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ON PAIRS OF INDEPENDENT RANDOM VARIABLES WHOSE QUOTIENTS FOLLOW SOME KNOWN DISTRIBUTION

BY

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1. Introduction. Some important probability distributions encountered in mathematical statistics are defined as probability distributions of a quotient of two independent random variables. So are the distributions of Student as well as those of Fisher's variance ratios. A question suggests itself if the distributions of nominators and denominators of the quotients in question are determined uniquely, up to a multiplication by a constant factor or to a passage to reciprocals of the random variables involved by the distribution of the quotient. The simplest problem is connected with the Cauchy distribution, the probability density of which is

$$\frac{1}{\pi}\frac{1}{1+x^2} \qquad (-\infty < x < \infty).$$

It turns out to be the distribution of a quotient of two independent random variables having the same normal distribution symmetrical about zero, or, in other words, to be Student's distribution with one degree of freedom. This problem has been studied by Mauldon [7], Laha [4, 5, 6], Steck [8], Kotlarski [3] to the effect that there is no such uniqueness — there are many non-normal distributions symmetrical about zero such that a quotient of two independent random variables having such distribution has Cauchy distribution. Another case — where nominators and denominators have gamma distributions — has been considered by Mauldon [7]. He has shown that also in this case there is no uniqueness of the above mentioned kind and thus has shown the ambiguity phenomenon for Fisher's, or, as some say, Snedecor's F distributions.

In this paper we are considering the more general case of quotients $U = X_1^{q_1}:X_2^{q_2}$, where X_1 and X_2 are independent random variables having gamma distributions and q_1, q_2 are real numbers not equal to 0.

Our aim is to show that, whatever be q_1, q_2 and the parameters of the distributions of X_1 and X_2, there exist independent positive random variables Y_1 and Y_2 with distributions intrinsically other than those of $X_1^{q_1}$ and $X_2^{q_2}$ such that the quotient $Y_1:Y_2$ has the same distribution as U. Our result, eventually with evident changes consisting in replacement of positive random variables by those which are distributed symmetrically about zero, show the ambiguity phenomenon spoken of not only in the case of Cauchy distribution or of F distributions, but also for Student distributions.

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2. Formulating the problem. Let us have two independent random variables X_1, X_2 having gamma distributions with densities

$$(1)\qquad f_r(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ \dfrac{a_r^{p_r}}{\Gamma(p_r)} x^{p_r-1} e^{-a_r x} & \text{for } x > 0, \end{cases} \qquad (r = 1, 2)$$

where $p_r > 0$, $a_r > 0$. Write

$$(2)\qquad Z_r = X_r^{q_r} \quad (r = 1, 2),$$

where $q_r \neq 0$, and

$$(3)\qquad U = \frac{Z_1}{Z_2}.$$

Let V be a positive random variable. Denote by V^* the symmetrical random variable, for which $|V^*|$ has the same distribution as V. In this paper we shall mark positive random variables by capital letters without asterisks, and the corresponding symmetrized random variables by capital letters with asterisks. It is easy to see that their distribution functions $F(x)$ and $F^*(x)$ are connected by the formula

$$(4)\qquad F^*(x) = \begin{cases} \frac{1}{2}[1 - F(|x|)] & \text{for } x \leqslant 0, \\ \frac{1}{2}[1 + F(|x|)] & \text{for } x > 0. \end{cases}$$

If one of the random variables V, V^* has a density function, so does the other, and they are connected by the formula

$$(5)\qquad f^*(x) = \tfrac{1}{2} f(|x|) \quad (-\infty < x < +\infty).$$

We denote by $\mathcal{Y}$ the set of pairs of independent positive random variables Y_1, Y_2, whose quotients follow the same distribution as $U = Z_1:Z_2$ given by (3). If, moreover, Y_1 and Y_2 have the same distribution, we shall mark their set by $\mathcal{Y}_*$.

We denote by $\mathcal{Y}^*$ the set of pairs of independent random variables Y_1^*, Y_2, where Y_1^* is symmetrical about the origin and Y_2 is positive, whose quotients follow the same distribution as $U^* = Z_1^*:Z_2$.

We denote by $\mathcal{Y}^{**}$ the set of pairs of independent random variables Y_1^*, Y_2^*, both symmetrical, whose quotients follow the same distribution as $U^{**} = Z_1^*:Z_2^*$. If, moreover, Y_1^* and Y_2^* have the same distribution, we shall mark their set by $\mathcal{Y}_*^{**}$.

