagonal D. Now one observes here that in view of C_1 being primitive, $(\delta(SC_1), (p^{\mu+1}))$ divides $(\delta(S), (p^{\mu+1}))$ and therefore $(|D|, (p^{\mu+1})) = (\delta(SC_1), (p^{\mu+1}))$ divides (p^λ). The rest of the proof goes through exactly as in [1] and one has

$$A(C_1;\mu+1) = p^{4m-4}A(C_1;\mu)$$

leading finally to (115).

Remark. In [8], while proving a similar lemma (Lemma 5, p.470) for a quadratic form over the rational number field, we had imposed on it the restriction that it be a diagonal form in the ring of 2-adic integers over the rational number field. If we adopt the method of Lemma 11 above, it is clear that the said condition is unnecessary even there.

Before we consider representations of T by S modulo (p^μ) which are not primitive, we shall state the following

LEMMA 12. *If S is an integral m-rowed hermitian matrix and if S represents a primitively modulo (p^μ) (with $\mu \geqslant 1$), then S represents $T^{(2)} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ primitively modulo (p^μ), provided that $m \geqslant 5$.*

The proof goes through word for word as in Lemma 6 of [8], except that we have to fill a lacuna by demonstrating the validity of the following

THEOREM 8. *Any m-rowed non-singular integral hermitian matrix represents 0 primitively modulo (p^μ) for every prime p and $\mu \geqslant 1$, provided that $m \geqslant 3$.*

Proof. It suffices clearly to prove the assertion of the theorem for $m = 3$. If $S = S^{(3)}$ is indefinite hermitian, one can show that S represents 0 non-trivially in k, by going over to the 6-rowed rational symmetric indefinite matrix $A = \frac{1}{2}\begin{pmatrix} E^{(3)} & E \\ \overline{\omega}E & \omega E\end{pmatrix}\begin{pmatrix} 0 & S \\ S' & 0\end{pmatrix}\begin{pmatrix} E & \overline{\omega}E \\ E & \omega E\end{pmatrix}$. Let then $S[X] = 0$ with $X \neq 0$ in $\{\mathfrak{O}\}_{3,1}$. By multiplying X by a suitable number in k, it is possible to find $Y \in \{\mathfrak{O}\}_{3,1}$ with the greatest common divisor of elements of Y coprime to p such that $S[Y] = 0$. Now $S[Y] \equiv 0 \left(\operatorname{mod}(p^\mu)\right)$ $(\mu \geqslant 1)$ gives the required primitive representation of 0 modulo p^μ.

Let $S^{(3)}$ be not indefinite. If at least two of its characteristic roots are different, then taking aS (with large enough integral $a > 0$, $p \nmid a$) instead of S, we can find a rational integer λ such that $S_1 = aS - \lambda p^\mu E^{(3)}$ is indefinite. By the foregoing, S_1 represents 0 primitively modulo (p^μ) and since $aS \equiv S_1 \left(\operatorname{mod}(p^\mu)\right)$, we see that aS and therefore S represents 0 primitively modulo (p^μ).

Finally, let $S^{(3)}$ be definite (in fact, positive-definite, without loss of generality) and let all its characteristic roots be equal. Then $S = bE^{(3)}$ with rational integral b. To complete the proof of the theorem, it suffices to show that $E^{(3)}$ represents 0 primitively modulo (p_μ), for $\mu \geqslant 1$. First, let $(d/p) = +1$ and $(p) = \mathfrak{p}\overline{\mathfrak{p}}$. Let $\pi \in \mathfrak{O}$ divisible just by $\mathfrak{p}$ and not by $\overline{\mathfrak{p}}$. Then $E^{(3)}\left[\begin{pmatrix}\pi^\mu \\ \overline{\pi}^\mu \\ 0\end{pmatrix}\right] \equiv 0 \left(\operatorname{mod}(p^\mu)\right)$ gives a primitive representation of 0 modulo (p^μ). Now, let $(d/p) = -1$ or 0. In the latter case, let $(p) = \mathfrak{p}^2$ and $\pi \in \mathfrak{O}$ divisible by $\mathfrak{p}$ but not by $\mathfrak{p}^2$. Now we know that the rational symmetric matrix $A = \frac{1}{2}\begin{pmatrix} E^{(3)} & E \\ \overline{\omega}E & \omega E\end{pmatrix}\begin{pmatrix} 0 & E^{(3)} \\ E^{(3)} & 0\end{pmatrix}\begin{pmatrix} E & \overline{\omega}E^{(3)} \\ E & \omega E\end{pmatrix}$ represents 0 non-trivially in the ring of p-adic integers over the rational number field. Therefore $E^{(3)}$ represents 0 non-trivially in the ring R_p of p-adic integers over k (every element of R_p is of the form $a + \omega\beta$ where a and β are p-adic integers over the rational number field). Let then $E^{(3)}[X] = 0$ with elements of X in R_p. Since $X \neq 0$, we can write $X = p^\nu X_1$ or $\pi^{\nu'} X_1$ with primitive X_1 according as $(d/p) = -1$ or 0. In any case $E[X_1] = 0$ with X_1 primitive. Hence $E^{(3)}$ represents 0 primitively modulo (p^μ) for every $\mu \geqslant 1$. Our theorem is therefore completely proved.

We prove finally

LEMMA 13. *If S is a non-singular m-rowed integral hermitian matrix and T is a 2-rowed integral hermitian matrix and if S represents T modulo (p^ν) where $\nu = 6\lambda+7$, $6\lambda+7$ or $6\lambda+8$ according as $(d/p) = +1, -1$, or 0 and $p^\lambda \| |S|$, then $a_p(S, T) \geqslant p^{-\nu(4m-4)}$, provided that $m \geqslant 5$.*

Proof. Let $S[G] \equiv T \left(\operatorname{mod}(p^\nu)\right)$. If G is primitive modulo (p^ν) or if $T \equiv 0 \left(\operatorname{mod}(p^\nu)\right)$, we see that the lemma is true, in view of Lemmas 11 and 12 and Theorem 8. Let us therefore assume that G is not primitive modulo (p^ν) and $T \not\equiv 0 \left(\operatorname{mod}(p^\nu)\right)$.

Let $(d/p) = +1$, $(p) = \mathfrak{p}\overline{\mathfrak{p}}$, $\mathfrak{p} \neq \overline{\mathfrak{p}}$, $(\pi) = \mathfrak{p}\mathfrak{q}$ for $\pi \in k$ and $\left((p), \mathfrak{q}\right) = \mathfrak{O}$. There exist unimodular matrices U, V such that

$$UGV \equiv \begin{pmatrix} a\pi^{a_1}\overline{\pi}^{a_2} & 0 \\ 0 & b\pi^{\beta_1}\overline{\pi}^{\beta_2} \\ & 0 \end{pmatrix} \left(\operatorname{mod}(p^\nu)\right) \quad \text{with} \quad (a, p) = \mathfrak{O} = (b, p)$$

and at least one of $a_1, a_2, \beta_1, \beta_2$ is different from zero. By changing from T to $T[V]$ (which certainly does not affect $a_p(S, T)$) we may assume therefore that $G = C\begin{pmatrix}\pi^{a_1}\overline{\pi}^{a_2} & 0 \\ 0 & \pi^{\beta_1}\overline{\pi}^{\beta_2}\end{pmatrix} \left(\operatorname{mod}(p^\nu)\right)$ with primitive $C^{(m,2)}$ modulo (p^ν).

