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A REMARK ON THE CURVATURE OF NON-PLANE CURVES

BY

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Introduction. It is well known that assuming the existence of tangent t to a given curve Γ ($\Gamma \subset R^n$, $n \geqslant 2$) at all its points P except its end point P_0 and convergence of t to a straight line t_0 as P tends to P_0, the straight line t_0 is (one-sided) tangent to Γ at P_0. The problem arises whether the analogue for a curvature is true. (If P_0 is not an end point of Γ the answer is trivially negative. Consider the curve $\beta = \alpha^2 \operatorname{sgn} \alpha$, consisting of two semi-parabolas, at the point $\alpha = 0$, $\beta = 0$ which has no curvature but there exists the limit of the curvature at $\alpha = 0$.) The answer to the problem is positive for $n = 2$. It was proved in [2], p. 98, under additional assumptions imposed on the curve Γ. A proof under weaker assumptions will be published later.

In this note we shall give an example showing that for $n \geqslant 3$ the answer is negative. We shall construct a curve Γ ($\Gamma \subset R^3$) having no (Menger or even Alt [1], [3]) curvature at its end point P_0 and such that there exists the limit of the curvature at P as P tends to P_0.

1. Consider the surface Σ obtained by rotation of the parabola $a_2 = a_1^2$ around the axis a_1 in the three-dimensional space R^3 with the coordinates a_1, a_2, a_3. In the complex notation the equation of surface Σ is

$$a = a(\alpha, \beta) = \alpha^2 e^{i\beta}, \tag{1}$$

where α, β are real and a is complex, $\alpha = a_1$, $a = a_2 + ia_3$. Obviously the point P_0: $a_1 = 0$, $a_2 = 0$, $a_3 = 0$ is a singular point of Σ.

Putting $\beta = \beta(\alpha)$, where $\beta(\alpha)$ is a continuous function for positive α and defined arbitrarily for $\alpha = 0$ (the value $\beta(0)$ is irrelevant) we obtain on the surface Σ a curve Γ with the end point P_0 having an equation of the form

$$a = a(\alpha) = \alpha^2 e^{i\beta(\alpha)}. \tag{2}$$

We shall choose the function $\beta(\alpha)$ for an example.

We have by (1) the equality $|a(\alpha,\beta)/\alpha| = \alpha$, therefore $a(\alpha,\beta)/\alpha \to 0$ as $\alpha \to 0$, and it follows that the surface Σ and consequently the curve Γ (for any function $\beta(\alpha)$) is tangent to axis a_1 at the point P_0.

2. To compute the curvature $\varkappa(\alpha)$ of the curve Γ for $\alpha > 0$, assume that $\beta(\alpha)$ is of class C^2 for $\alpha > 0$. Equation (2) can be rewritten in real form $a_1 = a_1(\alpha)$, $a_2 = a_2(\alpha)$, $a_3 = a_3(\alpha)$, where $a_1(\alpha) = \alpha$, $a(\alpha) = a_2(\alpha) + ia_3(\alpha)$. We have for $\alpha > 0$

$$\varkappa(\alpha) = \big((a_2'a_3'' - a_2''a_3')^2 + (a_3'a_1'' - a_3''a_1')^2 + (a_1'a_2'' - a_1''a_2')^2\big)^{1/2}(a_1'^2 + a_2'^2 + a_3'^2)^{-3/2}$$

or, in complex notation,

$$\varkappa(\alpha) = \big(|a''|^2 + O(a'^2a''^2)\big)^{1/2}(1+|a'|^2)^{-3/2}, \tag{3}$$

where $O(f(\alpha))$ denotes such a function $g(\alpha)$ that $|g(\alpha)/f(\alpha)|$ is bounded for small positive α. We have by (2)

$$a' = (2\alpha + i\alpha^2\beta')e^{i\beta}, \tag{4}$$

$$a'' = (2 + 4i\alpha\beta' - \alpha^2\beta'^2 + i\alpha^2\beta'')e^{i\beta}. \tag{5}$$

Hence

$$|a''|^2 = (2 - \alpha^2\beta'^2)^2 + (4\alpha\beta' + \alpha^2\beta'')^2. \tag{6}$$

