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ON A NOTION OF UNIFORMITY FOR L-SPACES OF FRÉCHET

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The purpose of this paper* is to introduce a notion of uniformity into Fréchet spaces (spaces with a given convergence of sequences). Our definition is related to the notion of δ-spaces of Efremovič [1] but, in general, a Fréchet space with uniformity in our sense is not a topological one, and even if it is, its topology may not be completely regular, while every δ space, as well as the uniform spaces of A. Weil, must be completely regular.

For our spaces, called $\mathscr{U}\mathscr{L}$-spaces, we define the notions of uniform continuity of functions and of uniform convergence which are generalizations of the corresponding notions for metric spaces. A similar theory could be developed for sets instead of sequences.

1. DEFINITIONS AND EXAMPLES

1.0. Let X be an abstract space; its elements will be denoted by $x, x', \ldots, a, b, \ldots$; sequences of elements by $\{x_n\}$, $\{x_n'\}, \ldots$ or by small Greek letters $\xi, \xi', \eta, \ldots, \alpha, \ldots$; $\{x\}$ will denote the constant sequence, i. e. the sequence $\{x_n\}$ in which $x_n = x$ for every n. If necessary, the index of the sequence will specially be indicated, e. g. $\{x_n^{(i)}\}_n$ denotes for each constant i a sequence with index n, $\{x_n^{(i)}\}_i$ denotes for each constant n a sequence with index i.

Sequences of natural numbers which appear in this paper are supposed to be increasing.

1.1. Let us consider in X a relation $\mathbf{n}$ between sequences of elements of X; $\mathbf{n}$ is called a *nearness relation* if it satisfies the following conditions:

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(1) See [1]; a detailed account of δ-spaces is given in Smirnov [6].

(i) $\xi\mathbf{n}\xi$;
(ii) if $\xi\mathbf{n}\xi'$, then $\xi'\mathbf{n}\xi$;
(iii) if $\xi\mathbf{n}\xi'$ and $\xi'\mathbf{n}\xi''$, then $\xi\mathbf{n}\xi''$;
(iv) $\{x\}\mathbf{n}\{x'\}$ if and only if $x = x'$;
(v) if $\{x_i\}\mathbf{n}\{x_i'\}$, then $\{x_{i_n}\}\mathbf{n}\{x'_{i_n}\}$ for each sequence $\{i_n\}$ of indices;

and eventually the condition

(vi) if each sequence $\{i_k\}$ of natural numbers contains a subsequence $\{j_k\}$ for which the relation $\{x_{j_n}\}\mathbf{n}\{x'_{j_n}\}$ holds, then $\{x_i\}\mathbf{n}\{x_i'\}$.

A set X in which a relation $\mathbf{n}$ satisfying the conditions (i)-(v) is defined is called a *$\mathscr{UL}$-space*. If, moreover, the condition (vi) is satisfied, X with the relation $\mathbf{n}$ is called a *$\mathscr{UL}^*$-space*. We denote $\mathscr{UL}$-spaces ($\mathscr{UL}^*$-spaces) by $(X, \mathbf{n})$ or, if no confusion can arise, simply by X.

Examples

1.2. $\{x_n\}\mathbf{n}\{x_n'\}$ if $x_n = x_n'$ for sufficiently large n. $(X, \mathbf{n})$ is a $\mathscr{UL}^*$-space. It is called *trivial* $\mathscr{UL}^*$-space.

1.3. X is a metric space with the distance function ϱ. $\{x_n\}\mathbf{n}\{x_n'\}$ if $\lim\limits_{n\to\infty}\varrho(x_n, y_n) = 0$. $(X, \mathbf{n})$ is a $\mathscr{UL}^*$-space. When speaking of metric spaces we shall always regard them as $\mathscr{UL}^*$-spaces with the nearness relation just defined.