We have now to consider various cases.

Case 1. $a_1, a_2, \beta_1, \beta_2 \leqslant 2\lambda+2$. Setting $T_1 = T\left[\begin{pmatrix}\pi^{-a_1}\overline{\pi}^{-a_2} & 0 \\ 0 & \pi^{-\beta_1}\overline{\pi}^{-\beta_2}\end{pmatrix}\right]$ we have a primitive representation of T_1 by S viz. $S[C] \equiv T_1 \pmod{p^{2\lambda+3}}$. Applying Lemma 11, we see that there are at least $p^{r(4m-4)}$ primitive representations C_1 of T_1 by S incongruent modulo $(p^{2\lambda+3+r})$. Corresponding to each such C_1, $C = C_1\begin{pmatrix}\pi^{a_1}\overline{\pi}^{a_2} & 0 \\ 0 & \pi^{\beta_1}\overline{\pi}^{\beta_2}\end{pmatrix}$ provides a representation of T by S modulo $(p^{2\lambda+3+r})$. If $r > 4\lambda+4$, at least $p^{(r-4\lambda-4)(4m-4)}$ of these C's are incongruent modulo $(p^{2\lambda+3+r})$. Hence it follows that $a_p(S, T) \geqslant p^{-(6\lambda+7)(4m-4)}$.

Case 2. If $a_2, \beta_2 \geqslant 2\lambda+3$, then $T \equiv 0 \left(\operatorname{mod}(p^{2\lambda+3})\right)$. By using Lemmas 11 and 12 and Theorem 8, we see that $a_p(S, T) \geqslant p^{-(2\lambda+3)(4m-4)}$.

Case 3. Not both of a_2, β_2 are $\leqslant 2\lambda+2$ i.e. without loss of generality, we may assume $a_2 \leqslant 2\lambda+2$, $\beta_2 \geqslant 2\lambda+3$. Further let $\beta_1 \geqslant 2\lambda+3$, $a_1 \leqslant 2\lambda+2$. Then, if we set $T_2 = T\left[\begin{pmatrix}\pi^{-a_1}\overline{\pi}^{-a_2} & 0 \\ 0 & 1\end{pmatrix}\right] \equiv \begin{pmatrix} t_{11} & 0 \\ 0 & 0\end{pmatrix} \left(\operatorname{mod}(p^{2\lambda+3})\right)$, we have $S\left[C\begin{pmatrix}1 & 0 \\ 0 & \pi^{\beta_1}\overline{\pi}^{\beta_2}\end{pmatrix}\right] \equiv T_2 \left(\operatorname{mod}(p^{2\lambda+3})\right)$. Hence S represents t_{11} primitively modulo $(p^{2\lambda+3})$ and by Lemma 12, also $\begin{pmatrix} t_{11} & 0 \\ 0 & 0\end{pmatrix} \equiv T_2$ primitively modulo $(p^{2\lambda+3})$. Hence, by the same arguments as above, for $r > 4\lambda+4$, there are at least $p^{(r-4\lambda-4)(4m-4)}$ incongruent modulo $(p^{2\lambda+3+r})$ representations of T by S. Thus we obtain $a_p\,(S, T) \geqslant p^{-(6\lambda+7)(4m-4)}$.

Case 4. Let $a_2 \leqslant 2\lambda+2$, $\beta_2 \geqslant 2\lambda+3$, $a_1 \geqslant 2\lambda+3$, $\beta_1 \geqslant 2\lambda+3$. We are now again in the old situation $T \equiv 0 \left(\operatorname{mod}(p^{2\lambda+3})\right)$ and therefore $a_p(S, T) \geqslant p^{-(2\lambda+3)(4m-4)}$.

Case 5. Let $\alpha_2 \leqslant 2\lambda+2$, $\beta_2 \geqslant 2\lambda+3$, $\alpha_1 \geqslant 2\lambda+3$, $\beta_1 < 2\lambda+3$. Let $T_3 = T\left[\begin{pmatrix}\overline{\pi}^{-\alpha_2} & 0\\ 0 & \pi^{-\beta_1}\end{pmatrix}\right]$; then $S\left[C\begin{pmatrix}\pi^{\alpha_1} & 0\\ 0 & \overline{\pi}^{\beta_2}\end{pmatrix}\right] \equiv T_3 \,(\bmod\,(p^{4\lambda+5}))$. Observe that T_3 is not necessarily congruent to 0 modulo $(p^{2\lambda+3})$. In fact, $T_3 \equiv \begin{pmatrix}0 & \overline{\pi}^\mu a\\ \pi^\mu\overline{a} & 0\end{pmatrix}$ $(\bmod\,(p^\mu))$ with $\mu = 2\lambda+3$, for some $a \in \mathfrak{O}$ of the form $\pi^a\cdot b$ with $a \geqslant 0$ and $b\in\mathfrak{O}$ and $(b,p)=\mathfrak{O}$. Consider $T_4 = T_3\left[\begin{pmatrix}1&0\\1&1\end{pmatrix}\right] = \begin{pmatrix}t_{11}^* & *\\ * & *\end{pmatrix}$ equivalent to T_3. Then $t_{11}^* = \pi^\mu\overline{a} + \overline{\pi}^\mu a$. Further, if $C=(C_1 C_2)$, then $C\begin{pmatrix}\pi^{\alpha_1}&0\\0&\overline{\pi}^{\beta_2}\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix} = (C_1^* \; \overline{\pi}^{\beta_2}C_2)$ where $C_1^* = \pi^{\alpha_1}C_1 + \overline{\pi}^{\beta_2}C_2$ is of the form $r\pi^\mu C_1 + s\overline{\pi}^\mu C_2$ with $r,s\in\mathfrak{O}$ coprime to p. It is clear that C_1^* is primitive modulo (p), for if $\mathfrak{p}$ divides all elements of C_1^*, then $\mathfrak{p}$ must divide all elements of C_2, which is not true, since C_2 is primitive modulo (p). Thus $\mathfrak{p}$ and similarly $\overline{\mathfrak{p}}$, cannot divide all elements of C_1^*. Now $T_4 = \begin{pmatrix}t_{11}^* & \overline{\pi}^\mu a\\ \pi^\mu\overline{a} & 0\end{pmatrix}$. We can find $y\in\mathfrak{O}$ such that $t_{11}^*\overline{\pi}^\mu y \equiv \overline{\pi}^\mu a \,(\bmod\,(p^\mu))$; in fact, y may be taken to be a solution of $\overline{\pi}^{2\mu}y \equiv \overline{\pi}^\mu \,(\bmod\,(p^\mu))$. Consider now $T_5 = T_4\left[\begin{pmatrix}1 & -\overline{\pi}^\mu y\\ 0 & 1\end{pmatrix}\right]$ $\equiv\begin{pmatrix}t_{11}^*&0\\0&0\end{pmatrix}\,(\bmod\,(p^\mu))$. We know that S represents t_{11}^* primitively modulo (p^μ) and hence, by Lemma 12, also T_5 primitively modulo (p^μ). Since T_5 is equivalent to T_3 modulo (p^μ), we have at least $p^{r(4m-4)}$ representations of T_3 by S, incongruent modulo $(p^{\mu+r})$. It is now easy to deduce that $\alpha_p(S,T) \geqslant p^{-(4\lambda+5)(4m-4)}$.