The question arises whether it is possible to obtain a characterisation of the set $\mathcal{Y}$ of pairs of random variables by the distribution of their quotients. This problem can be more precisely formulated as follows: Let Y_1, Y_2 be two independent positive random variables. Let the quotient $Y_1:Y_2$ have the same distribution as $U = Z_1:Z_2$ given by (3). Have Y_1 and Y_2 the same distributions as Z_1 and Z_2 respectively?

The same question can be formulated about the sets $\mathcal{Y}_*, \mathcal{Y}^*, \mathcal{Y}^{**}, \mathcal{Y}_*^{**}$.

This inverse problem is not true, because of

$$(6)\qquad \begin{cases} \dfrac{Z_1}{Z_2} = \dfrac{aZ_1}{aZ_2} = \dfrac{1/Z_2}{1/Z_1} = \dfrac{a/Z_2}{a/Z_1}, & a > 0, \\[2ex] \dfrac{Z_1^*}{Z_2} = \dfrac{aZ_1^*}{aZ_2} = \dfrac{1/Z_2^*}{1/Z_1} = \dfrac{a/Z_2^*}{a/Z_1}, & a > 0, \\[2ex] \dfrac{Z_1^*}{Z_2^*} = \dfrac{aZ_1^*}{aZ_2^*} = \dfrac{1/Z_2^*}{1/Z_1^*} = \dfrac{a/Z_2^*}{a/Z_1^*}, & a \neq 0. \end{cases}$$

Hence we see, that if $(Z_1, Z_2) \in \mathcal{Y}$, then $(1/Z_2, 1/Z_1) \in \mathcal{Y}$ also; if $(Z_1^*; Z_2) \in \mathcal{Y}^*$, then $(1/Z_2^*, 1/Z_1) \in \mathcal{Y}^*$; if $(Z_1^*; Z_2^*) \in \mathcal{Y}^{**}$, then $(1/Z_2^*, 1/Z_1^*) \in \mathcal{Y}^{**}$; if $Z \in \mathcal{Y}_*$, then $1/Z \in \mathcal{Y}_*$; if $Z^* \in \mathcal{Y}_*^{**}$, then $1/Z^* \in \mathcal{Y}_*^{**}$.

In this paper we shall give two theorems about the sets $\mathcal{Y}, \mathcal{Y}_*, \mathcal{Y}^*, \mathcal{Y}^{**}, \mathcal{Y}_*^{**}$ and a way to obtain pairs of random variables belonging to these sets. The problem is solved in the same way as in [3] by using the Mellin transforms of random variables.

3. The Mellin transform of a positive random variable. We define the Mellin transform of a positive random variable V by the formula (see [2, 9])

$$(7)\qquad h(s) = \mathrm{E}[V^s] = \int_0^\infty x^s dF(x).$$

where s is a complex variable. The function $h(s)$ is given in a strip S: $c_1 < \operatorname{Re} s < c_2$ containing the imaginary axis and being parallel to it. The Mellin transform $h(s)$ defines the distribution of a random variable uniquely.

If the positive random variable V has a density $F'(x) = f(x)$ satisfying the condition $\int_0^\infty x^c f(x)\,dx < \infty$, where c is an arbitrary number safisfying the condition $c_1 < c < c_2$, then it is given in every point of its continuity by the formula

$$f(x) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} x^{-s-1} h(s)\,ds. \tag{8}$$

It should be noted, that if $h(s)$ is the Mellin transform of a positive random variable V, then the function $h(it)$ is the characteristic function $\varphi(t)$ of the variable $\ln V$, because

$$h(it) = \mathrm{E}[V^{it}] = \mathrm{E}[e^{it\ln V}] = \varphi_{\ln V}(t). \tag{9}$$

So we see that the function $h(s)$ satisfies the following conditions:

$$\begin{cases} h(it) \text{ is continuous along the whole axis } t, \\ h(0) = 1, \quad h(-it) = \overline{h(it)}. \end{cases} \tag{10}$$

The moment m_r of order r of the positive random variable V is given by its Mellin transform by the formula

$$m_r = \mathrm{E}[V^r] = h(r): \tag{11}$$

The moments of such random variables exist only for these r which lie in the strip S.

