REFERENCES

[1] E. Artin, *Coordinates in affine geometry*, Publication of the University of Notre Dame, Reports of a Mathematical Colloquium, Second Series, 2 (1940), p. 15-20.

[2] G. Birkhoff, *Lattice theory*, New York 1948.

[3] R. C. Bose, S. S. Shrikhande and E. T. Parker, *Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture*, Canadian Journal of Mathematic 12 (1960), p. 189-203.

[4] D. M. Johnson, A. L. Dulmage, and N. S. Mendelsohn, *Orthomorphismus of groups and orthogonal Latin squares I*, ibidem 13 (1961), p. 356-372.

[5] H. F. Mac Neish, *Euler squares*, Annals of Mathematics 23 (1923), p. 221-227.

[6] Th. Skolem, *Some remarks on the triple systems of Steiner*, Mathematica Scandinavica 6 (1958), p. 273-280.

[7] G. Tarry, *Le problème de 36 officiers*, Comptes Rendus de l'Association Française 1 (1900), p. 122-123, 2 (1901), p. 170-203.

[8] A. N. Whitehead, *Axioms of projective geometry*, Cambridge Tracts in Mathematics and Mathematical Physics 4 (1913), p. 7.

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PROBLEMS OF ORDER WITH RESPECT TO TETRAHEDRAL SEXTUPLES

BY

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Introduction. H. Steinhaus [3] has posed the following problem:

There exist numbers $a > b > c > d > e > f > 0$ for which there are 30 different tetrahedra with the edges a, b, c, d, e, f. Which other values besides 0 and 30 can be assumed by $N(a, b, c, d, e, f)$, i. e. the number of all different tetrahedra which have the edges a, b, c, d, e, f?

In the first section of this paper we solve this problem by showing that N can also assume all integral values between 0 and 30.

When T is a tetrahedron with the edges $a > b > c > d > e > f > 0$, we write $T = (x, y, z)$ if x is the edge opposite to a, y is the edge opposite to the largest edge which remains if a and x are omitted, z is the third side of the triangle which contains x and y. In the second section we show that there are sextuples for which the alphabetical order of the 30 tetrahedra (x, y, z) is also the order with respect to their size.

In the third section we study the semi-order of the 30 types of tetrahedra (x, y, z) with respect to their volume. It is seen that this semi-order is not quite the same for completely tetrahedral sextuples and for sextuples which are not necessarily completely tetrahedral. (A sextuple (a, b, c, d, e, f) is called *completely tetrahedral* [2] if all possible tetrahedra (x, y, z) exist.)

I. Determination of all possible $N(a, b, c, d, e, f)$. To prove that $N(a, b, c, d, e, f)$ can assume all integral values from 0 up to 30, we adopt a method developed by Blumenthal [1] for other purposes: we take $a = (t+5)^{1/2}$, $b = (t+4)^{1/2}$, $c = (t+3)^{1/2}$, $d = (t+2)^{1/2}$, $e = (t+1)^{1/2}$, $f = t^{1/2}$; then N becomes a function of t alone and is equal to zero if $t = 0$ and takes in succession all integral values up to 30 if t increases to infinity.

If A is the set a, b, c, d, e, f and if $a_i, b_j \in A$ for $i, j = 1, 2, 3$ with all a_i, b_j different from each other, then, if a_i and b_j are opposite edges, $T = (b_1, b_2, b_3)$ is a tetrahedron if and only if the following two conditions are satisfied:

1. At least one of the triples b_1, b_2, b_3; a_1, a_2, b_3; a_1, b_2, a_3; b_1, a_2, a_3 forms a triangle.

2. The determinant

$$D = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a_1^2 & a_2^2 & a_3^2 \\ 1 & a_1^2 & 0 & b_3^2 & b_2^2 \\ 1 & a_2^2 & b_3^2 & 0 & b_1^2 \\ 1 & a_3^2 & b_2^2 & b_1^2 & 0 \end{vmatrix}$$

(which for a tetrahedron gives the square of its volume multiplied by 288) has a positive value.

In our case the first condition is automatically satisfied since every triple which does not contain f forms a triangle. Hence $N(a, b, c, d, e, f)$ is equal to the number of determinants D with positive value in the complete set of 30 determinants.

For each tetrahedron of the set we have

$$D = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & t+\alpha & t+\beta & t+\gamma \\ 1 & t+\alpha & 0 & t+\delta & t+\varepsilon \\ 1 & t+\beta & t+\delta & 0 & t+\zeta \\ 1 & t+\gamma & t+\varepsilon & t+\zeta & 0 \end{vmatrix} = 4t^3+30t^2+At+\mathrm{B},$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ is some permutation of the numbers $0, 1, 2, 3, 4, 5$ and A and B are integral numbers.

The difference of two such polynomials is always a linear form in t with constant coefficients. Therefore if two of these polynomials have a common root, this root is necessarily rational.

Below we give in Table I the polynomials $U(t) = \frac{1}{2}D(t)-2t^3-15t^2$ and the values of $D(t)$ for $t = \frac{1}{2}$, $t = 1$, $t = \frac{3}{2}$ and $t = 2$. From this table it is seen that:

1. All $U(t)$ have integral coefficients.

2. For $t = 0$ all D_i are negative.

3. Since $\frac{1}{2}D_i = 2t^3+15t^2+\varrho_i t-\sigma_i$, with positive ϱ_i, σ_i, the D_i are monotone increasing for $t > 0$.

4. For $t = 2$ all D_i are positive. Hence all D_i have exactly one root between 0 and 2.

5. No D_i is equal to zero for $t = \frac{1}{2}$, $t = 1$ or $t = \frac{3}{2}$.

This completes the proof that all D_i have different positive roots.