1.4. Given a set X, let $\mathscr{F} = \{f\}$ be a family of mappings of X into the $\mathscr{UL}$-space ($\mathscr{UL}^*$-space) $(Y, \mathbf{n})$. Suppose that the family $\mathscr{F}$ separates the points of X, i. e. that for each two points x_1 and x_2 of X there exists a function $f\in\mathscr{F}$ for which $f(x_1) \neq f(x_2)$. Then a nearness relation in X can be defined by setting

$$\{x_i\}\mathbf{N}\{x_i'\} \text{ if } \{f(x_i)\}\mathbf{n}\{f(x_i')\} \text{ for each } f\in\mathscr{F}.$$

$(X, \mathbf{N})$ is a $\mathscr{UL}$-space ($\mathscr{UL}^*$-space); it is said to be *generated* by the family of mappings $\mathscr{F}$.

1.5. A subset Y of a $\mathscr{UL}$- ($\mathscr{UL}^*$-) space $(X, \mathbf{n})$ with the relation $\mathbf{n}|Y$ (the restriction of $\mathbf{n}$ to Y) is a $\mathscr{UL}$- ($\mathscr{UL}^*$-) space itself. It is called a *$\mathscr{UL}$- ($\mathscr{UL}^*$-) subspace of* X.

1.6. Given two $\mathscr{UL}$- ($\mathscr{UL}^*$-) spaces $(X, \mathbf{n})$ and $(Y, \mathbf{m})$, let us define a nearness relation $\mathbf{n}\times\mathbf{m}$ between sequences of elements of $X\times Y$ by setting

$$\{(x_n, y_n)\}\mathbf{n}\times\mathbf{m}\{x_n', y_n')\} \text{ if } \{x_n\}\mathbf{n}\{x_n'\} \text{ and } \{y_n\}\mathbf{m}\{y_n'\}.$$

$(X\times Y, \mathbf{n}\times\mathbf{m})$ is a $\mathscr{UL}$- ($\mathscr{UL}^*$-) space called the *product* of $(X, \mathbf{n})$ and $(Y, \mathbf{m})$.

Similarly, the product of an arbitrary family of $\mathscr{UL}$- ($\mathscr{UL}^*$-) spaces $(X_t, \mathbf{n}_t)_{t\in T}$ can be defined. In particular, the space X^T of all mappings of T into the $\mathscr{UL}$- ($\mathscr{UL}^*$-) space X can be considered as a $\mathscr{UL}$- ($\mathscr{UL}^*$-) space.

1.7. Let $\mathscr{F}$ be a non-empty family of one-to-one mappings of Y into X, and let $(Y, \mathbf{n})$ be a $\mathscr{UL}$-space. Suppose that for each pair of mappings f and $g\in\mathscr{F}$ there exists such a mapping $h\in\mathscr{F}$ that $\{h^{-1}(x_n)\}\mathbf{n}\{h^{-1}(x_n')\}$ whenever $\{f^{-1}(x_n')\}\mathbf{n}\{f^{-1}(x_n)\}$ or $\{g^{-1}(x_n)\}\mathbf{n}\{g^{-1}(x_n')\}$, and let us define a nearness relation $\mathbf{N}$ in X by setting $\{x_n\}\mathbf{N}\{x_n'\}$ if there exists a mapping $f\in\mathscr{F}$ satisfying the condition $\{f^{-1}(x_n)\}\mathbf{n}\{f^{-1}(x_n)\}$. Then $(X, \mathbf{N})$ is a $\mathscr{UL}$-space.

Indeed, conditions (i), (ii), (v) are evident, (iv) is satisfied because every $f\in\mathscr{F}$ is one-to-one; (iii) follows from our supposition about $\mathscr{F}$.

In particular, Y may be a subset of X, and the family $\mathscr{F}$ may be supposed to include the injection mapping. In this case the relation $\{y_n\}\mathbf{n}\{y_n'\}$ implies $\{y_n\}\mathbf{N}\{y_n'\}$.

2. $\mathscr{UL}$-SPACES AND CONVERGENCE

2.1. A sequence $\{x_n\}$ of elements of a $\mathscr{UL}$-space $(X, \mathbf{n})$ is said to *converge to* $a\in X$, in symbols

$$x_n \to a \quad \text{or} \quad a = \lim x_n,$$

if $\{x_n\}\mathbf{n}\{a_n\}$; a is called the *limit* of the sequence $\{x_n\}$. This convergence is said to be *generated* by the nearness relation $\mathbf{n}$.