If we have n independent positive random variables $X_1, X_2, \ldots, X_n$ with their Mellin transforms $h_1(s), h_2(s), \ldots, h_n(s)$, then the Mellin transform of the variable

$$c\cdot X_1^{q_1}\cdot X_2^{q_2}\cdot\ldots\cdot X_n^{q_n}, \tag{12a}$$

where $c > 0$, $q_1, q_2, \ldots, q_n \neq 0$, is

$$c^s\cdot h_1(q_1 s)\cdot h_2(q_2 s)\cdot\ldots\cdot h_n(q_n s). \tag{12b}$$

4. The functional equation for Mellin transforms of any pair of random variables Y_1, Y_2 belonging to $\mathcal{Y}$. The Mellin transform of the random variables given by (1) is

$$a_r^{-s}\frac{\Gamma(p_r+s)}{\Gamma(p_r)} \quad (\mathrm{Re}\, s > -p_r;\ r = 1, 2). \tag{13}$$

The Mellin transforms of the random variables Z_1, Z_2 given by (2) are

$$a_r^{-q_r s}\frac{\Gamma(p_r+q_r s)}{\Gamma(p_r)} \quad (p_r + q_r\cdot\mathrm{Re}\, s > 0;\ r = 1, 2). \tag{14}$$

Taking into account formulae (12) we see that the Mellin transform of the variable U given by (3) is

$$a_1^{-q_1 s}\frac{\Gamma(p_1+q_1 s)}{\Gamma(p_1)}\cdot a_2^{-q_2 s}\frac{\Gamma(p_2-q_2 s)}{\Gamma(p_2)} \quad \begin{pmatrix} p_1+q_1\cdot\mathrm{Re}\, s > 0, \\ p_2-q_2\cdot\mathrm{Re}\, s > 0 \end{pmatrix}. \tag{15}$$

Hence we see that for a pair of independent positive random variables to belong to the set $\mathcal{Y}$, it is necessary and sufficient that their Mellin transforms $h_1(s), h_2(s)$ satisfy the functional equation

$$h_1(s)\cdot h_2(-s) = b^s\frac{\Gamma(p_1+q_1 s)}{\Gamma(p_1)}\cdot\frac{\Gamma(p_2-q_2 s)}{\Gamma(p_2)} \quad \begin{pmatrix} p_1+q_1\cdot\mathrm{Re}\, s > 0, \\ p_2-q_2\cdot\mathrm{Re}\, s > 0 \end{pmatrix}, \tag{16}$$

where $b = a_2^{q_2} : a_1^{q_1}$.

5. Solving equation (16). Let us put

$$h_1(s) = b^s\frac{\Gamma(p_1+q_1 s)}{\Gamma(p_1)}\cdot e^{\gamma_1(s)}, \qquad h_2(s) = \frac{\Gamma(p_2+q_2 s)}{\Gamma(p_2)}\cdot e^{\gamma_2(s)}. \tag{17}$$

Putting formula (17) into equation (16) we see that

$$\gamma_1(s)+\gamma_2(-s) = 0. \tag{18}$$

Putting $\gamma_1(s) = \gamma(s)$ we obtain $\gamma_2(s) = -\gamma(-s)$, and so we may write

$$h_1(s) = b^s\frac{\Gamma(p_1+q_1 s)}{\Gamma(p_1)}e^{\gamma(s)}, \qquad h_2(s) = \frac{\Gamma(p_2+q_2 s)}{\Gamma(p_2)}e^{-\gamma(-s)}. \tag{19}$$

Now we shall put $s = it$. Presenting the function $\gamma(it)$ as the sum of its real part $\alpha(t)$ and its imaginary part $\beta(t)$,

$$\gamma(it) = \alpha(t)+i\beta(t), \tag{20}$$

we can write

$$h_1(it) = b^{it}\frac{\Gamma(p_1+q_1 it)}{\Gamma(p_1)}e^{\alpha(t)+i\beta(t)}, \qquad h_2(it) = \frac{\Gamma(p_2+q_2 it)}{\Gamma(p_2)}e^{-\alpha(-t)-i\beta(-t)}. \tag{21}$$