TABLE I

Order in which the polynomials pass 0	i	$x\ y\ z$	U	$D(\frac{1}{2})$	$D(1)$	$D(\frac{3}{2})$	$D(2)$	root
26	1	$b\ d\ e$	$4t-53$	-94	-64	-13	62	1,6001
25	2	$b\ d\ f$	$4t-52$	-92	-62	-11	64	1,5852
28	3	$b\ e\ d$	$7t-77$	-139	-106	-52	26	1,8506
27	4	$b\ e\ f$	$7t-73$	-131	-98	-44	34	1,8017
30	5	$b\ f\ d$	$8t-84$	-152	-118	-63	16	1,9107
29	6	$b\ f\ e$	$8t-81$	-146	-112	-57	22	1,8760
16	7	$c\ d\ e$	$7t-17$	-19	14	68	146	0,8220
11	8	$c\ d\ f$	$7t-13$	-11	22	76	154	0,7021
23	9	$c\ e\ d$	$11t-41$	-63	-26	32	114	1,2484
19	10	$c\ e\ f$	$11t-29$	-39	-2	56	138	1,0211
24	11	$c\ f\ d$	$13t-49$	-77	-38	22	106	1,3374
22	12	$c\ f\ e$	$13t-41$	-61	-22	38	122	1,2060
6	13	$d\ c\ e$	$8t-4$	8	42	97	176	0,3111
1	14	$d\ c\ f$	$8t-1$	14	48	103	182	0,1043
21	15	$d\ e\ c$	$16t-36$	-48	-6	57	144	1,0564
7	16	$d\ e\ f$	$16t-9$	6	48	111	198	0,4025
20	17	$d\ f\ c$	$17t-37$	-49	-6	58	146	1,0554
8	18	$d\ f\ e$	$17t-13$	-1	42	106	194	0,5148
14	19	$e\ c\ d$	$13t-19$	-17	22	82	166	0,7492
9	20	$e\ c\ f$	$13t-11$	-1	38	98	182	0,5168
18	21	$e\ d\ c$	$17t-27$	-29	14	78	166	0,8603
2	22	$e\ d\ f$	$17t-3$	19	62	126	214	0,1549
13	23	$e\ f\ c$	$19t-23$	-19	26	92	182	0,7381
5	24	$e\ f\ d$	$19t-7$	13	58	124	214	0,2964
17	25	$f\ c\ d$	$16t-25$	-26	16	79	166	0,8354
12	26	$f\ c\ e$	$16t-20$	-16	26	89	176	0,7190
15	27	$f\ d\ c$	$19t-25$	-23	22	88	178	0,7823
4	28	$f\ d\ e$	$19t-5$	17	62	128	218	0,2228
10	29	$f\ e\ c$	$20t-20$	-12	34	101	192	0,6527
3	30	$f\ e\ d$	$20t-5$	18	64	131	222	0,2145

II. It is possible to construct sextuples for which the volumes are in alphabetic order. We have

THEOREM 1. $D_{2n} > D_{2n-1}$.

If $T_{2n} = (b_1, b_2, b_3)$, b_i opposite to a_i, then

$$\tag{1} \tfrac{1}{2}D_{2n} = \Big(\sum a_i^2+\sum b_i^2\Big)\sum a_i^2 b_i^2-2\sum a_i^2 b_i^2(a_i^2+b_i^2)-b_1^2 b_2^2 b_3^2- \\ -a_1^2 a_2^2 b_3^2-a_1^2 b_2^2 a_3^2-b_1^2 a_2^2 a_3^2.$$

Then $T_{2n-1} = (b_1, b_2, a_3)$, a_i opposite to b_i. Hence $\frac{1}{2}(D_{2n}-D_{2n-1}) = (a_1^2-b_1^2)(a_2^2-b_2^2)(a_3^2-b_3^2)$, where $a_i > b_i$.

Now let $b_i^2 = t+\beta_i$, $a_i^2 = t+\alpha_i$. Then we hawe $\frac{1}{2}(D_{2n}-D_{2n-1}) = (\alpha_1-\beta_1)(\alpha_2-\beta_2)(\alpha_3-\beta_3)$, independent of t. Hence for sufficient large t the 30 determinants D_i form 15 couples; the order of these couples depends on the coefficients of t in the third degree polynomials $D(t)$.

If we take $\alpha_i+\beta_i = \lambda_i$, $\alpha_i\beta_i = \mu_i$, $\sum\lambda_i = \lambda$, $\sum\mu_i = \mu$, then

$$\tfrac{1}{2}D = 2t^3+t^2+t\left\{\tfrac{1}{2}\lambda^2-\tfrac{3}{2}\sum(\alpha_i^2+\beta_i^2)-\mu\right\}+\tfrac{1}{2}D(0).$$

If we permute the a_i and b_i, this does not influence the values of λ and $\sum(a_i^2+b_i^2)$. Hence the order of the couples is the order of the coefficients $-\mu = -\sum\alpha_i\beta_i$.

Now we write

$$\begin{aligned} a^2 &= t+\alpha, & b^2 &= t+\beta, & c^2 &= t+\gamma,\\ d^2 &= t+\delta, & e^2 &= t+\varepsilon, & f^2 &= t; \end{aligned}$$

$\alpha>\beta>\gamma>\delta>\varepsilon>0$. If μ_n is the coefficient μ which occurs in the determinant D_{2n}, then the list of the μ_n is given in Table II.

TABLE II

$$\begin{array}{llll}
\mu_1 = \alpha\beta+\gamma\delta & & & \\
\mu_2 = \alpha\beta+\gamma\varepsilon & \mu_1-\mu_2 & = & \gamma(\delta-\varepsilon)>0\\
\mu_3 = \alpha\beta+\delta\varepsilon & \mu_2-\mu_3 & = & \varepsilon(\gamma-\delta)>0\\
\mu_4 = \alpha\gamma+\beta\delta & \mu_3-\mu_4 & = & \alpha(\beta-\gamma)-\delta(\beta-\varepsilon)\\
\mu_5 = \alpha\gamma+\beta\varepsilon & \mu_4-\mu_5 & = & \beta(\delta-\varepsilon)>0\\
\mu_6 = \alpha\gamma+\delta\varepsilon & \mu_5-\mu_6 & = & \varepsilon(\beta-\delta)>0\\
\mu_7 = \alpha\beta+\beta\gamma & \mu_6-\mu_7 & = & \gamma(\alpha-\beta)-\delta(\alpha-\varepsilon)\\
\mu_8 = \alpha\delta+\beta\varepsilon & \mu_7-\mu_8 & = & \beta(\gamma-\varepsilon)>0\\
\mu_9 = \alpha\delta+\gamma\varepsilon & \mu_8-\mu_9 & = & \varepsilon(\beta-\gamma)>0\\
\mu_{10} = \alpha\varepsilon+\beta\gamma & \mu_9-\mu_{10} & = & \alpha(\delta-\varepsilon)-\gamma(\beta-\varepsilon)\\
\mu_{11} = \alpha\varepsilon+\beta\delta & \mu_{10}-\mu_{11} & = & \beta(\gamma-\delta)>0\\
\mu_{12} = \alpha\varepsilon+\gamma\delta & \mu_{11}-\mu_{12} & = & \delta(\beta-\gamma)>0\\
\mu_{13} = \beta\gamma+\delta\varepsilon & \mu_{12}-\mu_{13} & = & \varepsilon(\alpha-\delta)-\gamma(\beta-\delta)\\
\mu_{14} = \beta\delta+\gamma\varepsilon & \mu_{13}-\mu_{14} & = & (\beta-\varepsilon)(\gamma-\delta)>0\\
\mu_{15} = \beta\varepsilon+\gamma\delta & \mu_{14}-\mu_{15} & = & (\beta-\gamma)(\delta-\varepsilon)>0
\end{array}$$

If $\beta>\gamma>\delta>\varepsilon>0$ are given, then we can choose α larger than the largest of β, $\delta(\beta-\varepsilon)(\beta-\gamma)^{-1}$, $(\beta\gamma-\delta\varepsilon)(\gamma-\delta)^{-1}$, $\gamma(\beta-\varepsilon)(\delta-\varepsilon)^{-1}$, $\delta+\gamma(\beta-\delta)\varepsilon^{-1}$. Then μ_i decreases if i increases. If we take t sufficiently large, then the 30 tetrahedra are ordered with respect to their volume if they are ordered alphabetically.