It is quite obvious that:

THEOREM. *The set X with the convergence defined above is an $\mathscr{L}$-space of Fréchet. If, moreover, $(X, \mathbf{n})$ is a $\mathscr{UL}^*$-space, X with this convergence is an $\mathscr{L}^*$-space* [2].

Therefore, we may apply to $\mathscr{UL}$-spaces all the notions which are defined for Fréchet spaces as, for example, that of derived set, density, compactness, continuity of functions etc.

Examples

2.2. In example 1.2 the only convergent sequences are constant for sufficiently large indices.

2.3. In example 1.3 the convergence coincides with the usual convergence in metric spaces.

2.4. In example 1.4 the formula $a = \lim x_n$ is equivalent to $f(a) = \lim f(x_n)$ for every $f\in\mathscr{F}$. In the particular case of a Banach space and the family $\mathscr{F}$ of all linear functionals this convergence coincides with the well-known *weak convergence*.

[2] In what concerns $\mathscr{L}^*$-spaces we follow Kuratowski [4], ch. II, § 14.

2.5. In example 1.5 precisely those sequences are convergent in Y which are convergent in X to a limit belonging to Y.

2.6. In examples 1.6 the convergence is equivalent to the convergence of the projections on all axes.

2.7. In the particular case of example 1.7 the convergence coincides with the "operational convergence" of J. Mikusiński ([5], Appendix, ch. IV, § 2).

3. $\mathcal{UL}$-STRUCTURES IN A GIVEN $\mathcal{L}$-SPACE

3.1. Given an $\mathcal{L}$- ($\mathcal{L}^*$-) space X, a nearness relation $\mathbf{n}$ is said to be *compatible* with the convergence of the given space if the convergence generated by $\mathbf{n}$ coincides with the given convergence. In this case we also say that $\mathbf{n}$ determines a $\mathcal{UL}$- (or $\mathcal{UL}^*$-) structure in the given $\mathcal{L}$- ($\mathcal{L}^*$-) space.

There may exist, of course, more nearness relations compatible with the same convergence.

3.2. A nearness relation $\mathbf{n}$ is said to *majorize* the relation $\mathbf{n}'$, both relations being defined in the same abstract set X, in symbols

$$\mathbf{n} \geqslant \mathbf{n}' \quad \text{or} \quad \mathbf{n}' \leqslant \mathbf{n},$$

if for every two sequences ξ and ξ' the relation $\xi \mathbf{n}' \xi'$ implies $\xi \mathbf{n} \xi'$; $\mathbf{n}'$ is said to *minorize* $\mathbf{n}$.

The relation $\leqslant$ is a partial order relation in the set of all $\mathcal{UL}$-structure defined in X.

3.3. Given a set X, let us consider an arbitrary family $(\mathbf{n}_t)_{t\epsilon T}$ of $\mathcal{UL}$-structures in X, and define a nearness relation $\mathbf{n}$ by setting $\xi \mathbf{n} \xi'$ if $\xi \mathbf{n}_t \xi'$ for each $t \epsilon T$.

It is easy to verify that $\mathbf{n}$ *is the largest $\mathcal{UL}$-structure minorizing all the $\mathbf{n}_t$'s.* We denote it by

$$\mathbf{n} = \bigwedge_{t\epsilon T} \mathbf{n}_t$$

or, if T is finite, by

$$\mathbf{n} = \mathbf{n}_1 \wedge \mathbf{n}_2 \wedge \ldots \mathbf{n}_k \wedge .$$

It is clear that if all the $\mathbf{n}_t$'s generate the same convergence in X, *the $\mathcal{UL}$-structure $\wedge \mathbf{n}_t$ is compatible with it.*

If all the spaces $(X, \mathbf{n}_t)$ are $\mathcal{UL}^*$-spaces $(X, \bigwedge_{t\epsilon T} \mathbf{n}_t)$ is also a $\mathcal{UL}^*$-space.