It should be noted, that $h_1(it)$ and $h_2(it)$ are characteristic functions (see formula (9)), and so they satisfy the conditions (10). Hence we see, that the functions $\alpha(t)$ and $\beta(t)$ satisfy the following conditions:

$$\begin{cases} \alpha(t),\ \beta(t) \text{ are real and continuous along the axis } t, \\ \alpha(0) = 0,\ \alpha(-t) = \alpha(t),\ \beta(-t) = -\beta(t). \end{cases} \tag{22}$$

Thus the functions (21) are

$$(23)\quad h_1(it) = b^{it}\frac{\Gamma(p_1+q_1 it)}{\Gamma(p_1)}e^{\alpha(t)+i\beta(t)}, \qquad h_2(it) = \frac{\Gamma(p_2+q_2 it)}{\Gamma(p_2)}e^{-\alpha(t)+i\beta(t)},$$

where $\alpha(t)$, $\beta(t)$ satisfy conditions (22).

6. Theorems. The result of parts 4 and 5 may be formulated as the following two theorems.

THEOREM 1. *For a pair of independent positive random variables Y_1, Y_2 to belong to the set $\mathcal{Y}$, it is necessary and sufficient that their Mellin transforms $h_r(s) = \mathrm{E}[Y_r^s]$ $(r = 1, 2)$ should be given on the imaginary axis in the form* (23), *where $\alpha(t)$, $\beta(t)$ satisfy the conditions* (22).

Remark 1. Taking Y_1^* symmetrical instead of positive Y_1 and $h_1(s) = \mathrm{E}[|Y_1|^s]$ we obtain the conditions necessary and sufficient for belonging to $\mathcal{Y}^*$.

Remark 2. Taking Y_1^*, Y_2^* both symmetrical instead of positive Y_1, Y_2 and $h_r(s) = \mathrm{E}[|Y_r|^s]$ we obtain the conditions necessary and sufficient for belonging to $\mathcal{Y}^{**}$.

In the particular case when Z_1 and Z_2 have the same distribution, and we want Y_1 and Y_2 to have the same distribution, we put

$$h_1(s) = h_2(s) = h(s), \quad p_1 = p_2 = p, \quad q_1 = q_2 = q$$

into the formula (23) and we obtain $b = 1$ and $\alpha(t) \equiv 0$. So we obtain the following theorem for belonging to $\mathcal{Y}_*$:

THEOREM 2. *For a pair of independent identically distributed positive random variables Y_1, Y_2 to belong to the set $\mathcal{Y}_*$ it is necessary and sufficient that their common Mellin transform $h(s) = E[X_r^s]$ should be given on the imaginary axis in the form*

$$(24)\qquad h(it) = \frac{\Gamma(p+qit)}{\Gamma(p)}e^{i\beta(t)},$$

where $\beta(t)$ is a real odd function continuous along the axis t.

Remark 3. Taking Y_r^* symmetrical instead of positive Y_r and $h(s) = E[|Y_r^s|]$ we obtain the conditions necessary and sufficient for belonging to $\mathcal{Y}_*^{**}$.

7. The construction of diverse pairs of independent random variables Y_1, Y_2 belonging to $\mathcal{Y}, \mathcal{Y}^*, \mathcal{Y}^{**}$. The conditions given in theorems 1 and 2 are to be regarded as a reduction of the question how to

characterize the classes $\mathcal{Y}$ and $\mathcal{Y}_*$ to the question for what functions $\alpha(t)$ and $\beta(t)$ satisfying condition (22) the right sides of formulae (23) or of formula (24) are Mellin transforms of some random variables. It is, however, hard to see by the aid of that reduction that there are in $\mathcal{Y}$ and $\mathcal{Y}_*$ other elements than those described in section 2. Therefore we are going to vizualize this by some examples. In order to do that, we shall apply the known formula for the function $\Gamma(s)$

$$(25)\qquad \Gamma(nw) = \frac{n^{nw-\frac{1}{2}}}{(2\pi)^{(n-1)/2}}\prod_{k=1}^{n}\Gamma\left(w+\frac{k-1}{n}\right) \qquad (\mathrm{Re}\, w > 0)$$

in equation (16). We can write this equation in the form

$$(26)\qquad h_1(s)h_2(-s) = b^s\prod_{k=1}^{n}\frac{\Gamma\left(\frac{p_1+k-1+q_1 s}{n}\right)}{\Gamma\left(\frac{p_1+k-1}{n}\right)}\prod_{l=1}^{m}\frac{\Gamma\left(\frac{p_2+l-1-q_2 s}{m}\right)}{\Gamma\left(\frac{p_2+l-1}{m}\right)}.$$