Case 6. Let $\alpha_2\leqslant 2\lambda+2$, $\beta_2 \geqslant 2\lambda+3$, $\alpha_1\leqslant 2\lambda+2$, $\beta_1\leqslant 2\lambda+2$. Now we have $S\left[C\begin{pmatrix}1&0\\0&\overline{\pi}^{\beta_2}\end{pmatrix}\right] \equiv T_6\left(=T\left[\begin{pmatrix}\pi^{-\alpha_1}\overline{\pi}^{-\alpha_2} & 0\\ 0 & \pi^{-\beta_1}\end{pmatrix}\right]\right) (\bmod\,(p^{2\lambda+3}))$. Then T_6 $\equiv \begin{pmatrix}t_{11} & \overline{\pi}^\mu a\\ \pi^\mu\overline{a} & 0\end{pmatrix}(\bmod\,(p^\mu))$ with $\mu=2\lambda+3$. If $p\nmid t_{11}^* = t_{11}+\overline{\pi}^\mu a + \pi^\mu \overline{a}$, then $T_7 = T_6\left[\begin{pmatrix}1&0\\1\cdot 1\end{pmatrix}\begin{pmatrix}1 & -t^*\overline{\pi}^\mu a\\0&1\end{pmatrix}\right] \equiv \begin{pmatrix}t_{11}^*&0\\0&0\end{pmatrix}(\bmod\,(p^\mu))$ where $t^* t_{11}^{**} \equiv 1 \,(\bmod\,p^\mu)$. Now $S\left[C\begin{pmatrix}1\\ \overline{\pi}^{\beta_2}\end{pmatrix}\right]\equiv t_{11}^*\,(\bmod\,(p^\mu))$ and since $p\nmid t_{11}^*$, this is a primitive representation modulo (p^μ). By Lemma 12, S represents T_7 and hence T_6, primitively modulo (p^μ). We may now easily conclude that $\alpha_p(S,T)$ $\geqslant p^{-(6\lambda+7)(4m-4)}$.

Suppose now $\mathfrak{p}^k \| t_{11}^*$ $(k>0)$ and $\mathfrak{p}^l\|t_{11}$. We may assume that $k<\mu$, $l<\mu$ (for, if $l=\mu$, then T_6 is itself of the form $\begin{pmatrix}0&\overline{\pi}^\mu a\\ \pi^\mu\overline{a}&0\end{pmatrix}$ modulo (p^μ)

and if $k=\mu$, then $T_6\left[\begin{pmatrix}1&0\\1&1\end{pmatrix}\right]$ is of the same form modulo (p^μ) and therefore, we are in the same situation as in case 5 above and we could conclude that $\alpha_p(S,T)\geqslant p^{-(6\lambda+7)(4m-4)}$). Now, since $\mathfrak{p}^k\|t_{11}^*$ we have $t_{11}\equiv -\overline{\pi}^\mu a \,(\bmod\,\mathfrak{p}^k)$. Thus $((a),\mathfrak{p}^k) = ((t_{11}),\mathfrak{p}^k)$. If $l\leqslant k$, then we can consider $T_8 = T_6\left[\begin{pmatrix}1 & -\overline{\pi}^\mu y\\0&1\end{pmatrix}\right]$ where y satisfies $t_{11}y\equiv a\,(\bmod\,(p^\mu))$. Then $T_8\equiv\begin{pmatrix}t_{11}&0\\0&0\end{pmatrix}(\bmod\,(p^\mu))$. Since S represents t_{11} primitively modulo (p^μ), S represents T_8 and hence T_6 primitively modulo (p^μ). We may now conclude, as before, that $\alpha_p(S,T)$ $\geqslant p^{-(6\lambda+7)(4m-4)}$. If $l>k$, then we take $T_9 = T_6\left[\begin{pmatrix}1&0\\1&1\end{pmatrix}\right] = \begin{pmatrix}t_{11}^* & \overline{\pi}^\mu a\\ \pi^\mu\overline{a} & 0\end{pmatrix}$. We claim that $((t_{11}^*),\mathfrak{p}^\mu) = \mathfrak{p}^k$ is the same as $((a),\mathfrak{p}^\mu)$. For, if $\mathfrak{p}^{k+1}|a$, then, since $l>k_2$ and $\mathfrak{p}^{k+1}|t_{11}$, we should necessarily have $\mathfrak{p}^{k+1}|t_{11}^*$, which is a contradiction. Thus $((t_{11}^*),\mathfrak{p}^\mu) = ((a),\mathfrak{p}^\mu)$ and therefore, we can find $y\in\mathfrak{O}$ such that $t_{11}^*y\equiv a\,(\bmod\,\mathfrak{p}^\mu)$ i.e. $\overline{\pi}^\mu t_{11}^* y \equiv \overline{\pi}^\mu a\,(\bmod\,(p^\mu))$. Now taking $T_{10} = T_9\left[\begin{pmatrix}1 & -\overline{\pi}^\mu y\\0&1\end{pmatrix}\right]$, we have $T_{10}\equiv\begin{pmatrix}t_{11}^*&0\\0&0\end{pmatrix}(\bmod\,(p^\mu))$. Consider the column $C_1^* = C_1+\overline{\pi}^{\beta_2}C_2$, where $C=(C_1C_2)$. It is clear that $\overline{\mathfrak{p}}$ cannot divide all the elements of C_1^* for otherwise, $\overline{\mathfrak{p}}$ will have to divide all the elements of the primitive column C_1. We observe now that $\mathfrak{p}$ too cannot divide all the elements of C_1^*. For, if we set $G_1 = C\begin{pmatrix}1&0\\0&\overline{\pi}^{\beta_2}\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}$ $= (C_1^*\;\overline{\pi}^{\beta_2}C_2)$, then, from the form of G_1, we see that $(\delta(G_1),(p^\mu)) = \overline{\mathfrak{p}}^\mu$, whereas, if $\mathfrak{p}|C_1^*$, then we would be having $\mathfrak{p}\overline{\mathfrak{p}}^\mu$ dividing $(\delta(G_1),(p^\mu))$, from the form $(C_1^*\;\overline{\pi}^{\beta_2}C_2)$ of G_1. Thus C_1^* gives a primitive representation of t_{11}^* by S modulo (p^μ). Hence S represents T_{10} and therefore T_6 primitively modulo (p^μ). We can conclude once again that $\alpha_p(S,T)\geqslant p^{-(6\lambda+7)(4m-4)}$, as before.