Example. $\alpha = 11$, $\beta = 4$, $\gamma = 3$, $\delta = 2$, $\varepsilon = 1$; $\mu_1 = 50$, $\mu_2 = 47$, $\mu_3 = 46$, $\mu_4 = 41$, $\mu_5 = 37$, $\mu_6 = 35$, $\mu_7 = 34$, $\mu_8 = 26$, $\mu_9 = 25$, $\mu_{10} = 23$, $\mu_{11} = 19$, $\mu_{12} = 17$, $\mu_{13} = 14$, $\mu_{14} = 11$, $\mu_{15} = 10$.

The result is given in Table III.

TABLE III

i	$\frac{1}{2}D-2t^3-21t^2$	$\frac{1}{2}D(89)$	i	$\frac{1}{2}D-2t^3-21t^2$	$\frac{1}{2}D(89)$
1	$-56t-371$	1 570 924	16	$-32t-123$	1 573 308
2	$-56t-364$	1 570 931	17	$-31t-211$	1 573 309
3	$-53t-431$	1 571 131	18	$-31t-139$	1 573 381
4	$-53t-403$	1 571 159	19	$-29t-43$	1 573 655
5	$-52t-444$	1 571 207	20	$-29t-23$	1 573 675
6	$-52t-423$	1 571 228	21	$-25t-99$	1 573 955
7	$-47t-209$	1 571 887	22	$-25t-39$	1 574 015
8	$-47t-193$	1 571 903	23	$-23t-107$	1 574 125
9	$-43t-281$	1 572 171	24	$-23t-67$	1 574 165
10	$-43t-233$	1 572 219	25	$-20t-7$	1 574 492
11	$-41t-301$	1 572 329	26	$-20t+4$	1 574 503
12	$-41t-269$	1 572 361	27	$-17t-43$	1 574 723
13	$-40t-76$	1 572 643	28	$-17t+1$	1 574 767
14	$-40t-67$	1 572 652	29	$-16t-44$	1 574 811
15	$-32t-204$	1 573 227	30	$-16t-11$	1 574 844

We can formulate the result as follows:

THEOREM 2. *If the T_i are in alphabetic order and if $D_i > D_j$ for any choice of $a>b>c>d>e>f>0$, then $i>j$.*

IIIa. Semi-order in not necessarily completely tetrahedral sextuples. Let (a, b, c, d, e, f) be a sextuple, $a>b>c>d>e>f>0$. We form the triples $T_i = (x_i, y_i, z_i)$, as mentioned in the introduction, whether the corresponding tetrahedra exist or not. The T_i are supposed to be written in alphabetical order.

We will say that $T_i > T_j$ if, for any sextuple, we have $D_i > D_j$. If there are sextuples in which $T_i > T_j$ and other sextuples in which $T_j > T_i$, we will call T_i and T_j *incomparable.*

From theorem 1 we already know that $T_{2n} > T_{2n-1}$. From (1), by interchanging b_1 and b_2 we derive, for $b_1 \neq b$:

$$\tfrac{1}{2}\{D(b_2, b_1, b_3)-D(b_1, b_2, b_3)\} = (a_1^2-a_2^2)(b_1^2-b_2^2)(a_1^2+a_2^2+b_1^2+b_2^2-2a_3^2-b_3^2) \tag{2}$$

and for $b_1 = b$:

$$\tfrac{1}{2}\{D(b_2, a_2, a_3)-D(b_1, b_2, b_3)\} = (a_1^2-a_2^2)(b_1^2-b_2^2)(a_1^2+a_2^2+b_1^2+b_2^2-2a_3^2-b_3^2). \tag{3}$$

From (2) we obtain: $T_{13} > T_7$, $T_{14} > T_8$, $T_{19} > T_9$, $T_{20} > T_{10}$, $T_{25} > T_{11}$, $T_{26} > T_{12}$, $T_{21} > T_{15}$, $T_{22} > T_{16}$, $T_{27} > T_{17}$, $T_{28} > T_{18}$; from (3): $T_{13} > T_2$, $T_{19} > T_4$, $T_{25} > T_6$.

If we interchange b_1 and b_3 and if $b_1 = b$, we obtain

$$(4)\qquad \tfrac{1}{2}\{D(b_3, a_3, a_2) - D(b_1, b_2, b_3)\} = (a_1^2 - a_3^2)(b_1^2 - b_3^2)(a_1^2 + b_1^2 + a_3^2 + b_3^2 - 2a_2^2 - b_2^2),$$

from which we have $T_{17} > T_3$, $T_{27} > T_4$, $T_{15} > T_5$, $T_{21} > T_6$.

If we interchange b_1 and a_2, and $b_1 = b$, we obtain

$$(5)\qquad \tfrac{1}{2}\{D(a_2, b_2, b_3) - D(b_1, b_2, b_3)\} = (a_1^2 - b_2^2)(b_1^2 - a_2^2)(a_1^2 + b_1^2 + a_2^2 + b_2^2 - a_3^2 - 2b_3^2),$$

which furnishes $T_7 > T_1$, $T_8 > T_2$, $T_9 > T_3$, $T_{10} > T_4$, $T_{11} > T_5$, $T_{12} > T_6$.

If we interchange b_1 and a_3, and $b_1 = b$, we obtain

$$(6)\qquad \tfrac{1}{2}\{D(a_3, b_3, b_2) - D(b_1, b_2, b_3)\} = (a_1^2 - b_3^2)(b_1^2 - a_3^2)(a_1^2 + b_1^2 + a_3^2 + b_3^2 - a_2^2 - 2b_2^2),$$

which leads to $T_{18} > T_4$, $T_{16} > T_6$.

If we interchange b_1 and a_3, and $b_1 \neq b$, we obtain

$$(7)\qquad \tfrac{1}{2}\{D(a_3, b_2, b_3) - D(b_1, b_2, b_3)\} = (a_1^2 - b_3^2)(b_1^2 - a_3^2)(a_1^2 + b_1^2 + a_3^2 + b_3^2 - a_2^2 - 2b_2^2).$$

From (7) we obtain $T_{30} > T_9$, $T_{16} > T_{10}$, $T_{24} > T_{11}$, $T_{18} > T_{12}$, $T_{29} > T_{15}$, $T_{23} > T_{17}$.