Taking into account formulae (1), (2), (12) and (14), we see that the following pair of functions are Mellin transforms of positive random variables satisfying equation (26):

$$(27)\qquad \begin{aligned} h_1(s) &= (bc)^s\prod_{k=1}^{\nu}\frac{\Gamma\left(\frac{p_1+\alpha_k-1+q_1 s}{n}\right)}{\Gamma\left(\frac{p_1+\alpha_k-1}{n}\right)}\prod_{l=1}^{\mu}\frac{\Gamma\left(\frac{p_2+\beta_l-1-q_2 s}{m}\right)}{\Gamma\left(\frac{p_2+\beta_l-1}{m}\right)},\\ h_2(s) &= c^s\prod_{k=\nu+1}^{n}\frac{\Gamma\left(\frac{p_1+\alpha_k-1-q_1 s}{n}\right)}{\Gamma\left(\frac{p_1+\alpha_k-1}{n}\right)}\prod_{l=\mu+1}^{m}\frac{\Gamma\left(\frac{p_2+\beta_l-1+q_2 s}{m}\right)}{\Gamma\left(\frac{p_2+\beta_l-1}{m}\right)},\end{aligned}$$

where c is an arbitrary positive number; n, m are positive integers; ν, μ are positive integers satisfying the conditions $0 \leqslant \nu \leqslant n$, $0 \leqslant \mu \leqslant m$; $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_m$ are arbitrary permutations of the numbers $1, 2, \ldots, n$, and $1, 2, \ldots, m$ respectively. It it easy to see that the Mellin transforms (27) correspond to the quotients

$$(28)\qquad \begin{aligned} &bc\frac{X_{\alpha_1}\cdot X_{\alpha_2}\cdot\ldots\cdot X_{\alpha_\nu}}{X_{\beta_1}\cdot X_{\beta_2}\cdot\ldots\cdot X_{\beta_\mu}},\\ &c\frac{X_{\beta_{\mu+1}}\cdot X_{\beta_{\mu+2}}\cdot\ldots\cdot X_{\beta_m}}{X_{\alpha_{\nu+1}}\cdot X_{\alpha_{\nu+2}}\cdot\ldots\cdot X_{\alpha_n}},\end{aligned}$$

where X_{α_k} and X_{β_l} are independent positive random variables having densities

$$(29)\qquad \begin{aligned} f_{\alpha_k}(x) &= \frac{n}{|q_1|\Gamma\left(\frac{p_1+\alpha_k-1}{n}\right)} x^{\frac{p_1+\alpha_k-1}{q_1}-1} e^{-x^{n/q_1}}, \\ f_{\beta_l}(x) &= \frac{m}{|q_2|\Gamma\left(\frac{p_2+\beta_l-1}{m}\right)} x^{\frac{p_2+\beta_l-1}{q_2}-1} e^{-x^{m/q_2}} \end{aligned}$$

(see formulae (1), (2), (14)).

In the special case when $n/q_1 = m/q_2 = 1/q$, we can use Mayer's functions (see [1], p. 207)

$$(30)\qquad G^{m,n}_{p,q}\left(x\middle|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^s \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\,ds.$$

It is easy to derive that the densities corresponding to the Mellin transforms (27) may be expressed by the following formulae (for $b = c = 1$):

$$(31)\qquad \begin{aligned} f_1(x) &= \frac{1}{|q|\prod_{k=1}^{\nu}\Gamma\left(\frac{p_1+\alpha_k-1}{n}\right)\prod_{l=1}^{\mu}\Gamma\left(\frac{p_2+\beta_l-1}{m}\right)} \times \\ &\quad \times \frac{1}{x} G^{\nu,\mu}_{\mu,\nu}\left(x^{\frac{1}{q}}\middle|\begin{matrix}1-\frac{p_2+\beta_1-1}{m},\dots,1-\frac{p_2+\beta_\mu-1}{m}\\ \frac{p_1+\alpha_1-1}{n},\dots,\frac{p_1+\alpha_\nu-1}{n}\end{matrix}\right), \\ f_2(x) &= \frac{1}{|q|\prod_{k=\nu+1}^{n}\Gamma\left(\frac{p_1+\alpha_k-1}{n}\right)\prod_{l=\mu+1}^{m}\Gamma\left(\frac{p_2+\beta_l-1}{m}\right)} \times \\ &\quad \times \frac{1}{x} G^{m-\mu,n-\nu}_{n-\nu,m-\mu}\left(x^{\frac{1}{q}}\middle|\begin{matrix}1-\frac{p_1+\alpha_{\nu+1}-1}{n},\dots,1-\frac{p+\alpha_n-1}{n}\\ \frac{p_2+\beta_{\mu+1}-1}{m},\dots,\frac{p_2+\beta_m-1}{m}\end{matrix}\right). \end{aligned}$$