Case 7. $\alpha_2,\beta_2\leqslant 2\lambda+2$. First we have $S\left[C\begin{pmatrix}\pi^{\alpha_1}&0\\0&\pi^{\beta_1}\end{pmatrix}\right]\equiv T_{11}\,(\bmod\,(p^{4\lambda+5}))$ where $T_{11} = T\left[\begin{pmatrix}\overline{\pi}^{-\alpha_2}&0\\0&\overline{\pi}^{-\beta_2}\end{pmatrix}\right]$. We may first suppose that $\alpha_1\leqslant\beta_1$ without loss of generality. If $\alpha_1\geqslant 2\lambda+3$, then $\beta_1\geqslant 2\lambda+3$ too and $t_{11}\equiv 0\,(\bmod\,(p^{2\lambda+3}))$. By an easy argument, we can show that $\alpha_p(S,T)\geqslant p^{-(4\lambda+5)(4m-4)}$. If α_1 $\leqslant 2\lambda+2$, $\beta_1\geqslant 2\lambda+3$, then $S\left[C\begin{pmatrix}1&0\\0&\pi^{\beta_1}\end{pmatrix}\right]\equiv T_{12}\,(\bmod\,p^{2\lambda+3})$ where T_{12} $= T\left[\begin{pmatrix}\pi^{-\alpha_1}\overline{\pi}^{-\alpha_2}&0\\0&\overline{\pi}^{-\beta_2}\end{pmatrix}\right]$. We are now in the same situation as in case 6 above with π instead of $\overline{\pi}$. We could conclude as above that $\alpha_p(S,T)$ $\geqslant p^{-(6\lambda+7)(4m-4)}$. The case $\alpha_1\leqslant 2\lambda+2$, $\beta_1\leqslant2\lambda+2$ may now be ruled out, having been already subsumed under Case 1.

Thus, when $(d/p) = +1$, our lemma is completely proved. If $(d/p) = -1$, then, for unimodular matrices U, V we have $UGV \equiv C\begin{pmatrix} p^{\alpha} & 0 \\ 0 & p^{\beta}\end{pmatrix}$ $\pmod{p^{\nu}}$ with primitive C. Proceeding exactly as in Lemma 7 of [8], we can show that $\alpha_p(S, T) \geqslant p^{-(6\lambda+7)(4m-4)}$. The restriction on S to be in the diagonal form in R_p is seen to be unnecessary, since we have avoided it in the proof of Lemma 11 above.

Finally, let $(d/p) = 0$, $(p) = \mathfrak{p}^2$, $(\pi) = \mathfrak{p}\mathfrak{q}$ with $(\mathfrak{q}, (p)) = \mathfrak{O}$. There exist unimodular U and V such that $UGV \equiv C\begin{pmatrix} \pi^{a_1} & 0 \\ 0 & \pi^{a_2}\end{pmatrix} \pmod{p^{\nu}}$ with primitive C. We might suppose that $a_1 \leqslant a_2$ without loss of generality. We have now to consider three cases.

(i) Let $a_1, a_2 \leqslant 4\lambda+5$. We have then a primitive representation of $T\left[\begin{pmatrix} \pi^{-a_1} & 0 \\ 0 & \pi^{-a_2}\end{pmatrix}\right]$ by S modulo $(p^{2\lambda+3})$ and proceeding as in Lemma 7 of [8], we obtain $\alpha_p(S, T) \geqslant p^{-(6\lambda+8)(4m-4)}$.

(ii) If $a_1 \geqslant 4\lambda+6$, then $T \equiv 0 \pmod{p^{4\lambda+6}}$ and hence S represents T primitively modulo $(p^{2\lambda+3})$ and we may conclude as above that $\alpha_p(S, T) \geqslant p^{-(2\lambda+8)(4m-4)}$.

(iii) If $a_1 \leqslant 4\lambda+5$, $a_2 \geqslant 4\lambda+6$, then $T_1 = T\left[\begin{pmatrix} \pi^{-a_1} & 0 \\ 0 & 1\end{pmatrix}\right] \equiv \begin{pmatrix} t_{11} & 0 \\ 0 & 0\end{pmatrix}$ $\pmod{p^{2\lambda+3}}$. Since S represents t_{11} primitively modulo $(p^{2\lambda+3})$, we see that S represents T_1 primitively modulo $(p^{2\lambda+3})$, in view of Lemma 12. Proceeding as above, we obtain $\alpha_p(S, T) \geqslant p^{-(6\lambda+8)(4m-4)}$. Our lemma is thus completely proved.

We might finally prove the following analogue, for hermitian forms, of a theorem of Tartakowsky's (see [8]).

THEOREM 9. *Let S be an m-rowed positive integral matrix and T a 2-rowed integral hermitian matrix with $\min T \geqslant c|T|^{1/2}$ for a fixed constant $c > 0$. Then, for $m \geqslant 5$, there exists a constant $c_{72} > 0$ depending only on S, k and c such that for $|T| > c_{72}$, S represents T integrally if and only if S represents T modulo $(p^{6\lambda+8})$ for every rational prime p dividing $|d||S|$ with $p^{\lambda}\,\|\,|S|$.*

Proof. If S represents T integrally, then it is trivial to see that S represents T modulo $((p^{6\lambda+8}))$ for $p\,|\,|d||S|$. We shall now prove the sufficiency of the conditions. For $p \nmid |d||S|$, by Lemma 9, we have $\alpha_p(S, T) > 1-p^{-9/8}$ for odd p and $\alpha_2(S, T) > \frac{5}{9}$ for $2 \nmid |d||S|$. For $p\,|\,|d||S|$, we have, by Lemma 13, that $\alpha_p(S, T) \geqslant p^{-(6\lambda+8)(4m-4)}$. Thus there exists a constant $c_{73} > 0$ depending only on $|d|$ and $|S|$ such that the infinite product $\prod_p \alpha_p(S, T) \geqslant c_{73} > 0$ (p running over all rational primes) uniformly for all T satisfying the conditions of the theorem. We now see,

in formula (86) for $A(S, T)$, that the principal term is of a strictly higher order in $|T|$ than the error term. Therefore there exists a constant $c_{72} > 0$ (depending on S, k and c) such that if $|T| > c_{72}$ then the error term is strictly less than the principal term. Thus $A(S, T) \neq 0$ which means that S represents T integrally.

For integral $S^{(m)} > 0$, $T^{(2)} > 0$ and $m \geqslant 5$, we may deduce from Theorem 9 that either $\prod_p \alpha_p(S, T) = 0$ or $\prod_p \alpha_p(S, T) \geqslant c_{73} > 0$. For, if $\prod_p \alpha_p(S, T) \neq 0$, then by the absolute convergence of the infinite product (see [1], Hilfssatz 51), no factor $\alpha_p(S, T)$ can be zero. In particular, S represents T modulo $(p^{6\lambda+8})$ for every rational prime p with $p^{\lambda}\,\|\,|d||S|$ and from the working of the proof of Theorem 9, we conclude $\prod_p \alpha_p(S, T) \geqslant c_{73}$. Now, if $\prod_p \alpha_p(S, T) \neq 0$, then for every S^* in the genus of S ([1]), we have also $\prod_p \alpha_p(S^*, T) \neq 0$. Thus, for $|T| > c_{72}$ and $\min T \geqslant c|T|^{1/2}$, we may conclude that every matrix in the genus of S or none at all represents T integrally and under these conditions on T, the matrices T which are representable by S are precisely those belonging to certain congruence classes modulo $((|d||S|)^{14})$ (say), which are completely determined by S.