Less automatically we derive

$$\begin{aligned}\tfrac{1}{2}(D_{30} - D_{11}) &= \tfrac{1}{2}(D_{30} - D_{25}) + \tfrac{1}{2}(D_{25} - D_{11})\\ &= (b^2 - d^2)(c^2 - e^2)(b^2 + c^2 + d^2 + e^2 - a^2 - 2f^2) +\\ &\quad + (a^2 - b^2)(c^2 - f^2)(a^2 + b^2 + c^2 + f^2 - d^2 - 2e^2) > 0,\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(D_{29} - D_{17}) &= \tfrac{1}{2}(D_{29} - D_{23}) + \tfrac{1}{2}(D_{23} - D_{17})\\ &= (a^2 - b^2)(e^2 - f^2)(a^2 + b^2 + e^2 + f^2 - c^2 - 2d^2) +\\ &\quad + (a^2 - c^2)(d^2 - e^2)(a^2 + c^2 + d^2 + e^2 - b^2 - 2f^2)\\ &> -(a^2 - b^2)(e^2 - f^2)(d^2 - e^2) + (a^2 - c^2)(d^2 - e^2)(e^2 - f^2) > 0,\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(D_{29} - D_6) &= \tfrac{1}{2}(D_{29} - D_{15}) + \tfrac{1}{2}(D_{15} - D_5) - \tfrac{1}{2}(D_6 - D_5)\\ &> (a^2 - c^2)(d^2 - f^2)(a^2 + c^2 + d^2 + f^2 - b^2 - 2e^2) -\\ &\quad - (a^2 - b^2)(c^2 - f^2)(d^2 - e^2)\\ &> (a^2 - b^2)(d^2 - f^2)(c^2 + d^2 - 2e^2) - (a^2 - b^2)(d^2 - e^2)(c^2 - e^2) -\\ &\quad - (a^2 - b^2)(e^2 - f^2)(d^2 - e^2) > 0,\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(D_{29} - D_4) &= \tfrac{1}{2}(D_{29} - D_{10}) + \tfrac{1}{2}(D_{10} - D_4)\\ &= (a^2 - d^2)(c^2 - f^2)(a^2 + c^2 + d^2 + f^2 - 2b^2 - e^2) +\\ &\quad + (a^2 - e^2)(b^2 - c^2)(a^2 + b^2 + c^2 + e^2 - d^2 - 2f^2)\\ &> -(a^2 - d^2)(c^2 - f^2)(b^2 - c^2) + (a^2 - e^2)(b^2 - c^2)(c^2 - f^2) > 0,\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(D_{28} - D_{15}) &= \tfrac{1}{2}(D_{28} - D_{18}) + \tfrac{1}{2}(D_{18} - D_{17}) + \tfrac{1}{2}(D_{17} - D_{15})\\ &= (a^2 - b^2)(d^2 - f^2)(a^2 + b^2 + d^2 + f^2 - 2c^2 - e^2) +\\ &\quad + (a^2 - d^2)(b^2 - f^2)(c^2 - e^2) +\\ &\quad + (b^2 - c^2)(e^2 - f^2)(b^2 + c^2 + e^2 + f^2 - a^2 - 2d^2)\\ &> (a^2 - b^2)(d^2 - f^2)(b^2 - c^2) + (a^2 - d^2)(b^2 - f^2)(c^2 - e^2) -\\ &\quad - (a^2 - b^2)(e^2 - f^2)(b^2 - c^2) - (b^2 - c^2)(e^2 - f^2)(d^2 - e^2) > 0,\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(D_{28} - D_9) &= \tfrac{1}{2}(D_{28} - D_7) + \tfrac{1}{2}(D_7 - D_9)\\ &= (c^2 - f^2)(a^2 - e^2)(a^2 + c^2 + e^2 + f^2 - b^2 - 2d^2) +\\ &\quad + (b^2 - f^2)(d^2 - e^2)(2a^2 + c^2 - b^2 - d^2 - e^2 - f^2)\\ &> f^2\{(c^2 - f^2)(a^2 - e^2) - (b^2 - f^2)(d^2 - e^2)\} -\\ &\quad - (d^2 - e^2)(c^2 - f^2)(a^2 - e^2) + (d^2 - e^2)(b^2 - f^2)(a^2 - e^2)\\ &> f^2\{(c^2 - f^2)(a^2 - d^2) - (b^2 - c^2)(d^2 - e^2)\} > 0,\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(D_{27} - D_6) &= \tfrac{1}{2}(D_{27} - D_{21}) + \tfrac{1}{2}(D_{21} - D_6)\\ &= (a^2 - c^2)(e^2 - f^2)(a^2 + c^2 + e^2 + f^2 - b^2 - 2d^2) +\\ &\quad + (a^2 - d^2)(b^2 - e^2)(a^2 + b^2 + d^2 + e^2 - 2c^2 - f^2)\\ &> -(a^2 - c^2)(e^2 - f^2)(d^2 - e^2) + (a^2 - d^2)(b^2 - e^2)(e^2 - f^2) > 0,\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(D_{23} - D_6) &= \tfrac{1}{2}(D_{23} - D_{21}) + \tfrac{1}{2}(D_{21} - D_6)\\ &= (b^2 - c^2)(d^2 - f^2)(b^2 + c^2 + d^2 + f^2 - a^2 - 2e^2) +\\ &\quad + (a^2 - d^2)(b^2 - e^2)(a^2 + b^2 + d^2 + e^2 - 2c^2 - f^2)\\ &> -(b^2 - c^2)(d^2 - f^2)(a^2 - b^2) + (b^2 - e^2)(d^2 - f^2)(a^2 - d^2) > 0,\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(D_{23} - D_4) &= \tfrac{1}{2}(D_{23} - D_{17}) + \tfrac{1}{2}(D_{17} - D_3) - \tfrac{1}{2}(D_4 - D_3)\\ &= (a^2 - c^2)(d^2 - e^2)(a^2 + c^2 + d^2 + e^2 - b^2 - 2f^2) +\\ &\quad + (a^2 - f^2)(b^2 - d^2)(a^2 + b^2 + d^2 + f^2 - 2c^2 - e^2) -\\ &\quad - (a^2 - b^2)(d^2 - e^2)(d^2 - f^2) - (a^2 - b^2)(c^2 - d^2)(d^2 - f^2) > 0,\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(D_{21} - D_4) &= \tfrac{1}{2}(D_{27} - D_4) - \tfrac{1}{2}(D_{27} - D_{21})\\ &= (a^2 - d^2)(b^2 - f^2)(a^2 + b^2 + d^2 + f^2 - 2c^2 - e^2) -\\ &\quad - (a^2 - c^2)(e^2 - f^2)(a^2 + c^2 + e^2 + f^2 - b^2 - 2d^2)\\ &> (a^2 - d^2)(b^2 - f^2)(a^2 + b^2 + d^2 - 2c^2 - e^2) -\\ &\quad - (a^2 - c^2)(e^2 - f^2)(a^2 + c^2 + e^2 - b^2 - 2d^2)\\ &> (a^2 - d^2)(e^2 - f^2)(a^2 - c^2) -\\ &\quad - (a^2 - c^2)(e^2 - f^2)(a^2 - b^2 + c^2 - d^2) > 0,\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(D_{17}-D_5) &= \tfrac{1}{2}(D_{17}-D_3)-\tfrac{1}{2}(D_5-D_3)\\ &= (a^2-f^2)(b^2-d^2)(a^2+b^2+d^2+f^2-2c^2-e^2)-\\ &\quad -(c^2-d^2)(e^2-f^2)(c^2+d^2+e^2+f^2-a^2-2b^2)\\ &> f^2(a^2-f^2)(b^2-d^2)-f^2(c^2-d^2)(e^2-f^2)>0,\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(D_{15}-D_3) &= \tfrac{1}{2}(D_{15}-D_9)+\tfrac{1}{2}(D_9-D_3)\\ &= (a^2-f^2)(c^2-d^2)(a^2+c^2+d^2+f^2-2b^2-e^2)+\\ &\quad +(a^2-e^2)(b^2-c^2)(a^2+b^2+c^2+e^2-2d^2-f^2)\\ &> -(a^2-f^2)(c^2-d^2)(b^2-c^2)+(a^2-e^2)(b^2-c^2)(a^2-f^2)>0.\end{aligned}$$