Taking the corresponding symmetrical random variable instead of the positive one in the formula (28a) we obtain pairs of random variables Y_1^*, Y_2 belonging to $\mathcal{Y}^*$.

Taking the corresponding symmetrical random variables instead of the positive ones in the formulae (28) we obtain pairs of random variables Y_1^*, Y_2^* belonging to $\mathcal{Y}^{**}$.

8. The construction of pairs of random variables Y_1, Y_2 belonging to $\mathcal{Y}_*, \mathcal{Y}_*^{**}$. Applying formula (25) in the equation

$$(32)\qquad h(s)\cdot h(-s) = \frac{\Gamma(p+qs)}{\Gamma(p)}\cdot\frac{\Gamma(p-qs)}{\Gamma(p)}$$

(see equation (16) and the proof of the theorem 2), we obtain the equation

$$(33)\qquad h(s)\cdot h(-s) = \prod_{k=1}^{n}\frac{\Gamma\left(\frac{p+k-1+qs}{n}\right)}{\Gamma\left(\frac{p+k-1}{n}\right)}\cdot\prod_{k=1}^{n}\frac{\Gamma\left(\frac{p+k-1-qs}{n}\right)}{\Gamma\left(\frac{p+k-1}{n}\right)}.$$

Then we see that the function

$$(34a)\qquad h(s) = c^s\prod_{k=1}^{\nu}\frac{\Gamma\left(\frac{p+\alpha_k-1+qs}{n}\right)}{\Gamma\left(\frac{p+\alpha_k-1}{n}\right)}\prod_{k=\nu+1}^{n}\frac{\Gamma\left(\frac{p+\alpha_k-1-qs}{n}\right)}{\Gamma\left(\frac{p+\alpha_k-1}{n}\right)},$$

where c is an arbitrary positive number, n is a positive integer, ν is a positive integer satisfying the condition $0 \leqslant \nu \leqslant n$, and $\alpha_1, \alpha_2, \dots, \alpha_n$ is an arbitrary permutation of the numbers $1, 2, \dots, n$, is a Mellin transform of some positive random variable belonging to $\mathcal{Y}_*$. But the Mellin transform (34a) corresponds to the quotient

$$(34b)\qquad c\,\frac{X_{\alpha_1}\cdot X_{\alpha_2}\cdot\ldots\cdot X_{\alpha_\nu}}{X_{\alpha_\nu+1}\cdot\ldots\cdot X_{\alpha_n}},$$

where X_{α_k} are independent positive random variables having densities

$$(35)\qquad f_{\alpha_k}(x) = \frac{n}{|q|\Gamma\left(\frac{p+\alpha_k-1}{n}\right)} x^{\frac{p+\alpha_k-1}{q}-1} e^{-x^{\frac{n}{q}}}$$

(see formulae (1), (2), (14)).

Taking, for instance, $n = 2$, $\nu = 1$, $a_1 = 1$, $a_2 = 2$, we obtain a positive random variable Y belonging to $\mathcal{Y}_*$; its density is

$$f(x) = \frac{2^p}{|q|} \cdot \frac{\Gamma(p+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(p)} \cdot \frac{y^{\frac{p+1}{q}-1}}{(1+y^{\frac{2}{q}})^{p+\frac{1}{2}}}. \tag{36a}$$

For $n = 2$, $\nu = 1$, $a_1 = 2$, $a_2 = 1$ the corresponding density is

$$f(x) = \frac{2^p}{|q|} \cdot \frac{\Gamma(p+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(p)} \cdot \frac{y^{\frac{p}{q}-1}}{(1+y^{\frac{2}{q}})^{p+\frac{1}{2}}}. \tag{36b}$$

Using Mayer's functions (30) we see that the density corresponding to the Mellin transform (34) can for $c = 1$ be expressed by the formula

$$f(x) = \frac{n^{p+\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}|q|\Gamma(p)} \cdot \frac{1}{x} G^{\nu,n-\nu}_{n-\nu,\nu}\left(x^{\frac{n}{q}}\,\middle|\, \begin{matrix} 1-\frac{p+a_{\nu+1}-1}{n}, \ldots, 1-\frac{p+a_n-1}{n} \\ \frac{p+a_1-1}{n}, \ldots, \frac{p+a_\nu-1}{n} \end{matrix}\right). \tag{37}$$

For $n = 3$, $\nu = 1, 2$, and for particular permutations a_1, a_2, a_3 of the numbers $1, 2, 3$, we obtain the following densities of positive random variables belonging to $\mathcal{Y}_*$ see ([1], p. 216, formula (8)):

$$f(x) = \frac{3^{p+\frac{1}{2}}\Gamma\left(\frac{2p+1}{3}\right)\Gamma\left(\frac{2p+2}{3}\right)}{2\pi|q|\Gamma(p)} x^{\frac{p}{q}-1} e^{\frac{1}{2}x^{\frac{3}{q}}} W_{-\frac{2}{3}p,-\frac{1}{6}}(x^{\frac{3}{q}}),$$

$$f(x) = \frac{3^{p+\frac{1}{2}}\Gamma\left(\frac{2p+2}{3}\right)\Gamma\left(\frac{2p+3}{3}\right)}{2\pi|q|\Gamma(p)} x^{\frac{p-1}{q}-1} e^{\frac{1}{2}x^{\frac{3}{q}}} W_{-\frac{2p+1}{3},-\frac{1}{6}}(x^{\frac{3}{q}}), \tag{38}$$

$$f(x) = \frac{3^{p+\frac{1}{2}}\Gamma\left(\frac{2p+1}{3}\right)\Gamma\left(\frac{2p+3}{3}\right)}{2\pi|q|\Gamma(p)} x^{\frac{p-\frac{1}{2}}{q}-1} e^{\frac{1}{2}x^{\frac{3}{q}}} W_{-\frac{2p+\frac{1}{2}}{3},\frac{1}{3}}(x^{\frac{3}{q}}),$$

where $W_{k,m}(x)$ is Whittaker's function (see [1], p. 264).

To obtain new random variables of the same set we shall use the known formula for the function $\Gamma(s)$

$$\frac{1}{\Gamma(s)} = \lim_{n\to\infty} e^{\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)s} \cdot s \cdot \prod_{k=1}^{n}\left(1+\frac{s}{k}\right)e^{-\frac{s}{k}} \qquad (\operatorname{Re} s > 0). \tag{39}$$

We may write the function (14) for $a_r = 1$, $s = it$ in the form

$$h(it) = \frac{\Gamma(p+qit)}{\Gamma(p)} = \lim_{n\to\infty} e^{qit\ln n}\prod_{k=0}^{n}\left[1+\frac{qit}{p+k}\right]^{-1}. \tag{40}$$

We see, that every factor of this product is the characteristic function of some infinitely divisible random variable, so the function (14) is for $s = it$ the characteristic function of some infinitely divisible random variable also. Thus the function

$$h(s) = \left[\frac{\Gamma(p+qs)}{\Gamma(p)}\right]^d \tag{41}$$

is for every positive d the Mellin transform of some random variable. Evidently

$$h(s) = c^s\prod_{j=1}^{N}\left[\prod_{k=1}^{\nu_j}\frac{\Gamma\left(\frac{p+a_k^{(j)}-1+qs}{n_j}\right)}{\Gamma\left(\frac{p+a_k^{(j)}-1}{n_j}\right)}\prod_{k=\nu_j+1}^{n_j}\frac{\Gamma\left(\frac{p+a_k^{(j)}-1-qs}{n_j}\right)}{\Gamma\left(\frac{p+a_k^{(j)}-1}{n_j}\right)}\right]^{d_j}, \tag{42}$$

where $N, n_1, n_2, \ldots, n_N$ are arbitrary positive integers, $c, d_1, d_2, \ldots, d_N$ are arbitrary positive numbers satisfying the condition $\sum_{j=1}^{N} d_j = 1$, $a_1^{(j)}, a_2^{(j)}, \ldots$ $\ldots, a_{n_j}^{(j)}$ are arbitrary permutations of the numbers $1, 2, \ldots, n_j$, ν_j are positive integers satisfying the conditions $0 \leqslant \nu_j \leqslant n_j$, is the Mellin transform of the same set.