Let us now take $S = E^{(m)}$, $m \geqslant 5$ and $n = 2$ in (86). From Lemma 9, $\prod_{p \nmid |d|} \alpha_p(E^{(m)}, T) \geqslant \prod_{p \nmid |d|} (1-p^{-9/8})$ and from Lemma 10, we have $\prod_{p\,|\,|d|} \alpha_p(E^{(m)}, T) > \frac{1}{5} \prod_{2 \neq p\,|\,|d|} (\frac{1}{6}) > 0$. Thus $\prod_p \alpha_p(E^{(m)}, T)$ is bounded away from zero uniformly in T, for $m \geqslant 5$. We have hence, for $A(E^{(m)}, T)$, a *truly asymptotic formula* as $|T| \to \infty$, viz.

$$A(E^{(m)}, T) = |T|^{m-2} \frac{(2\pi)^{2m-1}|d|^{\frac{3}{2}-m}}{\Gamma(m)\Gamma(m-1)} \prod_p \alpha_p(E^{(m)}, T) + O(|T|^{(3m-4)/4})$$

provided $\min T \geqslant c|T|^{1/2}$ for a fixed constant $c > 0$. This is an interesting analogue of a well-known asymptotic formula of Hardy-Ramanujan [5] for the number of representations of a rational integeras sum of m (> 4) squares of rational integers.

§ 8. Concluding remarks. The main purpose of this section is to remark that the considerations of §§ 3-5 can be generalized to hermitian modular forms of degree n, analogous to the treatment given in [8].

Let $f(Z)$ be a hermitian modular form of degree n, dimension $-r$, stufe s and belonging to a multiplier-system $\{v(M)/M \in \mathfrak{M}_n(s)\}$. With $f(Z)$, we associate for $r > 2n$, the function $\varphi(Z; f)$ defined by

$$\varphi(Z; f) = \sum_{N_i \in \mathfrak{R}} a(0, N_i^{-1})|C_i Z + D_i|^{-r}, \qquad Z \in \mathfrak{H}_n,$$

where $N_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ runs over a complete system of representatives of the right cosets of $\mathfrak{M}_n$ modulo $\mathfrak{A}_n$ and $a(0, N_i^{-1})$ is the constant term in the Fourier expansion $f(Z)/N_i^{-1} = \sum_{T=\tilde{T}\geqslant 0} a(T, N_i^{-1})\eta(s^{-1}TZ)$. In the first place, we see that the definition of $\varphi(Z; f)$ is independent of the choice of the representatives N_i in each coset. For, if $A_0 = \begin{pmatrix} \tilde{U}^{-1} & * \\ 0 & U \end{pmatrix} \in \mathfrak{A}_n$, then writing $f(Z)/N_i^{-1}A_0^{-1}$ as $\big(f(z)/N_i^{-1}\big)/A_0^{-1}$ and comparing Fourier coefficients, we obtain $a(0, N_i^{-1}A_0^{-1}) = |U|^r a(0, N_i^{-1})$ and therefore $a(0, N_i^{-1}A_0^{-1})|U|^{-r}|C_i Z + D_i|^{-r} = a(0, N_i^{-1})|C_i Z + D_i|^{-r}$.

Further for $M \in \mathfrak{M}_n(s)$, writing $f(Z)/MN_i^{-1}$ as $\big(f(Z)/M\big)/N_i^{-1}$ and comparing the Fourier coefficients, we have

$$a(0, MN_i^{-1}) = v(M)a(0, N_i^{-1}). \tag{116}$$

Since $|v(M)| = 1$ and $\big(\mathfrak{M}_n : \mathfrak{M}_n(s)\big) < \infty$, it follows from (116) that $|a(0, N_i^{-1})| \leqslant c_{74}$ for a constant c_{74} depending only on $f(Z)$ in general. This together with the fact that the full Eisenstein series $\sum_{\{C_i, D_i\}} |C_i Z + D_i|^{-r}$ converges absolutely and uniformly for Z belonging to a compact subset of $\mathfrak{H}_n$, for $r > 2n$, shows that $\varphi(Z; f)$ is regular in $\mathfrak{H}_n$. Further, from (116), $\varphi(Z; f) \in \{n, s, -r, v\}$.

We now claim that for $f(Z)$ we can obtain estimates similar to (46) viz. for $N_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \in \mathfrak{M}_n$, $Z \in N_j^{-1}\langle \mathfrak{G}_n \rangle$ and $y_j = \min I(N_j\langle Z \rangle)$,

$$\begin{gathered} |f(Z)| \leqslant c_{75}\|C_j Z + D_j\|^{-r}, \\ \big|f(Z) - a(0, N_j^{-1})|C_j Z + D_j|^{-r}\big| \leqslant c_{76}\|C_j Z + D_j\|^{-r} e^{-c_{77} y_j} \end{gathered} \tag{117}$$

$a(0, N_j^{-1})$ being the constant term in $f(Z)/N_j^{-1}$ and c_{75}, c_{76}, c_{77} are constants depending only on k, n and $f(Z)$ in general and *not* on Z or N_j. For the proof, we may proceed as follows. Since $N_j\langle Z \rangle \in \mathfrak{G}_n$, we know that there exists $A_0 = \begin{pmatrix} \tilde{U} & * \\ 0 & U^{-1} \end{pmatrix} \in \mathfrak{A}_n$ such that $A_0\langle N_j\langle Z \rangle\rangle \in \mathfrak{F}_n$ and further by property b) of $\mathfrak{F}_n$ (see § 2), $I\big(A_0\langle N_j\langle Z \rangle\rangle\big) > \gamma_n E^{(n)}$. On the other hand,

$$f(Z) = |U|^{-r}|C_j Z + D_j|^{-r} \sum_{T=\tilde{T}\geqslant 0} a(T, N_j^{-1}A_0^{-1})\eta\big(s^{-1}TA_0\langle N_j\langle Z \rangle\rangle\big)$$

and

$$\begin{aligned} f(Z) - a(0, N_j^{-1})|C_j Z + D_j|^{-r} &= f(Z) - a(0, N_j^{-1}A_0^{-1})|U|^{-r}|C_j Z + D_j|^{-r} \\ &= \sum_{0\neq T=\tilde{T}\geqslant 0} a(T, N_j^{-1}A_0^{-1})\eta\big(s^{-1}TA_0\langle N_j\langle Z \rangle\rangle\big). \end{aligned}$$

