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ON A QUASI-ORDERING IN THE CLASS OF CONTINUOUS MAPPINGS OF A CLOSED INTERVAL INTO ITSELF

BY

J. MIODUSZEWSKI (WROCŁAW)

1. Introduction. Let us consider the class C of all continuous mappings of the closed interval $I = \langle 0, 1\rangle$ onto itself. C forms a semigroup with respect to the superposition of mappings with unit e which is an identity mapping. The operation of superposition induces in a natural manner a quasi-ordering in C (i. e. a relation which is reflexive and transitive). Let $f, g \in C$. We say that

(1) $f \prec g$ *if there exists an* $a \in C$ *such that* $fa = g$.

The reflexivity and transitivity of $\prec$ is obvious. We have $e \prec f$ for every $f \in C$. Let us consider the equivalence relation associated with $\prec$:

(2) $f \sim g$ *if* $f \prec g$ *and* $g \prec f$.

There are some difficulties in the investigation of these relations on the whole of C. Namely, there exist pairs of mappings (see, for instance, the example in [4], p. 340) which are incomparable with respect to $\prec$, but which are defined in the relation $\prec'$ for $f, g \in C$ as follows:

(3) $f \prec' g$ *if for every* $\varepsilon > 0$ *there exists an* $a \in C$ *such that* $fa \underset{\varepsilon}{=} g$.

The meaning of $f \underset{\varepsilon}{=} g$ is: $|f(x) - g(x)| \leqslant \varepsilon$ for every $x \in I$.

The reflexivity and transitivity of $\prec'$ is obvious. The equivalence relation associated with $\prec'$ will be denoted by $\sim'$.

However, on some subclasses of C, which are also subsemigroups, the situation is easier than on the whole of C. We shall distinguish the class $R \subset C$ of all mappings which are not constant on subintervals of I. We shall prove (Theorem 1) that on R the terms $\prec$ and $\sim$ are equivalent to the terms $\prec'$ and $\sim'$ respectively. We shall also distinguish the class $S \subset C$ of all simplicial (piece-wise linear) mappings. We shall prove that R and S form directed sets with respect to $\prec$, i. e. for every $f, g \in R$ (or $f, g \in S$) there exists an $h \in R$ (or $h \in S$) such that $f \prec h$ and $g \prec h$ (Theorem 2). This was proved, in fact, with some additional conditions in [1], [2] and [4]. The meaning of relation $\sim$ on R is explained by Theorem 3 and a corollary to it. Namely, $f \sim g$ implies the existence of homeomorphisms $\alpha, \beta \in C$ such that $f\alpha = g$ and $g\beta = f$.

Throughout this paper we shall use the method of double graphs of two mappings introduced in [1] and [4]. We shall often say that g is factorized by f instead of $f \prec g$. The finding of $\alpha \in C$ such that $f\alpha = g$ we shall call a *factorization* of g by f. The finding of $\alpha, \beta \in C$ such that $f\alpha = g\beta$ we shall call a *uniformization* of f and g.

2. Double graphs. Let $f, g \in C$. The subset of I^2

$$[f, g] = \{(x, y)\colon f(x) = g(y)\}$$

is said to be a *double graph* of f and g. The double graphs are convenient in the investigation of factorization and uniformization of mappings according to the following property, which is obvious:

(4) *Let $f, g \in C$. The curve $x = \alpha(t)$, $y = \beta(t)$, $t \in I$, lies in $[f, g]$ if and only if $f\alpha = g\beta$.*

The local structure of double graphs of mappings $f, g \in C$ is explained by the following lemma (see [3]):

(5) *For every $(x, y) \in [f, g]$ there exists a decreasing family of rectangles $P = I' \times I''$ having (x, y) in interiors and such that for each P the boundary of P intersects $[f, g]$ at vertices of P only, and the intersection of all P in question is the component of (x, y) in $\{(\xi, \eta)\colon f(\xi) = g(\eta) = a\}$, where a is the common value of $f(x)$ and $g(y)$.*

We define a family $\{P_\varepsilon\} = I'_\varepsilon \times I''_\varepsilon$ of rectangles, where $\varepsilon > 0$. Let $a' = \max(0, a - \varepsilon)$ and $a'' = \min(1, a + \varepsilon)$. We define I'_ε and I''_ε as the components of x and y in sets $\{\xi\colon a' < f(\xi) < a''\}$ and $\{\eta\colon a' < g(\eta) < a''\}$ respectively. It is easy to verify that rectangles P_ε have the required properties (for details, see [3]).

