according to (6). Hence we conclude that

$$\dim_p X \leqslant 0,$$

since W has been an arbitrarily taken open neighbourhood of p in X, and the proof of Theorem 1 is complete.

It is seen by Duda's example given at the beginning of this note that the local compactness of quasi-components is a necessary hypothesis in Theorem 1. In fact, the example shows that a peripherically compact space can be 1-dimensional and have only 0-dimensional quasi-components. However, the difference between the dimension of a peripherically compact metric space and the maximal dimension of its quasi-components cannot be greater than 1. This is a consequence of the following

THEOREM 2. *If every component C of a peripherically compact metric space X has the dimension* $\dim C \leqslant n$ (*where* $n = 0, 1, \ldots$), *then* $\dim X \leqslant n+1$.

Proof. For any point p of X there is an arbitrarily small open neighbourhood V of p in X such that the boundary $\mathrm{Fr}_X(V)$ is compact. Since each component K of this boundary is contained in a component C of X, we have

$$\dim K \leqslant \dim C \leqslant n,$$

whence $\dim \mathrm{Fr}_X(V) \leqslant n$ (see [3], p. 106). It follows that $\dim_p X \leqslant n+1$ and Theorem 2 is proved.

At last, the following question concerning some generalization of Theorem 1 on higher dimensions remains open:

P 373. Is it true that if every quasi-component Q of a peripherically compact separable metric space X is locally compact and has the dimension $\dim Q \leqslant n$ (where $n = 1, 2, \ldots$), then $\dim X \leqslant n$?

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ON THE PROBLEM OF EXISTENCE OF FINITE REGULAR PLANES

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1. Preliminaries. An ordered pair $P = \langle X, R_3\rangle$, where X is a set consisting of finitely many elements (points) and R_3 is a relation of three arguments which are points of X, is said to be a *finite plane* if the following conditions hold:

A1. If $a = b$, then $R_3(a, b, c)$.

A2. If $R_3(a, b, c)$, then $R_3(b, a, c)$ and $R_3(c, a, b)$.

A3. If $a \neq b$, $R_3(a, b, c)$ and $R_3(a, b, d)$, then $R_3(b, c, d)$.

A4. There exist points $a, b, c \in X$ such that $\sim R_3(a, b, c)$.

Let $a \neq b$. The set of points $x \in X$ satisfying the relation $R_3(a, b, x)$ is said to be a *straight line*, which will be denoted by $[a, b]$. We say that a finite plane is *regular* if

A5. All straight lines consist of the same number of points.

All planes in this paper will be finite and regular. They will be simply called *planes*.

Let $a \in X$. The number of all different straight lines $[a, x]$, $x \in X$, is said to be an *order of ramification of* a. It is easy to see that, *if the order of ramification of an arbitrarily chosen point of the plane P is i, then the order of ramification of any other point of P is also i.*

We denote by P_i^k the plane whose points have the order of ramification i and the number of points of every straight line is k. We have, evidently, $i, k \geqslant 2$.

Let $P_i^k = \langle X, R_3\rangle$. From A1-A5 it immediately follows that

1.1. *X consists of $s = (k-1)i+1$ points,*

1.2. *k is a divisor of the product is,*

1.3. *$i \geqslant k$.*

It is easy to see that the planes P_k^k satisfy the following condition:

A6$_0$. $[a, b] \frown [c, d] \neq 0$ for every pair of straight lines $[a, b]$ and $[c, d]$ in X.

Hence P_k^k are projective plane geometries in the sense of [8].

The planes P_{k+1}^k satisfy the following condition:

$A6_1$. *For every triple a, b, c for which $\sim R_3(a, b, c)$ there exists one point d such that $[a, b] \frown [c, d] = 0$, and $R_3(c, d, e)$ for every $e \in X$ if $[a, b] \frown [c, e] = 0$.*

The planes P_{k+1}^k are affine plane geometries (e. g. in the sense of [1]).