3.4. Supposing now that X is an $\mathcal{L}$-space and all the $\mathbf{n}_t$'s are compatible with the given convergence, let us set $\xi \mathbf{N} \xi'$ if there exist such a finite system of sequences $\eta_0 = \xi, \eta_1, \eta_2, \ldots, \eta_k = \xi'$ and an adequate system $t_1, t_2, \ldots, t_k$ of elements of T that

$$\eta_{i-1} \mathbf{n}_{t_i} \eta_i \quad (i = 1, 2, \ldots, k).$$

The relation $\mathbf{N}$ is a nearness relation which defines the least $\mathcal{UL}$-structure majorizing all the structures $\mathbf{n}_t$ for $t \epsilon T$. We denote it by

$$\mathbf{N} = \bigcup_{t\epsilon T} \mathbf{n}_t$$

or by

$$\mathbf{N} = \mathbf{n}_1 \cup \mathbf{n}_2 \cup \ldots \cup \mathbf{n}_k$$

if T consists of finilety many elements.

Evidently N *is compatible with the given convergence.*

3.5. Given an $\mathcal{L}$-space X, let us define a nearness relation $\mathbf{n}_0$ by setting $\{x_n\}\mathbf{n}_0\{x'_n\}$ if either $x_n = x'_n$ for every n or both sequences are convergent and $\lim x_n = \lim x'_n$.

It is evident that $\mathbf{n}_0$ *is the least $\mathcal{UL}$-structure compatible with the given convergence.*

Note that $(X, \mathbf{n}_0)$ is not necessarily a $\mathcal{UL}^*$-space even if the given space X was an $\mathcal{L}^*$-space.

3.6. Let us now set $\{x_n\}\mathbf{n}_I\{x'_n\}$ if for each sequence of natural numbers $\{i_n\}$ the sequences $\{x_{i_n}\}$ and $\{x'_{i_n}\}$ either are both divergent or both converge to the same limit.

It is easy to see that $(X, \mathbf{n}_I)$ *is a $\mathcal{UL}$-space, and $\mathbf{n}_I$ is the largest $\mathcal{UL}$-structure compatible with the given convergence.*

3.7. The results of 3.3, 3.4, 3.5 and 3.6 may be summarized as follows:

THEOREM. *The set of the $\mathcal{UL}$-structures of a given set X forms an absolutely multiplicative semilattice. The subset of the $\mathcal{UL}$-structures compatible with a given convergence in X forms a subsemilattice which is itself a lattice with the least element $\mathbf{n}_0$ and the largest element $\mathbf{n}_I$.*

3.8. Given a $\mathcal{UL}$-space $(X, \mathbf{n})$, let us set $\{x_n\}\,\mathbf{n}^*\{x'_n\}$ if every sequence $\{i_n\}$ of natural numbers contains a subsequence $\{j_n\}$ for which $\{x_{j_n}\}\mathbf{n}\{x'_{j_n}\}$.

It is easy to show that $(X, \mathbf{n}^*)$ *is a $\mathcal{UL}^*$-space and that $\mathbf{n}^*$ is the least $\mathcal{UL}^*$-structure majorizing the $\mathcal{UL}$-structure* $\mathbf{n}$.

In particular, *if X is a given $\mathcal{L}$-space and $\mathbf{n}_0$ the nearness relation defined in* 3.5, *then $(X, \mathbf{n}_0^*)$ is a $\mathcal{UL}^*$-space, and $\mathbf{n}_0^*$ is the least $\mathcal{UL}^*$-structure in X which preserves the convergence*, i. e. $x_n \to a$ implies $\{x_n\}\mathbf{n}_0^*\{a\}$.

However, $\mathbf{n}_0^*$ *is compatible with the given convergence if and only if the space X is an* $\mathcal{L}^*$-space.

Thus, we obtain a construction of an $\mathcal{L}^*$-convergence in X from a given $\mathcal{L}$-convergence, by passing through the $\mathcal{UL}^*$-structure $\mathbf{n}_0^*$ defined for that $\mathcal{L}$-convergence. It leads to the same $\mathcal{L}^*$-convergence as the procedures used by Urysohn [7] or Kisyński [3].

