$g\beta\alpha = g$. According to Theorem 3, $\alpha\beta$ and $\beta\alpha$ are continuous involutions. Hence, α and β are homeomorphisms.

This explains the meaning of the relation $\sim$ on R. Namely we have

COROLLARY. *If $f, g \in R$, then $f \sim g$ if and only if there exist homeomorphisms α and β such that $f\alpha = g$ and $g\beta = f$.*

There are other problems concerning the semigroup C. Note the following one: do there exist for every $f \in C$ a mapping $g \in C$, different from f and the identity, non-monotone if f is monotone, such that $fg = gf$? (**P 372**)

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ON THE DIMENSION OF QUASI-COMPONENTS IN PERIPHERICALLY COMPACT SPACES

BY

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It is well known that a compact metric space has a dimension at most n provided that all its components have dimensions at most n. An example given in 1927 by Mazurkiewicz [4] shows that there exists such a separable metric space of an arbitrary positive dimension that each of its components, even every quasi-component, consists of a single point. Recently, Engelking asked if there exists a space of this kind which simultaneously is *periphérically compact* (also called *semicompact*), i. e. satisfies the condition that each point has arbitrarily small neighbourhoods whose boundaries are compact. This question has partially been answered by Duda, who constructed the following

EXAMPLE. *There is such a separable metric space X that* $\dim X = 1$, *every component of X consists of a single point and X is peripherically compact.* Indeed, denoting by $\mathscr{I}$ the segment $0 \leqslant t \leqslant 1$ and by $\mathscr{C}$ the Cantor ternary set in $\mathscr{I}$, let A be a set which is obtained from a biconnected set by removing its "explosive" point and is contained in $\mathscr{C} \times \mathscr{I}$ in such a way that the intersection $A \cap (\{t\} \times \mathscr{I})$ is a point p_t for every $t \in \mathscr{C}$. Then $\dim A = 1$. Put $X = A \cup (\mathscr{C} \times \mathscr{R})$, where $\mathscr{R}$ is the set of all rational numbers in $\mathscr{I}$. Obviously, X is a 1-dimensional peripherically compact set and contains only degenerate connected subsets.

However, in this example the quasi-components of X are the sets $\{p_t\} \cup (\{t\} \times \mathscr{R})$ for $t \in \mathscr{C}$. As we show, it is impossible to find any space X possessing all the properties of Duda's example described above with the word "quasi-component" instead of "component". Namely, we have the following

THEOREM 1. *If every quasi-component Q of a peripherically compact separable metric space X is locally compact and has the dimension* $\dim Q \leqslant 0$, *then* $\dim X \leqslant 0$.

Proof. The space X being peripherically compact, we may assume by a theorem proved in 1942 by Freudenthal [1] that X is a subset of a compact metric space Y such that

(1) $$\dim(Y-X) \leqslant 0.$$

Let p be an arbitrary point of X and Q — the quasi-component of X, containing p. For each open neighbourhood W of p in X there is such an open set V in Y that $W = V \cap X$. Since Q is locally compact, there exists an open neighbourhood U of p in Y such that $\overline{U \cap Q} \cap Q$ is compact (*). Then $p \in U \cap V$ and it follows from the inequality $\dim Q \leqslant 0$ that there exists an open neighbourhood T of p in Y satisfying

$$\overline{T} \subset U \cap V \quad \text{and} \quad Q \cap \mathrm{Fr}_Y(T) = 0,$$

where $\mathrm{Fr}_Y(T)$ denotes the boundary of T in Y (see [2], p. 164 and 173).

Consequently,

$$\mathrm{Fr}_Y(T) \subset Y - (\overline{U \cap Q} \cap Q)$$

and since the set $\overline{U \cap Q} \cap Q$ is compact, and thus closed in Y, it follows from (1) that each point $q \in \mathrm{Fr}_Y(T)$ has an open neighbourhood $S(q)$ in Y satisfying

$$\overline{S(q)} \subset Y - (\overline{U \cap Q} \cap Q) \quad \text{and} \quad (Y-X) \cap \mathrm{Fr}_Y[S(q)] = 0$$

