Suppose now that the cardinals of V and T are arbitrary. If a is a theorem in $\mathcal{T}$, there exists an open subtheory $\mathcal{T}_0$ of $\mathcal{T}$ such that a is a theorem in $\mathcal{T}_0$, and the sets of all terms and individual variables in $\mathcal{T}_0$ are countable. By the part of (xxi) which has just been proved, there exists a proper Herbrand disjunction δ for a such that δ is a theorem in $\mathcal{T}_0$. Since $\mathcal{T}_0$ is a subtheory of $\mathcal{T}$, δ is also a theorem in $\mathcal{T}$.

(xxii). *Let $\mathcal{T}$ be an open theory. A formula a is a theorem in $\mathcal{T}$ if and only if a proper Herbrand disjunction for a is a theorem in $\mathcal{T}$.*

This follows immediately from (xviii) and (xxi).

(xxiii). *In order that a theory $\mathcal{T}$ be open it is necessary and sufficient that, for every formula a (in the prenex form (12)), a be a theorem in $\mathcal{T}$ if and only if a proper Herbrand disjunction for a is a theorem in $\mathcal{T}$.*

The necessity follows from (xxii). To prove the sufficiency let us associate with every theorem a in $\mathcal{T}$ a proper Herbrand disjunction δ_a which is also a theorem in $\mathcal{T}$. By (xviii) the implication $\bar{\delta}_a \to a$ is a tautology. This proves that the set of all open formulas δ_a is a set of axioms for $\mathcal{T}$. Thus $\mathcal{T}$ is open.

REFERENCES

[1] J. Herbrand, *Recherches sur la théorie de la démonstration*, Prace Towarzystwa Naukowego Warszawskiego, Wydział III, 33 (1930), p. 33-160.

[2] H. Rasiowa and R. Sikorski, *On the isomorphism of Lindenbaum algebras with fields of sets*, Colloquium Mathematicum 5 (1958), p. 143-158.

[3] L. Rieger, *On free $\aleph_\xi$-complete Boolean algebras*, Fundamenta Mathematicae 38 (1951), p. 35-52.

[4] — *O jedné základni větě matematické logiky*, Časopis pro pěstovani Matematiky 80 (1955), p. 217-231.

[5] R. Sikorski, *On Herbrand's theorem*, Colloquium Mathematicum 6 (1958), p. 55-58.

[6] — *Boolean algebras*, Berlin-Göttingen-Heidelberg 1960.

[7] — *A topological characterization of open theories*, Bulletin de l'Académie Polonaise des Sciences, Série des sci. math., astr. et phys., 9 (1961), p. 259-260.

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A KIND OF CATEGORICITY

BY

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The notion of categoricity has been introduced in order to characterize theories which intentionally have only one model. However, the most elaborated formalization of this notion (categoricity in power introduced by Łoś [2] and Vaught [4]) does not correspond to these intuitions. The arithmetic of natural numbers intentionally related to one model is not categorical in any power. The same can be said about the complete theory of real numbers. The aim of this paper is to define a notion of categoricity according to which the classical elementary theories of arithmetics and geometry (and not too many others) would be categorical.

1. DEFINITIONS

Let $\mathrm{Cn}(X)$ be the notion of consequence based on the first order functional calculus. Let $\{A_\Phi\}$ and $\{G_\Phi\}$ be two sequences of constants indexed by the formulas. We define the Skolem forms of a set X ($\mathrm{skl}(X)$) of formulas in the normal prenex form.

If Φ is a formula in the normal prenex form, then

$$\mathrm{skl}'(\Phi) = \begin{cases} \Psi(A_\Phi) \text{ if } \Phi \text{ has the shape } \bigvee x_v \Psi(x_v), \\ \bigwedge x_{k_1}, \ldots, x_{k_n} \Psi(G_\Phi(x_{k_1}, \ldots, x_{k_n})) \text{ if } \Phi \text{ has the shape} \\ \qquad\qquad \bigwedge x_{k_1}, \ldots, x_{k_n} \bigvee x_v \Psi(x_v), \\ \Phi \text{ in other cases.} \end{cases}$$

$\mathrm{skl}(\Phi) = \mathrm{skl}'^n(\Phi)$ for such n that $\mathrm{skl}'^n(\Phi) = \mathrm{skl}'^{n+1}(\Phi)$, $\mathrm{skl}(X)$ = the set of $\mathrm{skl}(\Phi)$ for $\Phi \in X$.