Note that if $f, g \in R$, then the intersection of P_ε is (x, y) and therefore $[f, g]$ is a set whose order of ramification is at most 4 at every point. Hence, every continuum contained in $[f, g]$ is locally connected. Note also that if $f, g \in S$, then $[f, g]$ is the sum of segments, rectangles with sides parallel to the axes and isolated points.

If $f|X$ and $g|Y$ are partial functions of f and g, then $[f|X, g|Y]$ is the common part of $[f, g]$ and $X \times Y$.

The total structure of $[f, g]$ will be easier if we consider some partial functions of f and g. Let $f \in C$. We shall consider two kinds of segments of I with partial functions on it.

1° Let $A = \langle a_1, a_2\rangle$ be such that $f(A) = I$ and $f(\mathrm{Int}\ A)$ does not contain 0 and 1. There are two kinds of such segments. The first kind are those for which $f(a_1) = 0$ and $f(a_2) = 1$, and the second kind are those for which $f(a_1) = 1$ and $f(a_2) = 0$ We shall denote these segments by A^+ and A^- respectively.

2° Let B^- or B^+ be the maximal segment $\langle b_1, b_2\rangle$ such that $f(\langle b_1, b_2\rangle) \neq I$ and $f(b_1) = f(b_2) = 0$ or $f(b_1) = f(b_2) = 1$ respectively.

Note that the segments of 1° and 2° have at most the ends in common and form a finite covering of $J \subset I$, the maximal segment J' of I such that f transforms the ends of J' into the ends of I.

We shall denote similar segments for g by A'^+, A'^-, B'^+ and B'^-. Consider a division of I^2 into rectangles of the form $A \times A'$, $A \times B'$, $B \times A'$ and $B \times B'$ with several combinations of signs $+$ and $-$. The rectangles in question form a finite covering of $J \times J'$. The following two lemmas describe the behaviour of $[f, g]$ in these rectangles:

(6) $[f, g] \cap \mathrm{Fr}(A \times A')$ *consists of two opposite vertices of $A \times A'$, which are joined in $[f, g] \cap (A \times A')$ with a continuum. The vertices in question are the upper right and the lower left if the signs at A and A' are the same, and the upper left and the lower right if the signs are different.*

(7) $[f, g] \cap \mathrm{Fr}(A \times B')$ *contains two adjacent vertices of $A \times B'$, which are joined in $[f, g] \cap (A \times B')$ by a continuum. The vertices in question lie on the right side or on the left side of $A \times B'$ according as the signs at A and B' are the same or different. Furthermore, that side of $A \times B'$ contains $[f, g] \cap \mathrm{Fr}(A \times B')$.*

The case of $B \times A'$ is symmetric to that of (7). The situation is illustrated in fig. 1. The proofs of (6) and (7) are standard. As an example, we prove assertion (7) in the case of $A^+ \times B'^+$.

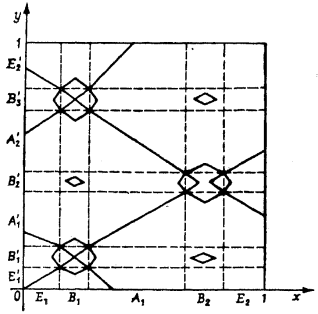

Fig. 1

Let $A^+ = \langle a_1, a_2\rangle$ and $B'^+ = \langle b'_1, b'_2\rangle$. We have $f(a_1) = 0$, $f(a_2) = 1$, $g(b'_1) = g(b'_2) = 1$. Hence $f(x) - g(y)$ is negative on $a_1 \times B'^+$, $(A^+ - a_2) \times b_1$ and $(A^+ - a_2) \times b_2$, i. e. on the lower left and upper left sides of $A^+ \times B'^+$, excluding vertices (a_2, b'_1) and (a_2, b'_2). $f(x) - g(y)$ is non-negative on $a_2 \times B'^+$, i. e. on the right side of $A^+ \times B'^+$. Hence $[f, g] \cap \mathrm{Fr}(A^+ \times B'^+) \subset a_2 \times B'^+$ and in addition (a_2, b'_1) and (a_2, b'_2) belong to $[f, g]$. According to the continuity of f and g, the double graph $[f, g]$ disconnects the rectangle between the first three sides and the last one. Therefore, it contains a continuum joining in $A^+ \times B'^+$ vertices (a_2, b'_1) and (a_2, b'_2).

We prove another simple lemma:

(8) *In every rectangle $A \times I$ there exists a continuum joining the upper side with the lower side of $A \times I$ and contained in $[f, g]$.*

In fact, let $A = \langle a_1, a_2\rangle$ and $f(a_1) = 0$ and $f(a_2) = 1$ (the case of A^+). The difference $f(x)-g(y)$ is negative or 0 on the left side $a_1\times I$ and is positive or 0 on the right side $a_2\times I$ of $A\times I$. According to the continuity of f and g, $[f, g]$ disconnects $A\times I$ between these two sides. Therefore, it contains a continuum joining in $A\times I$ the remaining two sides of $A\times I$.

3. Relations $\precsim$ and $\precsim'$ on R. In order to compare both these relations on R, let us consider equations $f\alpha \underset{1/n}{=} g$, $n = 1, 2, \ldots$, for $f, g\in R$. We prove that the solutions of these equations approximate the solutions of $f\alpha = g$. This is not true in general for C and even for S, because it may happen that the family of solutions of $f\alpha \underset{1/n}{=} g$ does not contain converging subsequences. We prove that

(9) *If $f, g\in R$ and $\{\alpha_n\}$ is a sequence of mappings in C such that $f\alpha_n \underset{1/n}{=} g$, $n = 1, 2, \ldots$, then $\{\alpha_n\}$ forms an equicontinuous sequence of mappings.*

To prove this, let $\varepsilon > 0$. Let us consider for every $(x, y)\in[f, g]$ a rectangle $P = I'\times I''$ of (5) with $\operatorname{diam} P\leqslant\varepsilon$. This is possible, because $f, g\in R$ (see the remarks which follow the proof of (5)). Rectangles P form a covering of $[f, g]$. We take a finite subcovering $P_1, P_2, \ldots, P_r$. The corresponding projections $I_1', I_2', \ldots, I_r'$ cover I. Let δ be the Lebesgue number of the last covering. Let us consider an η-neighbourhood of $[f, g]$ which contains no whole side of P_k for any $k = 1, 2, \ldots, r$. Such a neighbourhood exists according to the property of rectangles P_k described in (5). Clearly, $\eta\leqslant\varepsilon/3$. Let $x', x''\in I$ and $|x'-x''| < \delta$. Then there exists a j, $1\leqslant j\leqslant r$, such that $x', x''\in I_j'$. According to (4), the graphs of α_n lie, for sufficiently large n, in the η-neighbourhood of $[f, g]$. According to the definition of η, the graphs of $\alpha_n|I_j'$ lie for those n in the η-neighbourhood of a rectangle P_k, $k = 1, 2, \ldots$ or r, with the projection I_j'. Hence we have $|\alpha_n(x')-\alpha_n(x'')|\leqslant\varepsilon+2\eta < 2\varepsilon$ for these sufficiently large n. Thus (9) is proved.

THEOREM 1. *If $f, g\in R$, then $f\precsim' g$ implies $f\precsim g$.*

Proof. From $f\precsim' g$ it follows that there exists a sequence $\{\alpha_n\}$, $n = 1, 2, \ldots$, such that $f\alpha_n \underset{1/n}{=} g$. According to (9), $\{\alpha_n\}$ is equicontinuous. Let $\{\alpha_{k_n}\}$ be a subsequence of $\{\alpha_n\}$ which is uniformly convergent. According to (4), $\lim\alpha_{k_n}$ is a solution of $f\alpha = g$, because its graph lies in $[f, g]$. As a solution of $f\alpha = g$, it belongs to R. Hence $f\precsim g$ on R.

4. Uniformization. Let $f, g\in C$. In order to find $h\in C$ such that $f\precsim h$ and $g\precsim h$, it is sufficient to find $\alpha, \beta\in C$ such that $f\alpha = g\beta$. This is not always possible. For instance, the functions of the example quoted in § 1 (for details, see [4]) have no uniformization, or, in other words, no common majorization. This problem of uniformization was considered in [1], [2] and [4] for classes R and S with some inessential additional conditions, which will be omitted here. We express the theorem in terms of relation $\precsim$.

THEOREM 2. *R and S are directed sets with respect to $\precsim$.*

Proof. Let $f, g\in R$ or $f, g\in S$. Let $A\subseteq I$ and $A'\subset I$ be segments described in 1° of § 2 for f and g respectively. According to (8), there exist continua $K\subset(A\times I)\cap[f, g]$, joining the upper and lower sides of $A\times I$, and $K'\subset(I\times A')\cap[f, g]$ joining the right and left sides of $I\times A'$. Clearly, $K\cup K'$ is a continuum. If $f, g\in R$, then, according to the remarks following the proof of (5), $K\cup K'$ is locally connected. If $f, g\in S$, then, according to these remarks, $K\cup K'$ may be taken locally connected. In both cases $K\cup K'$ may be considered as a continuous image of the closed interval $0\leqslant t\leqslant 1$. Let $x = \alpha(t)$, $y = \beta(t)$ be the equations of $K\cup K'$. According to the definition of K and K', mappings α and β are onto. Hence, by (4), $f\alpha = g\beta$. The mapping $f\alpha = g\beta$ majorizes f and g.

Let $\varepsilon > 0$ and $f, g\in C$. We shall write, for convenience, $f\precsim_\varepsilon g$ if there exists an $\alpha\in C$ such that $f\alpha \underset{\varepsilon}{=} g$. We shall prove that

(10) *C is an approximatively directed set with respect to $\precsim$, i. e. for every $f, g\in C$ and $\varepsilon > 0$ there exists an $h_\varepsilon\in C$ such that $f\precsim_\varepsilon h_\varepsilon$ and $g\precsim_\varepsilon h_\varepsilon$.*

This follows from Theorem 2 and the following obvious implication: $f' \underset{\varepsilon}{=} f$ and $f'\precsim h$ imply $f\precsim_\varepsilon h$ for every $\varepsilon > 0$. In fact, let $f, g\in C$. Because R (and also S) is dense in C, there exist $f', g'\in R$ (or $f', g'\in S$) such that $f' \underset{\varepsilon}{=} f$ and $g' \underset{\varepsilon}{=} g$. According to Theorem 2, there exist α and β in R (or both in S) such that $f'\alpha = g'\beta$. Let $h_\varepsilon = f'\alpha = g'\beta$. We have $f'\precsim h_\varepsilon$ and $g'\precsim h_\varepsilon$. According to the implication mentioned above, we obtain $f\precsim_\varepsilon h$ and $g\precsim_\varepsilon h$, which proves (10) as ε was arbitrarily given.

5. A lemma concerning the equation $f\alpha = f$. We shall consider this equation for $f\in R$ only. Then $\alpha\in R$. Let $a_1, a_2\in I$ and $a_1 < a_2$. Let $\alpha(a_1) = b_1$ and $\alpha(a_2) = b_2$. We prove that

(11) *If $\alpha(b_1) = a_1$ and $\alpha(b_2) = a_2$, then $\alpha\alpha(x) = x$ for every x, $a_1\leqslant x\leqslant a_2$.*

To prove this, let us consider the graph Γ' of $\alpha|\langle a_1, a_2\rangle$ and the curve $x = \alpha(y)$, $b_1\leqslant y\leqslant b_2$, which we shall denote by Γ''. We consider the case $b_1 < b_2$ only, because the proof in the case $b_2 < b_1$ is the same. It is sufficient to prove that the curves Γ' and Γ'' coincide.

Suppose, on the contrary, that Γ' and Γ'' do not coincide. Hence there exist x_1 and x_2, $a_1\leqslant x_1 < x_2\leqslant a_2$, such that the points $(x_1, \alpha(x_1)) = p$ and $(x_2, \alpha(x_2)) = q$ lie on Γ'' and that this is not true for $(\xi, \alpha(\xi))$, where $x_1 < \xi < x_2$. Let us consider rectangles P and Q, having properties of (5), for p and q respectively. Because $f\in R$, rectangles P and Q may be taken such that their projections on x- and y-axis are disjoint. We shall consider the case $a(x_1) > a(x_2)$ only. By (5), curve Γ' meets $\mathrm{Fr}P$ at one of the right vertices of P, and meets $\mathrm{Fr}Q$ at one of the left vertices of Q. Similarly, Γ'' meets $\mathrm{Fr}P$ at one of the lower vertices of P, and it meets $\mathrm{Fr}Q$ at one of the upper vertices of Q. The situation is illustrated in fig. 2. Let D be the rectangle consisting of points whose coordinates lie between those of points of P and Q. We see that Γ' disconnects D between its left and right sides and that Γ'' disconnects D between its upper and lower sides. Then Γ' and Γ'' have a point in common in rectangle D. This is impossible, because the abscissa ξ of that point is such that $x_1 < \xi < x_2$, contrary to the definition of x_1 and x_2.

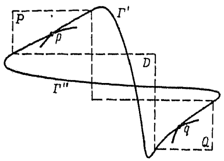

Fig. 2

6. The equation $fa = f$. The identity e is always a solution. But it is possible that there exist another solutions. If $f \in R$, this second solution, if it exists, is uniquely determined by f. The uniqueness fails if there exist intervals on which f is constant.

THEOREM 3. *If $f \in R$, then there exists at most one solution of the equation $fa = f$ different from the identity e. It is an involution which changes the orientation.*

Proof. Note first that if a is a solution such that $a(0) = 0$ and $a(1) = 1$, then a is an identity. In fact, according to (11) for $a_1 = 0$ and $a_2 = 1$, we have $aa(x) = x$ for every $x \in I$. Hence a is an involution on I with fixed points 0 and 1. It must be an identity.

If $f(0) = 0$ and $f(1) = 1$ or if $f(0) = 1$ and $f(1) = 0$, then for every solution a we have $a(0) = 0$ and $a(1) = 1$. In this case, according to the above remark, a is an identity.

In the general case, we consider the division of I into segments described in 1° and 2° in § 2. There is finite number of such segments and they may be enumerated and ordered as follows:

$$E_1 \leqslant B_1 \leqslant A_1 \leqslant B_2 \leqslant \ldots \leqslant A_{k-1} \leqslant B_k \leqslant E_2,$$

where the meaning of $X \leqslant Y$ is $x \leqslant y$ for all $x \in X$ and $y \in Y$. E_1 and E_2 are complementary segments to J, the sum of all A and B (see § 2). Recall that the graph of a is in $[f, f]$ and that a is onto. Hence, by (6) and (7), we have

(i) $a(a_j) = a_j$ and $a(a'_l) = a'_l$ for $j = 1, 2, \ldots, k$ and $l = 0, 1, \ldots, k-1$,

or

(ii) $a(a_j) = a'_{k-j}$ and $a(a'_l) = a_{k-l}$ for $j = 1, 2, \ldots, k$ and $l = 0, 1, \ldots, k-1$.

Here a_j and a'_j are the ends of A_j for $j = 1, 2, \ldots, k-1$, a'_0 is the right end of $E_1 = \langle 0, a'_0\rangle$, and a_k is the left end of $E_2 = \langle a_k, 1\rangle$. Hence a'_{j-1} and a_j, $j = 1, 2, \ldots, k$, are the ends of B_j.

In both cases, (i) and (ii), we have $aa(a_j) = a'_l$ and $aa(a'_l) = a'_l$ for all a_j and a'_l in question. Then we apply lemma (11). According to it, we obtain $aa(x) = x$ for every $x \in J$, i. e. $a'_0 \leqslant x \leqslant a_k$. In the case (i) a must be an identity on J, in the second — it is an involution on J, which changes the orientation.

The solutions on the whole of I are extensions of those on J. We have $a(J) = J$ and $0 \neq f(x) \neq 1$ for $x < a'_0$ and $x > a_k$. Hence, as a is onto, we obtain $a(E_1 \cup E_2) = E_1 \cup E_2$. We shall show that $aa(x) = x$ for $x \in E_1 \cup E_2$. According to (11), it is sufficient to show that $aa(0) = 0$ and $aa(1) = 1$. We show only the first equality. Denote by x_0 the infimum (which is, in fact, the minimum) of x such that $aa(x) = x$.

Suppose, on the contrary, that $x_0 \neq 0$. Then $a(x_0) \neq 0$ in case (i), and $a(x_0) \neq 1$ in case (ii). In case (i) this is obvious, as $a(x_0) = x_0$. In case (ii) we have $a(a'_0) = a_k$, $a(a_k) = a'_0$; if, in addition, $a(x_0) = 1$ and $a(1) = x_0$, then, according to (11), we should have $aa(x) = x$ for $x_0 \leqslant x \leqslant a'_0$ and $a_k \leqslant x \leqslant 1$, and a should transform the segment $a_k \leqslant x \leqslant 1$ onto $x_0 \leqslant x \leqslant a'_0$; but a is onto, and we have a contradiction. Hence $\langle 0, x_0\rangle$ and, in case (ii), $\langle a(x_0), 1\rangle$ are non-degenerate segments.

In case (i), a is an involution on $\langle x_0, a'_0\rangle$, in virtue of (11). As a is onto, we have $a(\langle 0, x_0\rangle) \supset \langle 0, x_0\rangle$. Then there exists an x, $0 \leqslant x < x_0$, such that $a(x) = x$ and, in consequence, $aa(x) = x$ which is in contradiction to the definition of x_0.

In case (ii), a is an involution on $\langle x_0, a'_0\rangle \cup \langle a_k, a(x_0)\rangle$, in virtue of (11). Then, because a is onto, we have $a(\langle 0, x_0\rangle) \supset \langle x_0, 1\rangle$ and $a(\langle x_0, 1\rangle) \supset \langle 0, x_0\rangle$. Then the curves $y = a(x)$, $0 \leqslant x \leqslant x_0$, and $x = a(y)$, $x_0 \leqslant y \leqslant 1$, intersect for some (x, y) with $x < x_0$. We have for such an x the equality $aa(x) = x$, contrary to the definition of x_0.

The uniqueness of the solution a, such that $a(0) = 0$ and $a(1) = 1$, was shown at the beginning of the proof; it is always an identity. The solution of case (ii) is also unique. In fact, we have for such an a the equalities $a(0) = 1$ and $a(1) = 0$. If a' is another solution of this kind, then we have $faa' = f$ and $fa'a = f$, where aa' and $a'a$ are a solution which does not change the orientation. Hence $aa' = e$ nad $a'a = e$. Therefore, $a' = a^{-1} = a$. This completes the proof.

7. Remarks. Note first the following

COROLLARY. *If $f, g \in R$ and $fa = g$ and $g\beta = f$, then a and β are homeomorphisms.*

In fact, from $fa = g$ and $g\beta = f$ follows immediately $fa\beta = f$ and

$g\beta\alpha = g$. According to Theorem 3, $\alpha\beta$ and $\beta\alpha$ are continuous involutions. Hence, α and β are homeomorphisms.

This explains the meaning of the relation $\sim$ on R. Namely we have

COROLLARY. *If $f, g \in R$, then $f \sim g$ if and only if there exist homeomorphisms α and β such that $f\alpha = g$ and $g\beta = f$.*

There are other problems concerning the semigroup C. Note the following one: do there exist for every $f \in C$ a mapping $g \in C$, different from f and the identity, non-monotone if f is monotone, such that $fg = gf$? (**P 372**)

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MATHEMATICAL INSTITUTE OF THE WROCŁAW UNIVERSITY

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ON THE DIMENSION OF QUASI-COMPONENTS IN PERIPHERICALLY COMPACT SPACES

BY

A. LELEK (WROCŁAW)

It is well known that a compact metric space has a dimension at most n provided that all its components have dimensions at most n. An example given in 1927 by Mazurkiewicz [4] shows that there exists such a separable metric space of an arbitrary positive dimension that each of its components, even every quasi-component, consists of a single point. Recently, Engelking asked if there exists a space of this kind which simultaneously is *peripherically compact* (also called *semicompact*), i. e. satisfies the condition that each point has arbitrarily small neighbourhoods whose boundaries are compact. This question has partially been answered by Duda, who constructed the following

EXAMPLE. *There is such a separable metric space X that* $\dim X = 1$, *every component of X consists of a single point and X is peripherically compact.* Indeed, denoting by $\mathscr{I}$ the segment $0 \leqslant t \leqslant 1$ and by $\mathscr{C}$ the Cantor ternary set in $\mathscr{I}$, let A be a set which is obtained from a biconnected set by removing its "explosive" point and is contained in $\mathscr{C} \times \mathscr{I}$ in such a way that the intersection $A \cap (\{t\} \times \mathscr{I})$ is a point p_t for every $t \in \mathscr{C}$. Then $\dim A = 1$. Put $X = A \cup (\mathscr{C} \times \mathscr{R})$, where $\mathscr{R}$ is the set of all rational numbers in $\mathscr{I}$. Obviously, X is a 1-dimensional peripherically compact set and contains only degenerate connected subsets.

However, in this example the quasi-components of X are the sets $\{p_t\} \cup (\{t\} \times \mathscr{R})$ for $t \in \mathscr{C}$. As we show, it is impossible to find any space X possessing all the properties of Duda's example described above with the word "quasi-component" instead of "component". Namely, we have the following

THEOREM 1. *If every quasi-component Q of a peripherically compact separable metric space X is locally compact and has the dimension* $\dim Q \leqslant 0$, *then* $\dim X \leqslant 0$.

Proof. The space X being peripherically compact, we may assume by a theorem proved in 1942 by Freudenthal [1] that X is a subset of