(ibidem), whence p does not belong to $\overline{S(q)}$ and $\mathrm{Fr}_Y[S(q)] \subset X$. According to the compactness of $\mathrm{Fr}_Y(T)$, let us take a finite cover $S_1, \ldots, S_m$ from the cover of $\mathrm{Fr}_Y(T)$ consisting of all the sets $S(q)$. So we have

(2) $$\overline{S}_1 \cup \ldots \cup \overline{S}_m \subset Y - (\overline{U \cap Q} \cap Q),$$

(3) $$\mathrm{Fr}_Y(S_i) \subset X \quad \text{for} \quad i = 1, \ldots, m$$

and the set

$$R = T - (\overline{S}_1 \cup \ldots \cup \overline{S}_m)$$

contains the point p and is open in Y. Furthermore, since the open sets $S_1, \ldots, S_m$ form a cover of $\mathrm{Fr}_Y(T)$, it is easily seen that each point belonging to the boundary of the set R also belongs to the boundary of the union $\overline{S}_1 \cup \ldots \cup \overline{S}_m$. Thus

(4) $$\mathrm{Fr}_Y(R) \subset \mathrm{Fr}_Y(\overline{S}_1 \cup \ldots \cup \overline{S}_m) \subset \mathrm{Fr}_Y(\overline{S}_1) \cup \ldots \cup \mathrm{Fr}_Y(\overline{S}_m) \subset X$$

(see [2], p. 29), according to (3). Moreover, we have

$$Q \cap \mathrm{Fr}_Y(R) \subset Q \cap \overline{R} \subset Q \cap \overline{T} \subset Q \cap U \subset \overline{U \cap Q},$$

whence

$$Q \cap \mathrm{Fr}_Y(R) = (\overline{U \cap Q} \cap Q) \cap \mathrm{Fr}_Y(R) \subset (\overline{U \cap Q} \cap Q) \cap \mathrm{Fr}_Y(\overline{S}_1 \cup \ldots \cup \overline{S}_m)$$
$$\subset (\overline{U \cap Q} \cap Q) \cap (\overline{S}_1 \cup \ldots \cup \overline{S}_m) = 0,$$

(*) $\overline{A}$ always denotes here the closure of a set A in the space Y.

by virtue of (2) and (4), i. e.

(5) $$Q \cap \mathrm{Fr}_Y(R) = 0.$$

Since there is a mapping $f\colon X \to \mathscr{C}$ of X into the Cantor set $\mathscr{C}$ such that the sets $f^{-1}(y)$, where $y \in f(X)$, coincide with the quasi-components of X (see [3], p. 93), one can find a decreasing sequence

$$G_1 \supset G_2 \supset \ldots$$

of subsets of X which satisfy

$$Q = \bigcap_{i=1}^{\infty} G_i$$

and are all both closed and open in X. For instance, it is sufficient to represent the point $f(Q)$ as the intersection of a decreasing sequence

$$[a_1, b_1] \supset [a_2, b_2] \supset \ldots$$

of segments on the real line such that no end point a_i, b_i belongs to $\mathscr{C}$, and take

$$G_i = f^{-1}([a_i, b_i])$$

for $i = 1, 2, \ldots$ Then, if all the sets $G_i \cap \mathrm{Fr}_Y(R)$ were non-empty ($i = 1, 2, \ldots$), they would form a decreasing sequence of non-empty closed subsets of $\mathrm{Fr}_Y(R)$, according to (4), and the compactness of the set $\mathrm{Fr}_Y(R)$ would imply that

$$0 \neq \bigcap_{i=1}^{\infty} G_i \cap \mathrm{Fr}_Y(R) = Q \cap \mathrm{Fr}_Y(R),$$

contrary to (5). Consequently, a positive integer j exists such that

(6) $$G_j \cap \mathrm{Fr}_Y(R) = 0.$$

Now, consider the set $G_j \cap R$. It is an open neighbourhood of p in X as $p \in Q \subset G_j$ and $p \in R$. Further, since $G_j \subset X$ and $R \subset T \subset V$, we have

$$G_j \cap R \subset X \cap V = W.$$

Moreover, the set G_j being both closed and open in X, we have

$$\mathrm{Fr}_X(G_j \cap R) \subset \overline{G_j \cap R} \cap X \subset \overline{G}_j \cap X = G_j$$

and the boundary $\mathrm{Fr}_X(G_j)$ of G_j in X is empty; thus

$$\mathrm{Fr}_X(G_j \cap R) = \mathrm{Fr}_X[G_j \cap (R \cap X)] \subset \mathrm{Fr}_X(G_j) \cup \mathrm{Fr}_X(R \cap X)$$
$$= \mathrm{Fr}_X(R \cap X) = \overline{R \cap X} \cap X \cap \overline{X - R} \subset \overline{R} \cap \overline{Y - R} = \mathrm{Fr}_Y(R)$$