Let X be a set of sentences (formulas without free variables) with extralogical constants: $O_1, \ldots, O_k$ (individual constants), $P_1, \ldots, P_n$ (predicates) and $F_1, \ldots, F_m$ (function-constants).

P_1 is the sign of identity. We suppose that all sets X of the sentences considered below contain the axioms of identity for P_1. We shall say that

1.1. $\mathfrak{M}$ is a *constructive model for* X iff:

1° $\mathfrak{M} = \langle M, o_1, \ldots, o_k, p_1, \ldots, p_n, f_1, \ldots, f_m, \{a_X\}, \{g_X\} \rangle$ is a model for $\mathrm{skl}(X)$, o_i, p_i, f_i are interpretations of extralogical constants of X and $\{a_X\}$, $\{g_X\}$ are interpretations of the added Skolem-constants (i. e. the constants occurring in $\mathrm{skl}(X)$ and not occurring in X) if they exist, and p_1 is the relation of identity,

2° M is the smallest set containing the individuals $o_1, \ldots, o_k$ and $\{a_X\}$ and closed under the functions $f_1, \ldots, f_m$ and $\{g_X\}$.

1.2. A set X of sentences in normal prenex form is *constructively categorical* iff X is consistent and every two constructive models of X are isomorphic.

We shall deal also with terms: 1. the individual constants $o_1, \ldots, o_k$, A_X are terms, 2. if $t_1, \ldots, t_r$ are terms and F_i is one of the primitive function-constants or Skolem function-constants, then $F_i(t_1, \ldots, t_r)$ is a term.

Let us notice the following easy theorem:

1.3. *Every consistent set of sentences has a constructive model.*

Proof. If X is consistent we can define an extension S of $\mathrm{skl}(X)$ as follows:

$$S = \bigcup Z_n, \quad \text{where} \quad Z_0 = \mathrm{skl}(X),$$

and

$$Z_{n+1} = \begin{cases} Z_n \cup \{\ulcorner \sim R^{(n)} \urcorner\} & \text{if} \quad R^{(n)} \notin \mathrm{Cn}(Z_n), \\ Z_n \cup \{\ulcorner R^{(n)} \urcorner\} & \text{in other cases;} \end{cases} \tag{1}$$

where $\{R^{(n)}\}$ is a sequence of all sentences of the form

$$P_i(t_1, \ldots, t_r),$$

where P_i is a predicate of the set X and $t_1, \ldots, t_r$ are terms. The extension S is consistent and determines a model on terms which satisfy definition 1.1.

1.4. *Every model $\mathfrak{M}$ for a set X of sentences contains a submodel $\mathfrak{M}' \subset \mathfrak{M}$ which is constructive for X.*

Proof. We choose from the model $\mathfrak{M}$ the interpretations of the Skolem-constants $\{A_X\}$ and using the axiom of choice we define the functions $\{G_X\}$.

Now for a syntactical characterization of the constructive categoricity we need the notion of decidability of a predicate:

1.5. A predicate P_i is *decidable* in the set Z of sentences iff for all the terms $t_1, \ldots, t_r$ of Z

$$\ulcorner P_i(t_1, \ldots, t_r) \urcorner \in \mathrm{Cn}(Z) \quad \text{or} \quad \ulcorner \sim P_i(t_1, \ldots, t_r) \urcorner \in \mathrm{Cn}(Z). \tag{2}$$

2. THE MAIN PROPERTY

The following theorem gives a criterion of categoricity:

2.1. *A set X is constructively categorical iff each predicate P_i of X is decidable in* $\mathrm{skl}(X)$.

Proof. If the predicate P_i is not decidable in $\mathrm{skl}(X)$, there is a sentence R of the form $P_i(t_1, \ldots, t_r)$ independent of $\mathrm{skl}(X)$. Starting from two sets, $Z_0' = \mathrm{skl}(X) \cup \{\ulcorner R \urcorner\}$ and $Z_0'' = \mathrm{skl}(X) \cup \{\ulcorner \sim R \urcorner\}$, by means of construction (1), we obtain two extensions, S' and S'', of $\mathrm{skl}(X)$ which determine two non-isomorphic constructive models on terms.