The present results can be found in table IV. Thus the relation $T_{15} > T_3$ is indicated by a black mark at the place (15, 3). From the above inequalities other relations can be derived. Thus from $T_{28} > T_{18}$, $T_{18} > T_{12}$, $T_{12} > T_{11}$ follows $T_{28} > T_{11}$. This relation is indicated in table IV by a cross at the place (28, 11). The above relations and those directly derivable from them are the only possible inequalities. All other couples are incomparable. Counterexamples are given in table IV and table V. So a 7 at the place (24, 15) in table IV signifies that in column 7 of table V there is an example of a tetrahedron for which $\frac{1}{2}D_{15} = 609 > 608 = \frac{1}{2}D_{24}$. Since in the example of table III we have $D_{24} > D_{15}$ the tetrahedra T_{15} and T_{24} are incomparable.

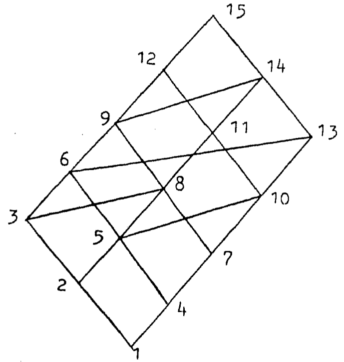


Fig. 1

Table IV contains also zero marks, grouped in square blocks of four. To explain what they mean we have to go back to table II. There we listed 15 coefficients μ_i occurring in the third degree polynomials $\frac{1}{2}D_i(t)$ which arise from $a^2 = t+\alpha$, $b^2 = t+\beta$, $c^2 = t+\gamma$, $d^2 = t+\delta$, $e^2 = t+\varepsilon$, $f^2 = t$.

If we can find numbers $\alpha, \beta, \gamma, \delta, \varepsilon$ with $\alpha > \beta > \gamma > \delta > \varepsilon > 0$ such that $\mu_i > \mu_j$, $i > j$, we have an example that both elements of the couple T_{2i}, T_{2i-1} are incomparable with T_{2j} and T_{2j-1}. They are even incomparable if we restrict ourselves to sextuples which are completely tetrahedral. For if we take t very large, all edges have nearly the same length; thus the set is completely tetrahedral. In section II we showed that the term $-\mu t$ was the only non-constant term which was different in D_{2i} and D_{2j}. Therefore

from $\mu_i > \mu_j$ follows $D_{2i-1} < D_{2i} < D_{2j-1} < D_{2j}$. Figure 1 gives the lattice of the couples $C_i = (T_{2i}, T_{2i-1})$, where we now define $C_i > C_j$ if $\mu_i < \mu_j$ for all quintuples $\alpha, \beta, \gamma, \delta, \varepsilon$. Again, if there are quintuples $\alpha, \beta, \gamma, \delta, \varepsilon$ such that $\mu_i > \mu_j$ for $i > j$, we call C_i and C_j incomparable.

The inequalities in table II give the lines in fig. 1 which are parallel to the short sides of the parallelogram.

To prove the other inequalities given in fig. 1 we have to complete table II by

TABLE IIa

$\mu_1-\mu_4 = (\alpha-\delta)(\beta-\gamma) > 0$	$\mu_7-\mu_{10} = \alpha(\delta-\varepsilon) > 0$
$\mu_2-\mu_5 = (\alpha-\varepsilon)(\beta-\gamma) > 0$	$\mu_8-\mu_{11} = (\alpha-\beta)(\delta-\varepsilon) > 0$
$\mu_3-\mu_6 = \alpha(\beta-\gamma) > 0$	$\mu_9-\mu_{12} = (\alpha-\gamma)(\delta-\varepsilon) > 0$
$\mu_4-\mu_7 = (\alpha-\beta)(\gamma-\delta) > 0$	$\mu_{10}-\mu_{13} = \varepsilon(\alpha-\delta) > 0$
$\mu_5-\mu_8 = \alpha(\gamma-\delta) > 0$	$\mu_{11}-\mu_{14} = \varepsilon(\alpha-\gamma) > 0$
$\mu_6-\mu_9 = (\alpha-\varepsilon)(\gamma-\delta) > 0$	$\mu_{12}-\mu_{15} = \varepsilon(\alpha-\beta) > 0$
$\mu_5-\mu_{10} = (\alpha-\beta)(\gamma-\varepsilon) > 0$	$\mu_6-\mu_{13} = \gamma(\alpha-\beta) > 0$
$\mu_3-\mu_8 = (\alpha-\varepsilon)(\beta-\delta) > 0$	$\mu_9-\mu_{14} = \delta(\alpha-\beta) > 0$