In the special case $N = 2$, $n_1 = n_2 = 1$, $\nu_1 = 1$, $\nu_2 = 0$, $a_1^{(1)} = a_1^{(2)} = 1$, $d_1 = d_2 = \frac{1}{2}$ we obtain

$$h(s) = \sqrt{\frac{\Gamma(p+qs)}{\Gamma(p)} \cdot \frac{\Gamma(p-qs)}{\Gamma(p)}}. \tag{43}$$

In the special case $p = \frac{1}{2}$ we obtain the Mellin transform

$$h(s) = \frac{1}{\sqrt{\cos\pi qs}} \tag{44}$$

and the corresponding density

$$f(x) = \frac{\left|\Gamma\left(\frac{1}{4}+i\frac{\ln x}{2\pi q}\right)\right|^2}{(2\pi)^{3/2}\pi|q|x}, \qquad x > 0. \tag{45}$$

Taking the corresponding symmetrical random variables instead of the positive ones in the formulae (35), (37), (38), (42), (43), (45) we obtain random variables belonging to $\mathcal{Y}_*^{**}$.

9. Particular cases. We shall now list the specifications which are to be made in order to see the connection of our considerations with probability distributions encountered in mathematical statistics.

a. If we put $a_1 = a_2 = 1/2$, $p_1 = m/2$, $p_2 = n/2$, $q_1 = q_2 = 1$ in (1) and (2), then Z_1 and Z_2 will have chi-square distributions with m and n degrees of freedom, respectively, and $\frac{n}{m} U$ will have Snedecor's F distribution with (m, n) degrees of freedom.

b. If we put $a_1 = p_1 = q_1 = q_2 = 1/2$, $a_2 = p_2 = n/2$ in (1) and (2), then Z_1^* will be normal with zero mean and unit variance, nZ_2^2 will have chi-square distribution with n degrees of freedom and $U^* = Z_1^*:Z_2$ will have Student distribution with n degrees of freedom.

c. If we put $a_1 = a_2 = p_1 = p_2 = q_1 = q_2 = 1/2$ in (1) and (2), then Z_1^* and Z_2^* will be normally distributed with zero mean and unit variance and $U^{**} = Z_1^*:Z_2^*$ will have Cauchy distribution.

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P R O B L È M E S

P 235, R 2. M. Fréchet, l'auteur du problème (1), vient de signaler qu'il l'a résolu par la négative (2) et que l'espace en question n'est que ce qu'il appelle un *semi-espace de Banach* (3).

VI, p. 36; VIII, p. 289.

(1) envisagé par lui dans les publications: *Sur deux problèmes d'Analyse non résolus*, Colloquium Mathematicum 6 (1958), p. 33-40, *Supplément à l'article „Sur deux problèmes d'Analyse non résolus"*, ibidem 7 (1960), p. 201-204, *L'espace des courbes est-il un espace de Banach?*, Buletinul Institutului Politechnic din Iaşi 5 (IX) (1959), p. 31-34, et *L'espace des courbes est-il un espace de Banach?*, Comptes rendus des séances de l'Académie des Sciences de Paris 250 (1960), p. 248 et 249.

(2) M. Fréchet, *L'espace des courbes n'est pas un espace de Banach*, ibidem 250 (1960), p. 2787-2790, et *Simplification d'une démonstration donnée dans une Note précédente*, ibidem 251 (1960), p. 9.

(3) M. Fréchet, *L'espace dont chaque élément est une courbe n'est qu'un semi-espace de Banach*, ibidem 251 (1960), p. 1258-1260, et *Exemples de semi-espaces de Banach*, ibidem 251 (1960), p. 1702-1703.

P 309, R 1. La réponse est négative (1).

VII.2, p. 311.

(1) A. Pełczyński and V. N. Sudakov, *Remark on non-complemented subspaces of the space m(S)*, Colloquium Mathematicum, ce fascicule, p. 85-88.

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P 348. Formulé dans la communication de L. Szamkołowicz, *Remarks on the Cartesian products of two graphs*.

Ce fascicule, p. 47.

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P 349. Formulé dans la communication *A local property of pointwise periodic homeomorphisms*.

Ce fascicule, p. 65.