Conversely, if two models $\mathfrak{M}$ and $\mathfrak{M}'$ are non-isomorphic, then for any one-one mapping of $|\mathfrak{M}|$ onto $|\mathfrak{M}'|$ there must be elements $m_1, \ldots, m_r \in |\mathfrak{M}|$ such that for a relation p_i

$$\sim \big(p_i(m_1, \ldots, m_r) \wedge p_i'(m_1', \ldots, m_r')\big). \tag{3}$$

Let us consider the following mapping $'$: $n' = m$ if $n \in |\mathfrak{M}|$, $m \in |\mathfrak{M}'|$ and there exists a term t_i such that n and m are interpretations of the term t_i in models $\mathfrak{M}$ and $\mathfrak{M}'$ respectively. $'$ is a well-defined one-one mapping because of the decidability of P_1 in $\mathrm{skl}(X)$. p_1 is the relation of identity (see definition 1.1). Hence if n is the interpretation of two different terms, t_1 and t_2, then from the two possibilities, $\ulcorner P_1(t_1, t_2) \urcorner \in \mathrm{Cn}(\mathrm{skl}(X))$ or $\ulcorner \sim P_1(t_1, t_2) \urcorner \in \mathrm{Cn}(\mathrm{skl}(X))$, only the first can occur, and thus the terms t_1 and t_2 must have a unique interpretation in every model. For the mapping $'$ defined above it is evident that condition (3) may be satisfied only if the sentence $P_i(t_1, \ldots, t_r)$ for some terms $t_1, \ldots, t_r$ is independent of $\mathrm{skl}(X)$.

3. EXAMPLES

The notion of constructive categoricity is non-extensional. This means that there are two axiom systems Q and Q' such that Q and Q' are logically equivalent, Q' is categorical and Q is not. Such is the case of the arithmetic Q of natural numbers considered by R. M. Robinson (1). The original axiom system for Q is non-categorical, as may easily be seen.

Let Q' be the following axiom system:

$$Sx = Sy \rightarrow x = y; \quad 0 \neq Sx; \quad \bigwedge x \bigvee y \big((x = 0 \wedge y = 0) \vee (x = Sy)\big);$$
$$x + 0 = x; \quad x + Sy = S(x+y); \quad x \cdot 0 = 0; \quad x \cdot Sy = (x \cdot y) + x.$$

(1) For the theory Q see [3], p. 51.

3.1. *The system Q' is constructively categorical.*

Proof. The unique axiom of Q' containing an existential quantifier is the third one. Its Skolem-form has the shape

$$\bigwedge x\big((x = 0 \wedge Gx = 0) \vee x = S(Gx)\big). \tag{4}$$

By this sentence the function G is completely characterized as the function of predecessor. For every term t there is an $n \in N$ such that $\ulcorner t = \Delta_n \urcorner \in Cn\big(\mathrm{skl}(Q')\big)$. Hence the predicate $=$ is decidable on terms, and, by 2.1, Q' is constructively categorical.

3.2. *The complete arithmetic B of real numbers has a constructively categorical axioms system.*

Proof. In order to give a complete characterization of all existences all existentional axioms may be formulated in the form of the following scheme:

$$\bigwedge z_0, \ldots, z_{2n+1} \bigvee x[(z_{2n+1} = 0 \wedge x = 0) \vee \{z_{2n+1} \neq 0 \wedge z_0 + z_1 \cdot x + \ldots$$
$$\ldots + z_{2n+1} \cdot x^{2n+1} = 0 \wedge \bigwedge y(z_0 + z_1 \cdot y + \ldots + z_{2n+1} \cdot y^{2n+1} = 0 \to x \leqslant y)\}].$$

Hence $\mathrm{skl}(B)$ contains an infinite sequence of Skolem-functions: $G_1(z_0, z_1)$, $G_3(z_0, z_1, z_2, z_3), \ldots$ denoting the least roots of the polynomials with the coefficients $z_0, \ldots, z_{2n+1}$ if $z_{2n+1} \neq 0$ and 0 if $z_{2n+1} = 0$. Every function G_{2n+1} is formally definable in the theory B. All terms t_i, t_j of $\mathrm{skl}(B)$ are thus definable in B and all sentences of the form $t_i = t_j$ or $t_i < t_j$ are decided in $\mathrm{skl}(B)$ because of the completeness of B. Hence the predicates $=$ and $<$ are decidable in $\mathrm{skl}(B)$ on terms.

Theorem 3.2 can easily be generalized:

3.3. *If T is a complete axioms-system and Skolem-functions for T are definable in* $\mathrm{Cn}(T)$, *then T is constructively categorical.*

Proof. Analogically to the proof of 3.2.

If we cancel a number of axioms from the theories Q or B without reducing the number of primitive notions we obtain a theory which is non-categorical, but has a constructively categorical recursively enumerable extension. There are also axioms-systems which have no constructively categorical extensions.

3.4. *The set theory of E. Zermelo has an axioms-system with no consistent recursively enumerable categorical extensions.*

Proof. Let Z be the axioms-system of set theory with the familiar "Aussonderungs-axiom" and the axiom of infinity assuming the existence of the set of von Neumann's natural numbers: $\Lambda, \{\Lambda\}, \{\Lambda, \{\Lambda\}\}, \ldots$. Let A_1 be the term denoting in $\mathrm{skl}(Z)$ the set of natural numbers whose existence is assumed in the axiom of infinity. Let $\{\Delta_n\}$ be the sequence

of numerals denoting in $\mathrm{skl}(Z)$ the above-mentioned von Neumann natural numbers. Hence $\ulcorner \Delta_n \in A_1 \urcorner \in \mathrm{Cn}\big(\mathrm{skl}(Z)\big)$.

It is easy to see that the implication $\mathrm{skl}'(\Phi) \to \Phi$ is logically true for every formula Φ. Hence from the Skolem-form of the "Aussonderungs-axiom" we infer that for every formula Ψ of Z there exists a term t_Ψ of $\mathrm{skl}(Z)$ such that the formula

$$\bigwedge x\big[x \in t_\Psi \equiv \big(x \in A_1 \wedge \Psi(x)\big)\big] \tag{5}$$

is a theorem of $\mathrm{skl}(Z)$.

The set S of natural numbers is weakly representable by the formula Ψ in the set T of sentences iff for every natural number n

$$n \in S \equiv \ulcorner \Psi(\Delta_n) \urcorner \in \mathrm{Cn}(T). \tag{6}$$

The arithmetic of natural numbers is interpretable in Z. Hence there exists a formula Ψ of Z such that in every extension Z' of Z the formula Ψ represents a non-recursive set $S_{Z'}$ (²). Hence from (5) and (6) it follows that for every extension Z' of Z there exists a non-recursive set $S_{Z'}$ such that

$$n \in S_{Z'} \equiv \ulcorner \Delta_n \in t_\Psi \urcorner \in \mathrm{Cn}\big(\mathrm{skl}(Z')\big).$$

If Z' is recursively enumerable, the set of n such that $\ulcorner \Delta_n \in t_\Psi \urcorner \in \mathrm{Cn}\big(\mathrm{skl}(Z')\big)$ as well as the set of n such that $\ulcorner \sim(\Delta_n \in t_\Psi) \urcorner \in \mathrm{Cn}\big(\mathrm{skl}(Z')\big)$ are recursively enumerable. Thus if $S_{Z'}$ is non-recursive, the predicate $\in$ cannot be decidable in $\mathrm{skl}(Z')$. Hence, by 2.1, Z' is non-categorical.

In the last theorem by the extension of a set T of sentences we understand each set X of sentences such that $T \subset X$. The problem whether theorem 3.4 remains true if as an extension of T we mean sets X such that $T \subset \mathrm{Cn}(X)$ remains open (**P 371**).

REFERENCES

[1] A. Grzegorczyk, *Some proofs of undecidability of arithmetics*, Fundamenta Mathematicae 43 (1956), p. 178-189.

[2] J. Łoś, *On the categoricity in power of elementary deductive systems and some related problems*, Colloquium Mathematicum 3 (1954), p. 58-62.

[3] A. Tarski, *Undecidable theories*, Amsterdam 1953.

[4] R. L. Vaught, *Applications of the generalized Skolem-Löwenheim theorem to problems of completeness and decidability*, Bulletin of the American Mathematical Society 59 (1953), p. 396-397.

(²) For the discussion of such formulas see [1].

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