Therefore, we have, for $Z \in N_j^{-1}\langle \mathfrak{G}_n \rangle$,

$$|f(Z)| \leqslant \|C_j Z + D_j\|^{-r} \big|f\big(A_0\langle N_j\langle Z \rangle\rangle\big)/N_j^{-1}A_0^{-1}\big|. \tag{118}$$

Further, since $I\big(A_0\langle N_j\langle Z \rangle\rangle\big)$ is reduced in the sense of Humbert, we know that there exists a constant c_{11} depending only on k and n such that

$$I\big(A_0\langle N_j\langle Z \rangle\rangle\big) \geqslant c_{11} \min I\big(A_0\langle N_j\langle Z \rangle\rangle\big) E^{(n)} = c_{11} y_j E^{(n)}.$$

Hence

$$\begin{aligned} &\big|f(Z) - a(0, N_j^{-1})|C_j Z + D_j|^{-r}\big| \\ &\qquad \leqslant e^{-\pi c_{11} y_j/s} \sum_{0\neq T=\tilde{T}\geqslant 0} \Big|a(T, N_j^{-1}A_0^{-1})\eta\Big(\frac{1}{2s} T \cdot A_0\langle N_j\langle Z \rangle\rangle\Big)\Big|. \end{aligned} \tag{119}$$

Now, in view of Satz 2 of [3] which continues to be valid even under our assumptions on the multiplier-system, we know that $f(Z_1)/M$ is bounded for all $M \in \mathfrak{M}_n$ and $Z_1 \in \mathfrak{H}_n$ with $\operatorname{Im} Z_1 > \frac{1}{2}\gamma_n E^{(n)}$. Since $\big(\mathfrak{M}_n : \mathfrak{M}_n(s)\big) < \infty$, there are only finitely many distinct functions in the set $\{|f(Z_1)/M| \,/\, M \in \mathfrak{M}_n\}$. Thus from (118) and (119), we see that the estimates (117) follow immediately.

Proceeding exactly as in the proof of Theorem 4, we have, similar to (57), estimates for $\varphi(Z; f)$ too, viz. for $Z \in N_j^{-1}\langle \mathfrak{G}_n \rangle$,

$$\begin{gathered} |\varphi(Z; f)| \leqslant c_{78}\|C_j Z + D_j\|^{-r}, \\ \big|\varphi(Z; f) - a(0, N_j^{-1})|C_j Z + D_j|^{-r}\big| \leqslant c_{79} y_j^{1-r}\|C_j Z + D_j\|^{-r} \end{gathered} \tag{120}$$

the constants c_{78}, c_{79} depending only on k, n and $f(Z)$ in general.

Let, for fixed complex $Y = \tilde{Y} > 0$, $\mathfrak{C}^*(s, Y)$ denote the set of $Z \in \mathfrak{H}_n$ satisfying the following conditions, viz. $I(Z) = Y$ and if $R(Z) = \dot{X} + \omega\ddot{X}$ with real $\dot{X} = (\dot{x}_{ij})$ and $\ddot{X} = (\ddot{x}_{kl})$, then, for $1 \leqslant i \leqslant j \leqslant n$, $1 \leqslant k < l \leqslant n$, we have $0 \leqslant \dot{x}_{ij}, \ddot{x}_{kl} \leqslant s$. Now referring to the Remark on p. 65 we observe that for the Farey dissection the essential things were the estimates (46) and (57) and we have indeed analogous estimates (117) and (120) for $f(Z)$ and $\varphi(Z; f)$. In order to obtain an estimate similar to (83) for the Fourier coefficients $c(T)$ $(T > 0)$ of $f(Z) - \varphi(Z; f)$, we have only to carry out (with suitable modifications as in [8]), the 'generalized Farey dissection' of the 'cube' $\mathfrak{C}^*(s, T^{-1})$ with $f(Z)$ and $\varphi(Z; f)$. We then get

$$|c(T)| \leqslant c_{80}\big((\min T^{-1})^{(n-1)(n-r-1)} + |T|^{r-n}(\min T)^{(2n-r)/2}\big).$$

If we impose on T the restriction (84), then we have

$$c(T) = O(|T|^{r-n+(2n-r)/2n})$$

the constants in the O-estimate depending only on k, n and $f(Z)$ in general. Thus we have

THEOREM 10. *With* $f(Z) = \sum_{T=\tilde{T}\geqslant 0} a(T)\eta(s^{-1}TZ) \in \{n, s, -r, v\}$, *we can associate for* $r > 2n$, *a function* $\varphi(Z; f) = \sum_{T=\tilde{T}\geqslant 0} b(T)\eta(s^{-1}TZ) \in \{n, s, -r, v\}$ *such that for* $T > 0$, *we have the formula*

$$a(T) = b(T) + O(|T|^{r-n+(2n-r)/2n}) \tag{121}$$

provided that $|T| > c_{39}$ *and* $\min T \geqslant c|T|^{1/n}$, *the constants in the* O-*estimate depending only on* k, n, c *and* $f(Z)$ *in general.*

If $\psi(Z) = \sum_{T=\tilde{T}>0} c(T)\eta(s^{-1}TZ) \in \{n, s, -r, v\}$ is a cusp form, then we can obtain an analogue of Hecke's estimate for coefficients of cusp forms ($n = 1$) viz. $c(T) = O(|T|^{r/2})$. The proof is similar to that of Theorem 1 of [8]. If in Theorem 10, $f(Z) - \varphi(Z; f)$ is a cusp form, then this gives us an estimate for $a(T) - b(T)$ which is far sharper than (121). But for $n > 1$, $f(Z) - \varphi(Z; f)$ may not be a cusp form, in general.

We know from [3] that the space $\{n, s, -r, v\}$ is a finite dimensional vector-space over the field of complex numbers. The forms $\varphi(Z; f)$ corresponding to $f(Z) \in \{n, s, -r, v\}$ constitute a subspace of $\{n, s, -r, v\}$ which is generated by the so-called generalized Eisenstein series for $\mathfrak{M}_n(s)$, defined as follows.

Let M_i be a left coset representative of $\mathfrak{M}_n$ modulo $\mathfrak{M}_n(s)$. We say M_i is *admissible* with respect to $\{v(M)\}$ if, for every $A_0 = \begin{pmatrix} \tilde{U} & * \\ 0 & U^{-1} \end{pmatrix} \in \mathfrak{M}_n(s)$, we have $v(M_i^{-1}A_0M_i) = |U|^r$. It can be shown as in [8] that this definition is independent of the choice of M_i in its coset and further M_i is in fact admissible, if there exists at least one $f(Z) \in \{n, s, -r, v\}$ for which the constant term $a(0, M_i^{-1})$ in the Fourier expansion of $f(Z)/M_i^{-1}$ is different from zero.

Let $M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in \mathfrak{M}_n$ be an admissible left coset representative of $\mathfrak{M}_n$ modulo $\mathfrak{M}_n(s)$. Then, corresponding to M_i we define, for $Z \in \mathfrak{H}_n$, the generalized Eisenstein series

$$\psi_i(Z) = \sum_{\substack{M = M_iN = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ N \in \mathfrak{M}_n(s)}} v(N)^{-1}|CZ + D|^{-r} \qquad (r > 2n)$$

the summation being over a complete set of elements in $M_i\mathfrak{M}_n(s)$ which are not left-associated with respect to $\mathfrak{A}_n$. In view of the admissibility of M_i, one sees that $\psi_i(Z)$ is well-defined and belongs to $\{n, s, -r, v\}$. In the special case when $v(M) = 1$ for all $M \in \mathfrak{M}_n(s)$ and $|U|^r = 1$ for all U in the group $\Omega_n(s)$ of $U \in \Omega_n$ with $U \equiv E^{(n)} \pmod{(s)}$, we can verify as in [8] that the number of linearly independent $\psi_i(Z)$ is precisely

$(\mathfrak{M}_n : \mathfrak{M}_n(s))/\{s^{n^2}(\Omega_n : \Omega_n(s))\}$. We can obtain explicitly the Fourier expansion of the Eisenstein series $\psi_i(z)$ by using the correspondence $\{CD\} \leftrightarrow [G_0, L]$ given by Lemma 3 and a summation formula of H. Braun (formula (78), p. 849, [2, I]). For the Fourier coefficients $b_i(T)$ of $\psi_i(Z)$, we have the estimate $b_i(T) = O\big((\delta(T))^{r-t}\big)$ where $t = r(T)$. Now it is simple to verify that $\varphi(Z; f)$ is a linear combination of the Eisenstein series $\psi_i(Z)$. Hence for the Fourier coefficients $b(T)$ of $\varphi(Z; f)$, we have the estimate $b(T) = O(\delta(T)^{r-t})$ where $t = r(T)$. For $T > 0$, in particular, $b(T) = O(|T|^{r-n})$.

Let now $f(Z)$ be the theta-series $f(S, Z)$ associated with an m-rowed integral matrix $S \geqslant 0$, of rank r. Then we know from Theorem 2 that $f(S, Z) \in \{n, \gamma^2, -r, v\}$. In view of Lemma 5 and the remark on p. 50, it follows that $\varphi(Z; f)$ coincides with the 'analytic genus-invariant' associated with S ([2, II]) and has the Fourier expansion $\sum_{T=\tilde{T}\geqslant 0} B(S, T)\eta(TZ)$ where

$$B(S, T) = \prod_{j=r-t+1}^{r} \frac{(2\pi)^j}{\Gamma(j)|d|^{j/2}} \delta(T)^{r-t}\delta(S)^{-n} \prod_p \alpha_p(S, T). \tag{122}$$

In (122), $t = r(T)$ and $\prod_p \alpha_p(S, T)$ is the infinite product extended over all the rational primes p, of $\alpha_p(S, T)$, the p-adic density of representation of T by S, in the sense of [1] (see p. 139; see also Lemma 1, p. 97, [2, II]).

Let now $T = \tilde{T} \geqslant 0$ and $t = r(T) \leqslant n$. We wish to obtain, for the number $A(S, T)$ of E_S-reduced representations of T by S, an asymptotic formula similar to (86). By (5) and (6), there exists $Q \in \{\mathfrak{O}\}_{n,n}$, with

$$T[Q^{-1}] = \begin{pmatrix} T_1^{(t)} & 0 \\ 0 & 0 \end{pmatrix}, \quad T_1 > 0, \quad Q = \begin{pmatrix} A^{(t,n)} \\ * \end{pmatrix}, \tag{123}$$
$$N(\delta(A)) \leqslant c_2, \quad 0 < \|Q\| \leqslant c_3.$$

Further, as remarked earlier, if E_T is any r-unit of T and G an E_S-reduced representation of T by S, then necessarily $GE_T = G$.

Let $A_0 = \begin{pmatrix} Q^{-1} & 0 \\ 0 & \tilde{Q} \end{pmatrix}$. Then $A(S, T)$ is precisely the Fourier coefficient of $\eta(T_1Z_1)$ in the Fourier expansion of

$$f^*(Z_1) = \lim_{\lambda\to\infty} |\tilde{Q}|^{+r} f\left(S, \begin{pmatrix} Z_1 & 0 \\ 0 & i\lambda E^{(n-t)} \end{pmatrix}\right) \Big/ A_0, \quad Z_1 \in \mathfrak{H}_t.$$

For, the limit-process can be performed term-wise and if $\lim_{\lambda\to\infty} \eta\left(S[GQ^{-1}]\begin{pmatrix} Z_1 & 0 \\ 0 & i\lambda E^{(n-t)} \end{pmatrix}\right)$ were to be different from zero, then necessarily $S[GQ^{-1}]$ should have its last $n-t$ diagonal elements equal to zero and since $S[GQ^{-1}] \geqslant 0$, it should then be of the form $\begin{pmatrix} * & 0 \\ 0 & 0^{(n-t)} \end{pmatrix}$. Thus, if $\lim_{\lambda\to\infty} \eta\left(S[GQ^{-1}]\begin{pmatrix} Z_1 & 0 \\ 0 & i\lambda E\end{pmatrix}\right) = \eta(T_1 Z_1)$, then $S[GQ^{-1}] = \begin{pmatrix} T_1 & 0 \\ 0 & 0\end{pmatrix}$ i.e. $S[G] = T$. By a similar argument, $B(S, T)$ defined by (122) is the coefficient of $\eta(T_1 Z_1)$ in the Fourier expansion of

$$\varphi^*(Z_1) = \lim_{\lambda\to\infty} |\widetilde{Q}|^{+r} \varphi\begin{pmatrix} Z_1 & 0 \\ 0 & i\lambda E\end{pmatrix} \Big/ A_0 .$$

It is clear that $f(S, Z)/A_0$ and $\varphi(Z: f)/A_0$ are in $\{n, \gamma^2\|Q\|^2, -r, v_1\}$ where $v_1(M) = v(A_0 M A_0^{-1})$ for $M \in \mathfrak{M}_n(\gamma^2\|Q\|^2)$ (see [4]). Again, from [4], the functions $f^*(Z_1)$ and $\varphi^*(Z_1)$ obtained from the above by applying the Siegel operator, belong to $\{t, \gamma^2\|Q\|^2, -r, v^*\}$ where, for $M^* = \begin{pmatrix} A^* & B^* \\ C^* & D^*\end{pmatrix} \in \mathfrak{M}_t(\gamma^2\|Q\|^2)$, we define $v^*(M^*) = v_1(M)$ where

$$M = \begin{pmatrix} A^* & 0 & B^* & 0 \\ 0 & E & 0 & 0 \\ C^* & 0 & D^* & 0 \\ 0 & 0 & 0 & E \end{pmatrix}.$$

If we can show that $f^*(Z_1)/N_i^{*-1}$ and $\varphi^*(Z_1)/N_i^{*-1}$ for every $N_i^* = \begin{pmatrix} A_i^* & B_i^* \\ C_i^* & D_i^*\end{pmatrix} \in \mathfrak{M}_t$ have the same constant term in their Fourier expansions, then we can apply the working of Theorem 10 to $f^*(Z_1)$ and $\varphi^*(Z_1)$ and obtain an asymptotic formula for $A(S, T)$ which is the Fourier coefficient of $\eta(T_1 Z_1)$ in $f^*(Z_1)$ with $T_1 > 0$. This is very easy to prove, since by [4] (formula (1.8), p. 13), there exists $M_i \in \mathfrak{M}_n$ and $P = \begin{pmatrix} \widetilde{C}^{(n)} & * \\ 0 & C^{-1}\end{pmatrix} \in \{k\}_{2n,2n}$ such that

(124) $$A_0 N_i^{-1} = M_i P$$

where

$$N_i = \begin{pmatrix} A_i^* & 0 & B_i^* & 0 \\ 0 & E & 0 & 0 \\ C_i^* & E & D_i^* & 0 \\ 0 & 0 & 0 & E \end{pmatrix} \in \mathfrak{M}_n .$$

Now, the constant term in the Fourier expansion of $f^*(Z_1)/N_i^{*-1}$ is precisely

$$\begin{aligned} \lim_{\mu\to\infty} f^*(i\mu E^{(t)})/N_i^{*-1} &= \lim_{\lambda\to\infty} f(S, i\lambda E^{(n)})/A_0 N_i^{-1} \\ &= \lim_{\lambda\to\infty} f(S, i\lambda E^{(n)})/M_i P \qquad \text{(by (124))} \\ &= \left(\lim_{\lambda\to\infty} f(S, i\lambda E^{(n)})/M_i\right)|C|^r . \end{aligned}$$

Similarly the constant term in the Fourier expansion of $\varphi^*(Z_1)/N_i^{*-1}$ is just $|C|^r \lim_{\lambda\to\infty} \varphi(i\lambda E^{(n)}; f)/M_i$. But we know that

$$\lim_{\lambda\to\infty} f(S; i\lambda E^{(n)})/M_i = \lim_{\lambda\to\infty} \varphi(i\lambda E^{(n)}; f)/M_i$$

since both are just the constant terms in the Fourier expansions of $f(S, Z)/M_i$ and $\varphi(Z; f)/M_i$ respectively. Thus our assertion above is proved and we have the asymptotic formula

(125) $$A(S, T) = \prod_{j=r-t+1}^{r} \frac{(2\pi)^j}{\Gamma(j)|d|^{j/2}} \delta(T)^{r-t} \delta(S)^{-t} \prod_p \alpha_p(S, T) + O(|T_1|^{r-t+(2t-r)/2t})$$

provided $r > 2n$, $|T_1| > c_{81}$ (depending only on k and t) and further that

(126) $$\min T_1 \geqslant c' |T_1|^{1/t}$$

where c' is a fixed positive constant. The constants in the O-estimate in (125) depend only on c', k, t and $f(Z)$ in general. (They also depend on $\|Q\|^2$ but we know that $\|Q\|^2 \leqslant c_3^2$ by (123).) But now $\delta(T) = |T_1| N(\delta(A))$, by (123) again. Let, for integral $T = \widetilde{T} \geqslant 0$, $\min T$ (*the reduced minimum of* T) denote the smallest non zero rational integer represented by T. Then, it is easy to see that

(127) $$\begin{gathered} \min T_1 \geqslant \|Q\|^{-2} \min T \geqslant c_3^{-2} \min T \geqslant c_3^{-2} \min T_1 , \\ |T_1| \leqslant \delta(T) \leqslant c_2 |T_1| . \end{gathered}$$

In view of (127), condition (126) may be rewritten in an equivalent form

$$\min T \geqslant c'' \delta(T)^{1/t}$$

for a fixed constant $c'' > 0$. We have thus

THEOREM 11. *Let $S = \widetilde{S} \geqslant 0$ be an m-rowed integral matrix of rank r, E_S a fixed r-unit of S and $T = \widetilde{T} \geqslant 0$ an n-rowed integral matrix of rank $t > 0$. Then, for $A(S, T)$, the number of E_S-reduced representations of T by S, we have the asymptotic formula*

$$A(S, T) = \prod_{j=r-t+1}^{t} \left(\frac{(2\pi)^j}{\Gamma(j)} |d|^{-j/2}\right) \delta(T)^{r-t} \delta(S)^{-t} \prod_p \alpha_p(S, T) + O\left(\delta(T)^{r-t+(2t-r)/2t}\right)$$

provided $r > 2n$, $\delta(T) > c_{82}$ and $\min T \geqslant c(\delta(T))^{1/t}$ for a constant c_{82} depending only on k and n and c (a fixed positive constant).

Remark. If $S^{(m)} > 0$ and $T^{(n)} \geqslant 0$ are symmetric matrices with elements in Γ and $r(T) = t < n$, there exists a rational unimodular matrix

U such that $T[U] = \begin{pmatrix} T_1^{(l)} & 0 \\ 0 & 0 \end{pmatrix}$. Then the number of (rational) integral representations of T by S is the same as of T_1 by S and so the corresponding formula in [8] (Theorem 5) was easier to prove. For k, we can not always reduce T to this form by a unimodular matrix over k, since the class number of k is greater than 1, in general.

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Contributions to the theory of the distribution of prime numbers in arithmetical progressions III

by

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1. Continuing the research of [1] and [2] I shall prove in this paper some results concerning the distribution of primes $\equiv l_1 \pmod k$ in comparison with those $\equiv l_2 \pmod k$. Once more I shall need the conjecture

(1.1) *In the rectangle* $0 < \sigma < 1$, $|t| \leqslant \max(c_1, k^7)$, $s = \sigma + it$, *all* L-*functions* $\bmod k$ *may vanish only at points of the line* $\sigma = \frac{1}{2}$ ([1]).

Writing, as usually,

$$\pi(x, k, l) = \sum_{\substack{p \equiv l \pmod k \\ p \leqslant x}} 1, \quad p \text{ primes},$$

we shall establish the following

Theorem. *Let* $k \geqslant 3$, $0 < l_1$, $l_2 < k$, $l_1 \neq l_2$, $(l_1, k) = (l_2, k) = 1$ *and suppose* (1.1) *to be satisfied. Then*

$$\int_X^T \frac{|\pi(x, k, l_1) - \pi(x, k, l_2)|}{x} dx > T^{1/2} \exp\left(-7 \frac{\log T}{\log\log T}\right) \tag{1.2}$$

with

$$X = T \exp\left(-(\log T)^{3/4}\right)$$

for

$$T \geqslant \max(c_2, e^{e^k}) \text{ ([2])}. \tag{1.3}$$

Remark. In the particular case of $l_1 = 1$ one might prove a similar inequality without assuming (1.1). However, for general l_1, l_2 I have not been able to supply any lower bound (e.g. $T^{1/4}$, as it used to be in the investigation of $\psi(x, k, l_1) - \psi(x, k, l_2)$ performed in [2]) for

$$\int_X^T \frac{|\pi(x, k, l_1) - \pi(x, k, l_2)|}{x} dx$$

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[1] c_1 and further $c_2, c_3, \ldots$ stand for positive numerical constants throughout.

[2] Compare the similar, though weaker, Theorem 3 of [2].