3.9. Similarly, starting from the relation $\mathbf{n}_I$ of 3.6 we obtain the $\mathcal{UL}^*$-structure $\mathbf{n}_I^*$. $\mathbf{n}_I^*$ preserves, of course, the given convergence; more precisely, *it is the largest $\mathcal{UL}^*$-structure compatible with the convergence generated by* $\mathbf{n}_0^*$.

If X with the given convergence is an $\mathcal{L}^*$-space, then $\mathbf{n}_I^*$ *is the largest $\mathcal{UL}^*$-structure compatible with that convergence.*

3.10. Let X be an $\mathcal{L}^*$-space, $(\mathbf{n}_t)_{t\in T}$ a family of $\mathcal{UL}^*$-structures compatible with the convergence in X, and $\mathbf{N} = \bigcup_{t\in T}\mathbf{n}_t$ (cf. 3.4). We denote the relation $\mathbf{N}^*$ obtained from $\mathbf{N}$ by the procedure of 3.8 by

$$\mathbf{N}^* = \bigvee_{t\in T}\mathbf{n}_t$$

or, respectively,

$$\mathbf{N}^* = \mathbf{n}_1 \vee \mathbf{n}_2 \vee \ldots \vee \mathbf{n}_k.$$

$\mathbf{N}^*$ *is the least $\mathcal{UL}^*$-structure majorizing all the* $\mathbf{n}_t$*'s.* Evidently, it is *compatible with the given convergence.*

3.11. Combining the results of 3.3, 3.8, 3.9 and 3.10 we may state the following

THEOREM. *The set of all $\mathcal{UL}^*$-structures of a given set X forms an absolutely multiplicative semilattice (which is a subsemilattice of that of* 3.7). *The set of $\mathcal{UL}^*$-structures compatible with the same $\mathcal{L}^*$-convergence in X forms its subsemilattice, which itself is a lattice and contains the least element* $\mathbf{n}_0^*$ *and the largest element* $\mathbf{n}_I^*$.

However, the last lattice is not, in general, a sublattice of the corresponding lattice of 3.6.

3.12. An $\mathcal{L}$- ($\mathcal{L}^*$-) space X is called *compact* if each sequence $\{x_n\}$ of elements of X contains a convergent subsequence.

If X is compact, then the case of $\{x_{j_n}\}$ and $\{x'_{j_n}\}$ being both divergent may be omitted in the definitions of $\mathbf{n}_0^*$ and $\mathbf{n}_I^*$. Hence $\mathbf{n}_0^* = \mathbf{n}_I^*$ and by 3.7 *this is the only $\mathcal{UL}^*$-structure compatible with the convergence generated by* $\mathbf{n}_0^*$. As a corollary we obtain the following

THEOREM. *Is X a compact $\mathcal{L}^*$-space, there exists one and only one $\mathcal{UL}^*$-structure compatible with the given convergence.*

4. $\mathcal{US}$- ($\mathcal{US}^*$-) SPACES

4.1. According to Fréchet [2] an $\mathcal{L}$-space (respectively $\mathcal{L}^*$-space) X is called an *$\mathcal{S}$-space* (*$\mathcal{S}^*$-space*) if the convergence satisfies the following condition:

(s) if $\lim\limits_{i\to\infty} x_i^{(n)} = x^{(n)}$ and $\lim\limits_{n\to\infty} x^{(n)} = a$, then there exists such a sequence $\{i_n\}$ of natural numbers that $\lim\limits_{n\to\infty} x_{i_n}^{(n)} = a$.

If the *closure* $\bar{Z}$ of the subset Z of the $\mathcal{S}$-space X is defined as the set of all limits of sequences of elements of Z, then $\bar{\bar{Z}} = \bar{Z}$ *and X is a T_1 topological space.* The topological convergence coincides, however, with the given convergence if and only if X is an $\mathcal{S}^*$-space.