To prove that the system of relations is exhausted by those given in fig. 1 we have to prove $C_2\,\mathrm{I}\,C_7$, $C_3\,\mathrm{I}\,C_{10}$, $C_6\,\mathrm{I}\,C_{11}$, $C_8\,\mathrm{I}\,C_{13}$, $C_{12}\,\mathrm{I}\,C_{14}$, where $C_i\,\mathrm{I}\,C_j$ means that C_i and C_j are incomparable. These relations follow from

$$\mu_2-\mu_7 = \alpha(\beta-\delta)-\gamma(\beta-\varepsilon) < 0 \quad \text{for } \alpha = 6,\ \beta = 5,\ \gamma = 4,\ \delta = 3,\ \varepsilon = 1;$$

$$\mu_3-\mu_{10} = \beta(\alpha-\gamma)-\varepsilon(\alpha-\delta) < 0 \quad \text{for } \alpha = 11,\ \beta = 10,\ \gamma = 9,\ \delta = 8,\ \varepsilon = 7;$$

$$\mu_6-\mu_{11} = \alpha(\gamma-\varepsilon)-\delta(\beta-\varepsilon) < 0 \quad \text{for } \alpha = 10,\ \beta = 9,\ \gamma = 4,\ \delta = 3,\ \varepsilon = 2;$$

$$\mu_8-\mu_{13} = \delta(\alpha-\varepsilon)-\beta(\gamma-\varepsilon) < 0 \quad \text{for } \alpha = 6,\ \beta = 5,\ \gamma = 4,\ \delta = 2,\ \varepsilon = 1;$$

$$\mu_{12}-\mu_{14} = \varepsilon(\alpha-\gamma)-\delta(\beta-\gamma) < 0 \quad \text{for } \alpha = 6,\ \beta \sim = 5,\ \gamma = 4,\ \delta = 3,\ \varepsilon = 1.$$

From $C_2\,\mathrm{I}\,C_7$ follow $C_2\,\mathrm{I}\,C_4$, $C_3\,\mathrm{I}\,C_7$, $C_5\,\mathrm{I}\,C_7$, $C_6\,\mathrm{I}\,C_7$, $C_3\,\mathrm{I}\,C_4$.
From $C_3\,\mathrm{I}\,C_{10}$ follow $C_3\,\mathrm{I}\,C_5$, $C_6\,\mathrm{I}\,C_{10}$, $C_9\,\mathrm{I}\,C_{10}$.
From $C_6\,\mathrm{I}\,C_{11}$ follow $C_6\,\mathrm{I}\,C_8$, $C_9\,\mathrm{I}\,C_{11}$.
From $C_8\,\mathrm{I}\,C_{13}$ follow $C_8\,\mathrm{I}\,C_{10}$, $C_9\,\mathrm{I}\,C_{13}$, $C_{11}\,\mathrm{I}\,C_{13}$, $C_{12}\,\mathrm{I}\,C_{13}$.
From $C_i\,\mathrm{I}\,C_j$ follow $T_{2i}\,\mathrm{I}\,T_{2j}$, $T_{2i-1}\,\mathrm{I}\,T_{2j}$, $T_{2i}\,\mathrm{I}\,T_{2j-1}$, $T_{2i-1}\,\mathrm{I}\,T_{2j-1}$.

Then in table IV there are marks 0 at the corresponding places. We have proved that table IV gives the whole system of inequality relations and incomparability relations between the tetrahedra. For the inequality relations the corresponding graph is given in figure 2.

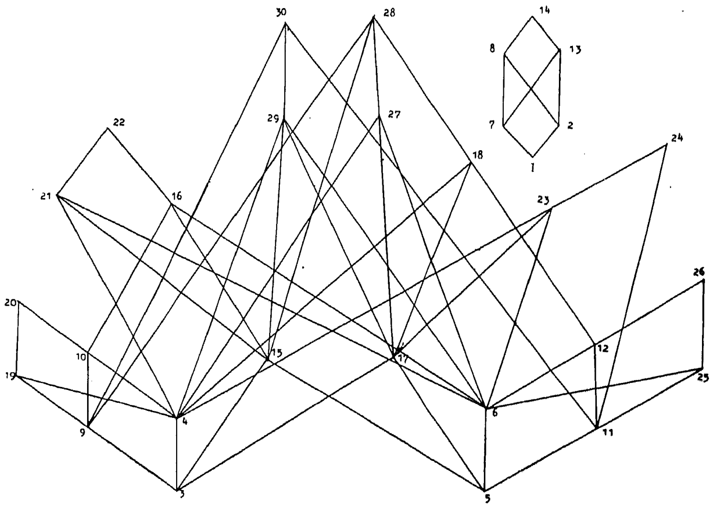


Fig. 2

IIIb. Semi-order in completely tetrahedral sextuples. In table IV there are 11 places marked 10 or 11. These are the places (30, 12), (30, 18), (29, 9), (29, 11), (29, 12), (27, 11), (24, 12), (24, 18), (23, 11), (23, 12), and (17, 11). They refer to column 10 and column 11 in table V, where the corresponding sextuples are seen to be incompletely tetrahedral. In this section we are going to prove that the scheme of relations for completely tetrahedral sextuples can be obtained from table IV by painting all these places black. Then the corresponding graph becomes that of figure 3.

From (1) we have:

$$\begin{aligned}\tfrac{1}{2}D_5 = {} & a^2b^2(c^2+d^2+e^2+f^2-a^2-b^2)-e^2(c^2-d^2)-d^2(e^2-f^2)-\\ & -a^2(c^2-f^2)(d^2-e^2)-(b^2-c^2)(c^2-d^2)(e^2-f^2)-\\ & -(a^2-f^2)(c^2-d^2)(e^2-f^2).\end{aligned}$$

In any case $\tfrac{1}{2}D_5 < a^2b^2(c^2+d^2+e^2+f^2-a^2-b^2)$. Now

$$\begin{aligned}\tfrac{1}{2}(D_{11}-D_{17}) &= (c^2-d^2)(a^2-e^2)(2b^2+f^2-a^2-c^2-d^2-e^2),\\ \tfrac{1}{2}(D_{12}-D_{23}) &= (c^2-e^2)(a^2-d^2)(2b^2+f^2-a^2-c^2-d^2-e^2),\\ \tfrac{1}{2}(D_{18}-D_{24}) &= (d^2-e^2)(a^2-c^2)(2b^2+f^2-a^2-c^2-d^2-e^2).\end{aligned}$$

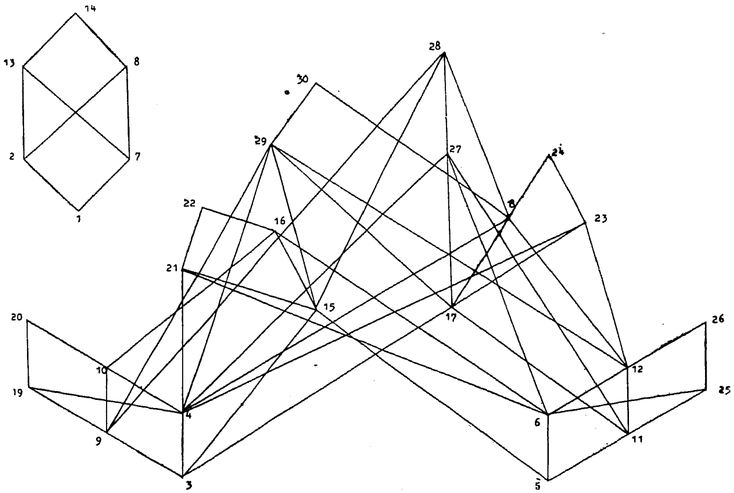
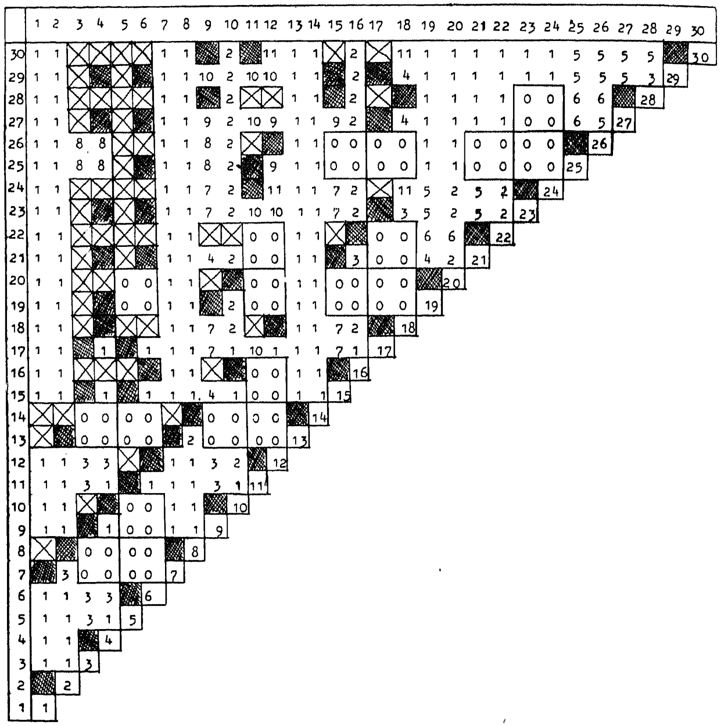


Fig. 3

If $2b^2+f^2-a^2-c^2-d^2-e^2 > 0$ it follows $b^2 > c^2+d^2$, $a^2 > e^2+f^2$, from which it is directly seen that $D_5 < 0$. Hence, for completely tetrahedral sextuples, we have $T_{17} > T_{11}$, $T_{23} > T_{12}$, $T_{24} > T_{18}$, and consequently also $T_{23} > T_{11}$, $T_{24} > T_{12}$, $T_{29} > T_{11}$, $T_{27} > T_{11}$. Further

$$\begin{aligned}\tfrac{1}{2}(D_{12}-D_{29}) &= \tfrac{1}{2}(D_{12}-D_{26})+\tfrac{1}{2}(D_{26}-D_{29})\\ &= -(a^2-b^2)(c^2-f^2)(a^2+b^2+c^2+f^2-2d^2-e^2)+\\ &\quad +(b^2-d^2)(c^2-e^2)(2a^2+f^2-b^2-c^2-d^2-e^2)\\ &= (b^2-d^2)(c^2-e^2)(b^2+f^2-c^2-d^2-e^2)-\\ &\quad -(a^2-b^2)\{(c^2-f^2)(a^2-b^2+c^2-e^2+f^2)+2(b^2-d^2)(e^2-f^2)\}\\ &< (b^2-d^2)(c^2-e^2)(b^2+f^2-c^2-d^2-e^2).\end{aligned}$$

TABLE IV



Thus if $D_{12} > D_{29}$ we have $b^2+f^2 > c^2+d^2+e^2$, and $D_5 < 0$. Hence, for completely tetrahedral sextuples, $T_{29} > T_{12}$ and $T_{30} > T_{12}$.

Next we have

$$\tfrac{1}{2}(D_{18}-D_{30}) = \tfrac{1}{2}(D_{18}-D_{28})+\tfrac{1}{2}(D_{28}-D_{30})$$

$$= (a^2-b^2)(d^2-f^2)(2c^2+e^2-a^2-b^2-d^2-f^2)+$$

$$+(b^2-c^2)(d^2-e^2)(2a^2+f^2-b^2-c^2-d^2-e^2)$$

$$= (b^2-c^2)(d^2-e^2)(b^2+f^2-c^2-d^2-e^2)-$$

$$-(a^2-b^2)\{(d^2-f^2)(a^2-b^2+d^2-e^2+f^2)+2(b^2-c^2)(e^2-f^2)\}$$

$$< (b^2-c^2)(d^2-e^2)(b^2+f^2-c^2-d^2-e^2).$$

TABLE V

	1	2	3	4	5	6	7	8	9	10	11
a, b	20,19	20,20	5,4	20,20	21,10	29,11	15,8	10,10	7,7	6,6	11,11
c, d	19,19	12,12	4,3	12,11	9,9	11,10	7,7	10,8	6,6	3,2	5,5
e, f	9,8	12,7	3,2	11,7	8,8	10,10	7,6	8,3	3,2	2,2	3,3
1	4301	900	52	205	627	309	473	608	105	−110	−702
2	4301	900	53	205	627	309	473	608	105	−110	−702
3	3861	900	52	205	609	309	473	608	57	−110	−770
4	3971	900	53	205	620	309	473	608	57	−110	−770
5	3861	900	51	129	609	309	473	598	57	−110	−770
6	3971	900	51	129	620	309	473	598	57	−110	−770
7	4301	3012	52	2653	927	309	609	608	123	22	126
8	4301	3332	53	2941	927	309	617	608	124	22	126
9	3861	3012	52	2653	895	309	609	608	93	22	46
10	3971	3332	53	2941	919	309	617	608	109	22	142
11	3861	3292	51	2833	895	309	608	598	92	22	46
12	3971	3292	51	2833	919	309	608	598	107	22	142
13	4301	3012	61	2653	927	867	609	608	123	22	126
14	4301	3332	61	2941	927	867	617	608	124	22	126
15	3861	3012	61	2640	895	841	609	650	93	18	46
16	3971	3332	65	3045	919	860	617	678	109	34	142
17	3861	3292	61	2896	895	841	608	650	92	18	46
18	3971	3292	65	3013	919	860	608	678	107	34	142
19	4181	3012	61	2653	1148	867	609	608	93	22	46
20	4181	3332	61	2941	1161	867	617	608	109	22	142
21	4181	3012	61	2640	1148	841	609	650	93	18	46
22	4181	3332	65	3045	1161	860	617	678	109	34	142
23	4181	3292	61	2896	1147	841	608	650	125	18	130
24	4181	3292	65	3013	1147	860	608	678	125	34	130
25	4180	3292	63	2833	1148	867	713	598	92	22	46
26	4180	3292	63	2833	1161	867	713	598	107	22	142
27	4180	3292	65	2896	1148	841	713	650	92	18	46
28	4180	3292	68	3013	1161	860	713	678	107	34	142
29	4180	3292	65	2896	1147	841	713	650	125	18	130
30	4180	3292	68	3013	1147	860	713	678	125	34	130

Thus again, if $D_{18}-D_{30} > 0$, then $D_5 < 0$. For completely tetrahedral sextuples $T_{30} > T_{18}$.

It remains to prove $D_5 < 0$ if $D_9 > D_{29}$.

$$\tfrac{1}{2}(D_9-D_{29}) = \tfrac{1}{2}(D_9-D_{10})+\tfrac{1}{2}(D_{10}-D_{29})$$

$$= -(a^2-c^2)(b^2-e^2)(d^2-f^2)-$$

$$-(a^2-d^2)(c^2-f^2)(a^2+c^2+d^2+f^2-2b^2-e^2)$$

$$= (b^2-d^2)(c^2-d^2)\{b^2-c^2-d^2+e^2-f^2(c^2-f^2)(c^2-d^2)^{-1}\}-$$

$$-(a^2-b^2)(c^2-f^2)(a^2-b^2+c^2-e^2+f^2)-$$

$$-(a^2-b^2)(b^2-e^2)(d^2-f^2)-(b^2-c^2)(d^2-e^2)(d^2-f^2).$$

Thus if $D_9 > D_{29}$, then $b^2+e^2 > c^2+d^2+\lambda f^2$, $\lambda = (c^2-f^2)(c^2-d^2)^{-1} > 1$. In that case

$$\begin{aligned}\tfrac{1}{2}D_5 = b^4(c^2+d^2+e^2+f^2-2b^2)-b^2(a^2-b^2)(a^2+2b^2-c^2-d^2-e^2-f^2)- \\ -(b^2-c^2)(c^2-d^2)(e^2-f^2)-e^2(c^2-d^2)^2-d^2(e^2-f^2)^2- \\ -(a^2-f^2)(c^2-d^2)(e^2-f^2)-a^2(c^2-f^2)(d^2-e^2) \\ < b^4(c^2+d^2+e^2+f^2-2b^2).\end{aligned}$$

Thus from $D_9 > D_{29}$ and $D_5 > 0$ it follows $3e^2 > c^2+d^2+(2\lambda-1)f^2$, whence $e^2-f^2 > c^2-d^2$, from which $\lambda > 2$.

Since b, e, f are sides of a triangle, they satisfy $b^2-e^2-f^2 < 2ef$. Now take $f = 1$, $d^2 = e^2+\delta$, $c^2 = e^2+\gamma$, $b^2-e^2-1 = \beta$. Then $\beta < 2e$, further $\beta > \gamma+\delta+\lambda-1 = \gamma+\delta+(e^2+\delta-1)(\gamma-\delta)^{-1} > 2\delta+2(e^2+\delta-1)^{1/2}$. From $2\delta+2(e^2+\delta-1)^{1/2} < \beta < 2e$ follows $\delta < 1$, and even $\delta < e-(e^2-1)^{1/2} = \{e+(e^2-1)^{1/2}\}^{-1}$. From $\beta > 2(e^2-1)^{1/2}$ follows $\beta = 2e-\theta$, $\theta < (e^2-1)^{1/2}$. From $\gamma-\delta+(e^2+\delta-1)(\gamma-\delta)^{-1} < 2e-2\delta$ follows $(\gamma-\delta)^2-2(e-\delta)(\gamma-\delta)+(e-\delta)^2 < 1-\delta+\delta^2-2e\delta < 1$, or $e-1 < \gamma < e+1$. From $e^2 > \gamma+\delta+(2\lambda-1) > e+2$ follows $e > 2$, $\delta < 2-3^{1/2}$, $\theta < 3^{-1/2}$. If we now compute $\frac{1}{2}D_5$ for $a^2 = b^2 = (e+1)^2-\theta$, $c^2 = e^2+e+\zeta$, $d^2 = e^2+\delta$, $|\zeta| < 1$, the result is $\frac{1}{2}D_5 = -4e^4-pe^3+qe^2+re+s$, where

$$\begin{aligned} p &= 16-9\theta-5\delta-4\zeta > 2+2\cdot 3^{1/2} > 5, \\ q &= 24\theta-20+11\delta+8\zeta-3\theta^2-\delta^2-\zeta^2-\delta\zeta-2\delta\theta-2\zeta\theta < 3, \\ r &= 19\theta-8+5\delta+2\zeta-11\theta^2-6\delta\theta-4\zeta\theta+2\delta\zeta+\delta^2 < 0, \\ s &= 1. \end{aligned}$$

Hence $\frac{1}{2}D_5 < -4e^4-5e^3+3e^2+1 < -91$ for $e > 2$.

REFERENCES

[1] L. M. Blumenthal, *A budget of curiosa metrica*, American Mathematical Monthly 66 (1959), p. 453-460.

[2] F. Herzog, *Completely tetrahedral sextuples*, ibidem 66 (1959), p. 460-464.

[3] H. Steinhaus, *Problem 309*, The New Scottish Book, Wrocław 1946-1958, p. 33.

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TRANSFORMATIONS OF COMPLEX SERIES

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Even the most recent editions of the book "Theorie und Anwendung der unendlichen Reihen" by K. Knopp do not mention certain problems concerning rearrangements and some other transformations of complex series, in particular, problems that have been solved since the first edition of the book had appeared. The intention of the present paper is to give a review of recent research in this domain and of its bibliography.

I. INTRODUCTION

1. Notation. Let S denote a series $S \equiv z_1+z_2+\ldots$ We assume that the sequence $\{z_n\}$ ($z_n = x_n+iy_n$ for $n = 1, 2, \ldots$) contains infinitely many terms different from 0, and that its limit equals 0.

A sequence differing from the natural sequence $1, 2, 3, \ldots$ at most in the order of terms will be called a *permutation* and denoted by $N \equiv \{N_n\}$.

By t we denote a sequence $t \equiv \{t_n\}$, each term of this sequence being a number of a fixed set of complex numbers T.

By St we mean the series $St \equiv t_1z_1+t_2z_2+\ldots$

$S(N)$ denotes a series which arises from the series S by rearrangement of its terms according to the permutation N.

Complex numbers will sometimes be treated as vectors and vice versa; the double notation, however, will not be introduced.

The "ordinary" complex plane will be denoted by H.

A space obtained from the plane H by joining to it a point at infinity will be denoted by H'.

H^* will denote a plane H with joined elements of the form (∞, φ), for which we assume $|(\infty, \varphi)| = \infty$, and $\arg(\infty, \varphi) = \varphi$, where $0 \leqslant \varphi < 2\pi$. By the *neighbourhood* of a point $z_0 \epsilon H^*$ we shall mean the interior of every circle with positive radius and centre at point z_0 if $|z_0| < \infty$. In the opposite case, by the neighbourhood of a point $z_0 \epsilon H^*$ we shall mean the set $A(z_0, \varepsilon)$ of elements satisfying (for $\varepsilon > 0$) the condition $1/\varepsilon < |z| \leqslant \infty$ and $|\arg z - \arg z_0| < \varepsilon$. It is easy to check that in the to-