In general, the planes P_{k+m}^k satisfy the following condition:

$A6_m$. *For every triple a, b, c for which $\sim R_3(a, b, c)$ there exist m different points $d_1, d_2, \ldots, d_m$ such that $[a, b] \frown [c, d_\alpha] = 0$ $(\alpha = 1, 2, \ldots, m)$, $\sim R_3(c, d_\alpha, d_\beta)$ for $\alpha \neq \beta$ $(\alpha, \beta = 1, 2, \ldots, m)$ and there is no set of $m+1$ points with the same property.*

We shall call these planes *L_m-planes* or *Lobatchevsky's m-planes*.

2. The problem of existence, for any k, of affine and projective geometries is well known (see, for instance, [2], [3], [4] and references in it) and remains in general still unsolved. The negative answer to the so-called Euler problem of 36 officers, which is equivalent to the problem of existence of the affine geometry P_7^6, was given by Tarry [7] as late as in 1901. The general problem is connected with a problem concerning the orthogonal Latin squares (a notion introduced by Euler). In 1923 McNeish [5] gave a construction of planes P_{k+1}^k when k is prime or is of the form p^n (p is prime). Recently, in [4], information has been given that P_{13}^{12} and P_{16}^{15} do not exist (this result has been obtained by machine computation).

Note that

2.1. *If there exists a plane $L_m = P_{k+m}^k$ $(m > 1)$, then k is a divisor of $m(m-1)$.*

In fact, from 1.1 it follows that the number of points of P_{k+m}^k is $s = (k-1)(k+m)+1$. According to 1.2, k is a divisor of $s(k+m) = k^3 + (2m-1)k^2 + (m-1)^2 k - m(m-1)$. Hence k is a divisor of $m(m-1)$.

From 2.1 it immediately follows that

2.2. *All L_2-planes are isomorphic to $L_2 = P_4^2$ (X consists of 5 points).*

It is known (see [5]) that there exist infinitely many affine and projective geometries (L_0- and L_1-planes) which are not isomorphic to each other.

The problem of the existence of the planes P_i^k in the general case, i. e. without restriction to affine and projective geometries seems to be interesting.

The construction of P_i^2, for every i, reduces to the construction of full graphs consisting of $i+1$ points. In the particular case of $k = 3$, i. e. for triple systems of Steiner, the problem of existence was solved

by M. Reiss in 1859, who showed that there exist P_i^3 for i of the form $6n+1$ and $6n+3$. Hence, our conditions 1.1-1.3 are sufficient for the existence of P_i^3.

In this paper I give another approach to the investigation of P_i^3, by constructing adequate algebras.

3. We call an *A_n^3-algebra* the ordered pair $\langle A, \circ \rangle$, where A is a set consisting of n points and $\circ$ is an operation having the following properties:

W1. $a \circ a = a$,

W2. $a \circ b = b \circ a$,

W3. $a \circ (a \circ b) = b$.

The existence of P_1^3 is equivalent, for $i \geqslant 3$, to the existence of an A_n^3-algebra $\langle A, \circ \rangle$, where A is a set consisting of $n = 2i+1$ points. In fact, we put $R_3(a, b, c)$ if and only if either two of the elements are identical or $a \circ b = c$. Then $\langle A, R_3 \rangle$ is a P_i^3 plane. Inversely, given a P_i^3 plane $\langle A, R_3 \rangle$, we put $a \circ a = a$ and, for $a \neq b$, $a \circ b = c$, where c is the unique point satisfying the relation $R_3(a, b, c)$. Then $\langle A, \circ \rangle$ is an A_n^3-algebra. The isomorphic planes correspond in this equivalence to isomorphic algebras.

The following theorems on algebras correspond to theorems 1.1 and 1.2 for $k = 3$, respectively:

3.1. *If n is even, then A_n^3-algebra does not exist.*

3.2. *If there exists the A_{2i+1}^3-algebra, then 3 is a divisor of $i(2i+1)$.*

We prove that

3.3. *The existence of the A_n^3-algebra implies the existence of the A_{2n+1}^3-algebra.*