(see [2], p. 29). It follows that

$$\mathrm{Fr}_X(G_j \cap R) \subset G_j \cap \mathrm{Fr}_Y(R) = 0,$$

according to (6). Hence we conclude that

$$\dim_p X \leqslant 0,$$

since W has been an arbitrarily taken open neighbourhood of p in X, and the proof of Theorem 1 is complete.

It is seen by Duda's example given at the beginning of this note that the local compactness of quasi-components is a necessary hypothesis in Theorem 1. In fact, the example shows that a peripherically compact space can be 1-dimensional and have only 0-dimensional quasi-components. However, the difference between the dimension of a peripherically compact metric space and the maximal dimension of its quasi-components cannot be greater than 1. This is a consequence of the following

THEOREM 2. *If every component C of a peripherically compact metric space X has the dimension* $\dim C \leqslant n$ (*where* $n = 0, 1, \ldots$), *then* $\dim X \leqslant n+1$.

Proof. For any point p of X there is an arbitrarily small open neighbourhood V of p in X such that the boundary $\mathrm{Fr}_X(V)$ is compact. Since each component K of this boundary is contained in a component C of X, we have

$$\dim K \leqslant \dim C \leqslant n,$$

whence $\dim \mathrm{Fr}_X(V) \leqslant n$ (see [3], p. 106). It follows that $\dim_p X \leqslant n+1$ and Theorem 2 is proved.

At last, the following question concerning some generalization of Theorem 1 on higher dimensions remains open:

P 373. Is it true that if every quasi-component Q of a peripherically compact separable metric space X is locally compact and has the dimension $\dim Q \leqslant n$ (where $n = 1, 2, \ldots$), then $\dim X \leqslant n$?

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ON THE PROBLEM OF EXISTENCE OF FINITE REGULAR PLANES

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1. Preliminaries. An ordered pair $P = \langle X, R_3 \rangle$, where X is a set consisting of finitely many elements (points) and R_3 is a relation of three arguments which are points of X, is said to be a *finite plane* if the following conditions hold:

A1. If $a = b$, then $R_3(a, b, c)$.

A2. If $R_3(a, b, c)$, then $R_3(b, a, c)$ and $R_3(c, a, b)$.

A3. If $a \neq b$, $R_3(a, b, c)$ and $R_3(a, b, d)$, then $R_3(b, c, d)$.

A4. There exist points $a, b, c \in X$ such that $\sim R_3(a, b, c)$.

Let $a \neq b$. The set of points $x \in X$ satisfying the relation $R_3(a, b, x)$ is said to be a *straight line*, which will be denoted by $[a, b]$. We say that a finite plane is *regular* if

A5. All straight lines consist of the same number of points.

All planes in this paper will be finite and regular. They will be simply called *planes*.

Let $a \in X$. The number of all different straight lines $[a, x]$, $x \in X$, is said to be an *order of ramification of* a. It is easy to see that, *if the order of ramification of an arbitrarily chosen point of the plane P is i, then the order of ramification of any other point of P is also i.*

We denote by P_i^k the plane whose points have the order of ramification i and the number of points of every straight line is k. We have, evidently, $i, k \geqslant 2$.

Let $P_i^k = \langle X, R_3 \rangle$. From A1-A5 it immediately follows that

1.1. *X consists of $s = (k-1)i+1$ points,*

1.2. *k is a divisor of the product is,*

1.3. *$i \geqslant k$.*

It is easy to see that the planes P_k^k satisfy the following condition:

A6$_0$. $[a, b] \frown [c, d] \neq 0$ for every pair of straight lines $[a, b]$ and $[c, d]$ in X.