4.2. We shall introduce a special kind of nearness in $\mathcal{S}$- ($\mathcal{S}^*$-) spaces $(X, \mathbf{n})$ is called a *$\mathcal{US}$-space* (*$\mathcal{US}^*$-space*) if $\mathbf{n}$ is a nearness relation satisfying the conditions (i)-(v) (respectively (i)-(vi)) and, moreover, the following condition:

(vii) Given two sequences $\{\xi_n\}$ and $\{\xi'_n\}$, where $\xi_n = \{x_i^{(n)}\}_i$ and $\xi'_n = \{x_i'^{(n)}\}_i$, if $\xi_n \mathbf{n}\, \xi'_n$ for every n then there exists such a sequence $\{i_n\}$ of natural numbers that $\{x_{j_n}^{(n)}\}\mathbf{n}\{x_{j_n}'^{(n)}\}$ for each sequence $\{j_n\}$ with $j_n \geqslant i_n$, $n = 1, 2, \ldots$

Examples. Metric spaces (example 1.3) are $\mathcal{US}^*$-spaces. The product of a finite or denumerable family of $\mathcal{US}^*$-spaces is a $\mathcal{US}^*$-space. Subspaces of $\mathcal{US}$- ($\mathcal{US}^*$-) spaces are $\mathcal{US}$- ($\mathcal{US}^*$-) spaces. However, the spaces of examples 1.4 and 1.7 are not, in general, $\mathcal{US}$-spaces.

4.3. Evidently, if $(X, \mathbf{n})$ is a $\mathcal{US}$- ($\mathcal{US}^*$-) space, then the *set X with the convergence generated by $\mathbf{n}$ is an $\mathcal{S}$- ($\mathcal{S}^*$-) space.*

As we have seen, for every $\mathcal{L}$- ($\mathcal{L}^*$-) space there exists a $\mathcal{UL}$- ($\mathcal{UL}^*$-) structure compatible with the given convergence. These $\mathcal{UL}$- ($\mathcal{UL}^*$-) structures are generally not $\mathcal{US}$- ($\mathcal{US}^*$-) structures even if the given space is an $\mathcal{S}$- ($\mathcal{S}^*$-) space. It seems that not for each $\mathcal{S}^*$-space a compatible $\mathcal{US}^*$-structure exists. The question of uniformisability of $\mathcal{S}^*$-spaces needs further investigations.

At any rate, if $(\mathbf{n}_t)_{t\in T}$ are $\mathcal{US}^*$-structures of the given X and card $T \leqslant \aleph_0$, then $\bigwedge_{t\in T}\mathbf{n}$ is a $\mathcal{US}^*$-structure.

5. UNIFORM CONTINUITY

5.1. Let $(X, \mathbf{n})$ and $(Y, \mathbf{N})$ be two $\mathcal{UL}$-spaces. A function f defined on X with values in Y is called $(\mathbf{n}, \mathbf{N})$*-uniformly continuous* (when no confusion can arise we simply say uniformly continuous) if for each two sequences $\{x_n\}$ and $\{x'_n\}$ of points of X

$$\{x_n\}\mathbf{n}\{x'_n\} \quad \text{implies} \quad \{f(x_n)\}\mathbf{N}\{f(x'_n)\}.$$

It is quite evident that every uniformly continuous function is continuous when X and Y are regarded as $\mathcal{L}$-spaces with the generated convergence.

5.2. It is clear that *if $\mathbf{n}_1 \leqslant \mathbf{n}_2$ and $f(x)$ is $(\mathbf{n}_2, \mathbf{N})$-uniformly continuous, it is also $(\mathbf{n}_1, \mathbf{N})$-uniformly continuous.*

Similarly, *if $\mathbf{N}_1 \leqslant \mathbf{N}_2$ every $(\mathbf{n}, \mathbf{N}_1)$-uniformly continuous function is $(\mathbf{n}, \mathbf{N}_2)$-uniformly continuous as well.*

5.3. THEOREM. *Let X be an $\mathcal{L}^*$-space and $(Y, \mathbf{N})$ a $\mathcal{UL}^*$-space, and let $\mathbf{n}_0^*$ denote the least $\mathcal{UL}^*$-structure in X compatible with the convergence* in X (cf. 3.8). *Then every continuous function in X with values in Y is* $(\mathbf{n}_0^*, \mathbf{N})$*-uniformly continuous.*

Indeed, suppose $\{x_n\}\mathbf{n}_0^*\{x_n'\}$; then each sequence $\{i_n\}$ of natural numbers contains a subsequence $\{j_n\}$ for which either $x_{j_n} = x'_{j_n}$ $(n = 1, 2, \ldots)$ or $\lim x_{j_n} = \lim x'_{j_n}$. The function f being continuous, the same holds for the sequences $\{f(x_{j_n})\}$, $\{f(x'_{j_n})\}$ and the nearness relation $\mathbf{N}$. That means that $\{f(x_n)\}\mathbf{N}_0^*\{f(x_n')\}$, where $\mathbf{N}_0^*$ is the least $\mathcal{UL}^*$-structure in Y compatible with the convergence generated by $\mathbf{N}$. Hence

$$\{f(x_n)\}\mathbf{N}\{f(x_n')\}.$$

Similarly, *if* $(X, \mathbf{n})$ *is a* $\mathcal{UL}^*$*-space and* Y *an* $\mathcal{L}^*$*-space, then every continuous function* $f\colon X \to Y$ *is* $(\mathbf{n}, \mathbf{N}_I^*)$*-uniformly convergent*, where $\mathbf{N}_I$ denotes the largest $\mathcal{UL}^*$*-structure* compatible with the convergence in Y (cf. 3.9).

Note that it may be proved by a similar argument that *every* $(\mathbf{n}, \mathbf{N})$*-uniformly continuous function is* $(\mathbf{n}^*, \mathbf{N}^*)$*-uniformly continuous*, where $\mathbf{n}$ and $\mathbf{N}$ are $\mathcal{UL}$-structures in X and Y respectively, and $\mathbf{n}^*$, $\mathbf{N}^*$ denote the corresponding $\mathcal{UL}^*$-structures defined as in (3.8).

5.4. Let $\{f_n\}$ be a sequence of functions defined in the $\mathcal{UL}$-space $(X, \mathbf{n})$ with values in the $\mathcal{UL}$-space $(Y, \mathbf{N})$. The sequence $\{f_n\}$ is called *Σ-continuous* if for each pair of sequences $\{x_n\}$ and $\{x_n'\}$ satisfying the condition $\{x_n\}\mathbf{n}\{x_n'\}$ there exists a sequence $\{i_n\}$ of natural numbers for which $\{f_n(x_{i_n})\}\mathbf{N}\{f_n(x'_{i_n})\}$.

Σ-continuity is of course a property of the sequence, not of single functions.

Setting $y_k^{(n)} = f_n(x_k)$ we immediately see that *if* $(Y, \mathbf{N})$ *is a* $\mathcal{US}$*-space, every sequence of uniformly continuous functions is* Σ*-continuous.*

6. *UNIFORM CONVERGENCE*

6.1. A sequence $\{f_n\}$ of functions defined on X with values in the $\mathcal{UL}$-space $(Y, \mathbf{N})$ is said to *converge uniformly* to the function f, in symbols $f_n \rightrightarrows f$, if for each sequence $\{x_n\}$ of elements of X

$$\{f_n(x_n)\}\mathbf{N}\{f(x_n)\}.$$

It is clear that the uniform convergence $f_n \rightrightarrows f$ implies the convergence $f_n(x) \to f(x)$ for every $x \in X$, and that a subsequence of a uniformly convergent sequence of functions is itself uniformly convergent to the same limit function. Thus, *the uniform convergence is an $\mathcal{L}$-convergence in the set of all functions defined on X with values in Y. If, moreover,* $\mathbf{N}$ *is a* $\mathcal{UL}^*$*-structure in* Y*, this convergence is of* $\mathcal{L}^*$*-type.*

6.2. From 5.4 and 6.1 we directly obtain the following

THEOREM. *The limit function of every uniformly convergent Σ-continuous sequence of functions is uniformly continuous. Is* $(Y, \mathbf{N})$ *a* $\mathcal{US}^*$*-space, then every uniformly convergent sequence of uniformly continuous functions has a uniformly continuous limit function.*

In particular, in the case when Y is a metric space, this coincides with the well-known fact about uniformly convergent sequences of continuous functions.

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