In fact, let $A = \{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}, \beta_1, \beta_2, \ldots, \beta_n\}$ be a given set consisting of $2n+1$ points and let us assume that the algebra operation $\circ$ is defined on the subset consisting of points $\beta_1, \beta_2, \ldots, \beta_n$. We extend this operation to the operation on the whole of A as follows:

$$\alpha_k \circ \alpha_k = \alpha_k \qquad (k = 1, 2, \ldots, n+1),$$

$$\alpha_k \circ \alpha_s = \beta_{(k+s-1) \bmod n} \qquad (k \neq s;\ k, s = 1, 2, \ldots, n),$$

$$\alpha_{n+1} \circ \alpha_k = \alpha_k \circ \alpha_{n+1} = \beta_{(2k-1) \bmod n} \qquad (k = 1, 2, \ldots, n),$$

$$\beta_{(k+s-1) \bmod n} \circ \alpha_k = \alpha_k \circ \beta_{(k+s-1) \bmod n} = \alpha_s \qquad (k \neq s;\ k, s = 1, 2, \ldots, n),$$

$$\beta_{(2k-1) \bmod n} \circ \alpha_k = \alpha_k \circ \beta_{(2k-1) \bmod n} = \alpha_{n+1} \qquad (k = 1, 2, \ldots, n).$$

It is easy to verify that all properties W1-W3 are satisfied. Now we prove that

3.4. *The existence of the A_m^3-algebra and the A_n^3-algebra implies the existence of the A_{mn}^3-algebra.*

In fact, $\{\beta_1, \beta_2, \ldots, \beta_m\}$ being the A_m^3-algebra and $\{\gamma^1, \gamma^2, \ldots, \gamma^n\}$ the A_n^3-algebra, let us consider the functions $p(i,j)$ $(i,j = 1, 2, \ldots, m)$ and $q(k,s)$ $(k, s = 1, 2, \ldots, n)$ such that $\beta_i \circ \beta_j = \beta_{p(i,j)}$, $\gamma^k \circ \gamma^s = \gamma^{q(k,s)}$. Let $A = \{a_1^1, \ldots, a_m^1; a_1^2, \ldots, a_m^2; \ldots; a_1^n, \ldots, a_m^n\}$ be a given set of mn elements. The operation $\circ$ is defined as follows:

$$a_i^k \circ a_j^s = a_{p(i,j)}^{q(k,s)} \qquad (k, s = 1, 2, \ldots, n;\ i, j = 1, 2, \ldots, m).$$

It is easy to verify that all the properties W1-W3 are satisfied for the operation $\circ$ on A defined above.

From 3.4 it follows immediately that

3.5. *The existence of* $A_{n_1}^3, A_{n_2}^3, \ldots, A_{n_k}^3$ *implies the existence of* $A_{n_1 \cdot n_2 \cdot \ldots \cdot n_k}^3$.

Let A_3^3 be an algebra given by the matrix

	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

The class of all A_n^3-algebras which may be obtained from A_3^3 in the way described in theorems 3.3 and 3.5 does not contain all algebras A_n^3. Other classes may be constructed using the results of Skolem [6]. For given $n \geqslant 13$ there exist several non-isomorphic A_n^3-algebras. The problem of determining their number remains for different n still open.

4. Problems. Let $A_{n_k}^3$ be the algebra obtained from A_3^3-algebra by the construction of theorem 3.3 applied k times. Each subalgebra of A_n^3 induced by two different elements consists of exactly three elements, which satisfy the relation R_3 of section 3 and are called *linearly dependent*. Let us consider subalgebras generated by three linearly independent elements x_1, x_2, x_3. It is easy to verify that every triple of independent elements of $A_7^3 = A_n^3$ generates the whole algebra. For this algebra satisfies condition

W4. If a, b, c are independent, then $a \circ (b \circ c) = (a \circ b) \circ c$.

The algebra $A_{15}^3 = A_{n_2}^3$ does not satisfy this condition. Indeed

$$a_1 \circ (a_2 \circ a_4) = a_1 \circ \beta_5 = a_5, \qquad (a_1 \circ a_2) \circ a_4 = \beta_2 \circ a_4 = a_6.$$

Moreover, the elements a_1, a_2, a_4 generate the whole algebra, and the elements $\beta_1, \beta_2, \beta_4$ generate only an A_7^3-subalgebra.