Let us now take

$$\beta(\alpha) = -\lambda \ln \alpha \quad \text{for} \quad \alpha > 0 \quad (\beta(0) \text{ arbitrary}), \tag{7}$$

where λ is a positive number. We obtain $\beta'(\alpha) = -\lambda/\alpha$, $\beta''(\alpha) = \lambda\alpha^{-2}$ and consequently

$$\alpha\beta' = -\lambda, \quad \alpha^2\beta'' = \lambda. \tag{8}$$

In virtue of (4), (5), (8) we get

$$|a'|^2 = O(\alpha^2), \quad O(a'^2a''^2) = O(\alpha^2),$$

and in virtue of (6), (8) we have $|a''|^2 = (2-\lambda^2)^2 + 9\lambda^2 = 4 + 5\lambda^2 + \lambda^4$ and therefore by (3) we obtain $\varkappa(\alpha) = (4 + 5\lambda^2 + \lambda^4)^{1/2} + O(\alpha^2)$. It follows the property

$$\varkappa(\alpha) \to (4 + 5\lambda^2 + \lambda^4)^{1/2} \quad \text{as} \quad \alpha \to 0. \tag{9}$$

For any number σ greater than two we can choose constant λ in (7) that $\varkappa(\alpha) \to \sigma$ as $\alpha \to 0$.

We can also choose $\beta(\alpha)$ in such a manner that $\varkappa(\alpha) \to 2$ as $\alpha \to 0$ and $\beta(\alpha) \to \infty$ as $\alpha \to 0$. It is enough to put $\beta(\alpha) = |\ln \alpha|^{1/2}$. Then we obtain $\alpha\beta' \to 0$, $\alpha^2\beta'' \to 0$ as $\alpha \to 0$ and by easy computations we get the desired property.

3. We will show now that (for any positive λ) the curve Γ has no curvature at the point P_0.

On the curve Γ consider the points A_n corresponding to $\alpha = e^{-2\pi n/\lambda}$, the points B_n corresponding to $\alpha = e^{-2\pi(2n+1)/\lambda}$ and the points C_n corresponding to $\alpha = e^{-\pi(2n+1)/\lambda}$. Points A_n, B_n, C_n lie on the plane $a_3 = 0$.

Points P_0, A_n, B_n lie on the parabola $a_2 = a_1^2$, consequently the limit ($n \to \infty$) of the radius of the circumference passing through points P_0, A_n, B_n ($n \geq 1$) equals the reciprocal of the curvature of the parabola at P_0 and therefore equals $\frac{1}{2}$. It follows from (9) that the Menger curvature does not exist at P_0. We will show that the same applies to the Alt curvature.

For this purpose consider the circumference passing through the points P_0, A_n, C_n. We compute its radius from the formula

$$R = \xi\eta\zeta(\xi+\eta+\zeta)^{-1/2}(\xi+\eta-\zeta)^{-1/2}(\xi+\zeta-\eta)^{-1/2}(\eta+\zeta-\xi)^{-1/2}, \tag{10}$$

where $\xi = |A_n - P_0|$, $\eta = |C_n - P_0|$, $\zeta = |C_n - A_n|$. We have

$$\xi = |A_n| = (e^{-4\pi n/\lambda} + e^{-8\pi n/\lambda})^{1/2} = e^{-2\pi n/\lambda}(1 + e^{-4\pi n/\lambda})^{1/2}.$$

Similarly

$$\eta = |C_n| = e^{-\pi(2n+1)/\lambda}(1 + e^{-2\pi(2n+1)/\lambda})^{1/2},$$

$$\zeta = e^{-2\pi n/\lambda}\big((1 - e^{-\pi/\lambda})^2 + (1 + e^{-2\pi/\lambda})^2 e^{-4\pi n/\lambda}\big)^{1/2}.$$

Denote for conciseness $\gamma = e^{-\pi/\lambda}$, $\varrho_n = e^{-2\pi n/\lambda}$. Notice that

$$0 < \gamma < 1, \tag{11}$$

$$\varrho_n \to 0 \quad \text{as} \quad n \to \infty. \tag{12}$$

Using these notations we get by (12)

$$\xi = \varrho_n(1+\varrho_n^2)^{1/2} = \varrho_n + \tfrac{1}{2}\varrho_n^3 + O(\varrho_n^5),$$

$$\eta = \gamma\varrho_n(1+\gamma^2\varrho_n^2)^{1/2} = \gamma\varrho_n + \tfrac{1}{2}\gamma^3\varrho_n^3 + O(\varrho_n^5),$$

$$\zeta = \varrho_n\big((1-\gamma)^2 + (1+\gamma^2)^2\varrho_n^2\big)^{1/2} = (1-\gamma)\varrho_n + \tfrac{1}{2}(1+\gamma^2)^2(1-\gamma)^{-1}\varrho_n^3 + O(\varrho_n^5),$$

where $O(\mu_n)$ denotes any sequence ν_n where $|\nu_n/\mu_n|$ is bounded. Hence by (12) we have

$$\xi+\eta+\zeta = 2\varrho_n + O(\varrho_n^3), \quad \xi+\eta-\zeta = 2\gamma\varrho_n + O(\varrho_n^3),$$

$$\xi+\zeta-\eta = 2(1-\gamma)\varrho_n + O(\varrho_n^3),$$

$$\eta+\zeta-\xi = \frac{1}{2}\,\frac{\gamma(\gamma+1)^2}{1-\gamma}\varrho_n^3 + O(\varrho_n^5), \quad \xi\eta\zeta = \gamma(1-\gamma)\varrho_n^3 + O(\varrho_n^5).$$

Therefore using formula (10) we obtain $R = \frac{1}{2}(1-\gamma)(1+\gamma)^{-1} + O(\varrho_n^2)$. In virtue of (12), (11) the radius of the circumference passing through P_0, A_n, C_n tends to $R_0 = \frac{1}{2}(1-\gamma)/(1+\gamma) < \frac{1}{2}$ as $n \to \infty$. We have proved before that the radius of the circumference passing through P_0, A_n, B_n tends to $\frac{1}{2}$. Therefore the radius of the circumference passing through points P_0, P_1, P_2, where $P_1 \epsilon \Gamma$, $P_2 \epsilon \Gamma$, $P_1 \neq P_0$, $P_2 \neq P_0$, $P_1 \neq P_2$, has no limit as $P_1 \to P_0$, $P_2 \to P_0$. This completes the proof that curve Γ does not possess the Alt curvature at P_0.

It is evident from (2), (7) that the curve Γ has no osculatory plane at P_0.

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INFORMATION WITHOUT PROBABILITY

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1. Introduction. Since the first definition of the notion of information given in its full generality by C. E. Shannon in 1948 (1), many mathematical investigations have been concerned with this notion (2). The general tendency of these investigations (initiated by Shannon himself (3)) has been to separate the definition of information, say H, from the explicit formula

$$H = -\sum_{i=1}^{n} p_i \log p_i, \tag{1}$$

adopted by Shannon from statistical physics (Boltzmann's formula for entropy). Here p_i denotes the probability of the i-th elementary event ($i = 1, \ldots, n$; we consider first a finite, or at any rate discrete, probability scheme, convergence of the sum in the case $n = \infty$ being assumed). It was felt from the beginning that such a formula as (1) should be rather a result than a starting point of the theory. Moreover, some investigators, as e. g. Rényi (4), considered (1) as too narrow to cover all possible applications of information theory and tried to generalize this formula. Of course, to get such a generalization in a natural way, it is necessary to have an abstract definition of information, i. e. by means of a set of axioms (this set may be subsequently diminished in the generalization process). Many such sets of axioms have so far been proposed (5) and their consequences as well as mutual interrelations have been investigated. All axiomatic definitions of information known to the present authors are equivalent to formula (1) (except Rényi's gene-

(1) Cf. Shannon [11]. The numbers in square brackets refer to the list of literature given at the end of this paper, p. 149-150.

(2) Cf., e. g., Khintchine [8], Feinstein [4], Rényi [10], where further references may be found.

(3) Cf. [11], p. 392, and Appendix 2, p. 419.

(4) Cf. Rényi [10].

(5) Cf. Shannon [11], Khintchine [8], Faddeev [3], Rényi [10].