It can be proved that if W4 is satisfied, then every triple of elements of the A_n^3-algebra generates an A_7^3-subalgebra.

P 374. Does the number n of elements of the A_n^3-algebra satisfying W4 belong to the sequence defined by the reccurent formula $a_{k+1} = 2a_k + 1$, $a_0 = 1$? Are these algebras isomorphic for a given n? Can the relation R be generalized for more than three elements? Find out the connections between such algebras and more dimensional regular finite projective spaces.

It is easy to prove that the algebra A_9^3 satisfies the condition

W4′. $(a \circ b) \circ (a \circ c) = a \circ (b \circ c)$.

P 375. Investigate the algebras satisfying W4′. Is the number of their elements of the form 3^{k-1}? Are they all isomorphic (for a given n)? Find out the connections between such algebras and more dimensional regular finite affine spaces.

We call an A_n^4-*algebra* an ordered pair $\langle A, \circ \rangle$, where A is a set consisting of n points and $\circ$ is an operation having the following properties:

W1′. $a \circ a = a$,

W2′. $a \circ (a \circ b) = b \circ a$,

W3′. $a \circ (a \circ (a \circ b)) = b$.

A subalgebra generated by two elements a and b consisting of four elements $a, b, c = a \circ b$ and $d = b \circ a$ is presented by the matrix

	a	b	c	d
a	a	c	d	b
b	d	b	a	c
c	b	d	c	a
d	c	a	b	d

obtained according to W1-W3.

In the same manner as in section 3, it can be shown that the existence of P_i^4 is equivalent, for $i \geqslant 4$, to the existence of the A_n^4-algebra $\langle A, \circ \rangle$, where A is a set consisting of $n = 3i + 1$ points.

P 376. Are implications, similar to that of 3.3 and 3.5, true for A_n^4-algebras?

Furthermore, it seems to be interesting to give an axiom scheme for the general case of A_n^k-algebras.

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PROBLEMS OF ORDER WITH RESPECT TO TETRAHEDRAL SEXTUPLES

BY

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Introduction. H. Steinhaus [3] has posed the following problem:

There exist numbers $a>b>c>d>e>f>0$ for which there are 30 different tetrahedra with the edges a, b, c, d, e, f. Which other values besides 0 and 30 can be assumed by $N(a, b, c, d, e, f)$, i. e. the number of all different tetrahedra which have the edges a, b, c, d, e, f?

In the first section of this paper we solve this problem by showing that N can also assume all integral values between 0 and 30.

When T is a tetrahedron with the edges $a>b>c>d>e>f>0$, we write $T=(x, y, z)$ if x is the edge opposite to a, y is the edge opposite to the largest edge which remains if a and x are omitted, z is the third side of the triangle which contains x and y. In the second section we show that there are sextuples for which the alphabetical order of the 30 tetrahedra (x, y, z) is also the order with respect to their size.

In the third section we study the semi-order of the 30 types of tetrahedra (x, y, z) with respect to their volume. It is seen that this semi-order is not quite the same for completely tetrahedral sextuples and for sextuples which are not necessarily completely tetrahedral. (A sextuple (a, b, c, d, e, f) is called *completely tetrahedral* [2] if all possible tetrahedra (x, y, z) exist.)

I. Determination of all possible $N(a, b, c, d, e, f)$. To prove that $N(a, b, c, d, e, f)$ can assume all integral values from 0 up to 30, we adopt a method developed by Blumenthal [1] for other purposes: we take $a=(t+5)^{1/2}$, $b=(t+4)^{1/2}$, $c=(t+3)^{1/2}$, $d=(t+2)^{1/2}$, $e=(t+1)^{1/2}$, $f=t^{1/2}$; then N becomes a function of t alone and is equal to zero if $t=0$ and takes in succession all integral values up to 30 if t increases to infinity.

If A is the set a, b, c, d, e, f and if $a_i, b_j \in A$ for $i, j = 1, 2, 3$ with all a_i, b_j different from each other, then, if a_i and b_j are opposite edges, $T=(b_1, b_2, b_3)$ is a tetrahedron if and only if the following two conditions are satisfied: