Thus if $D_9 > D_{29}$, then $b^2+e^2 > c^2+d^2+\lambda f^2$, $\lambda = (c^2-f^2)(c^2-d^2)^{-1} > 1$. In that case

$$\tfrac{1}{2}D_5 = b^4(c^2+d^2+e^2+f^2-2b^2)-b^2(a^2-b^2)(a^2+2b^2-c^2-d^2-e^2-f^2)-$$
$$-(b^2-c^2)(c^2-d^2)(e^2-f^2)-e^2(c^2-d^2)^2-d^2(e^2-f^2)^2-$$
$$-(a^2-f^2)(c^2-d^2)(e^2-f^2)-a^2(c^2-f^2)(d^2-e^2)$$
$$< b^4(c^2+d^2+e^2+f^2-2b^2).$$

Thus from $D_9 > D_{29}$ and $D_5 > 0$ it follows $3e^2 > c^2+d^2+(2\lambda-1)f^2$, whence $e^2-f^2 > c^2-d^2$, from which $\lambda > 2$.

Since b, e, f are sides of a triangle, they satisfy $b^2-e^2-f^2 < 2ef$. Now take $f=1$, $d^2 = e^2+\delta$, $c^2 = e^2+\gamma$, $b^2-e^2-1 = \beta$. Then $\beta < 2e$, further $\beta > \gamma+\delta+\lambda-1 = \gamma+\delta+(e^2+\delta-1)(\gamma-\delta)^{-1} > 2\delta+2(e^2+\delta-1)^{1/2}$. From $2\delta+2(e^2+\delta-1)^{1/2} < \beta < 2e$ follows $\delta < 1$, and even $\delta < e-(e^2-1)^{1/2} = \{e+(e^2-1)^{1/2}\}^{-1}$. From $\beta > 2(e^2-1)^{1/2}$ follows $\beta = 2e-\theta$, $\theta < (e^2-1)^{1/2}$. From $\gamma-\delta+(e^2+\delta-1)(\gamma-\delta)^{-1} < 2e-2\delta$ follows $(\gamma-\delta)^2-2(e-\delta)(\gamma-\delta)+(e-\delta)^2 < 1-\delta+\delta^2-2e\delta < 1$, or $e-1 < \gamma < e+1$. From $e^2 > \gamma+\delta+(2\lambda-1) > e+2$ follows $e > 2$, $\delta < 2-3^{1/2}$, $\theta < 3^{-1/2}$. If we now compute $\frac{1}{2}D_5$ for $a^2 = b^2 = (e+1)^2-\theta$, $c^2 = e^2+e+\zeta$, $d^2 = e^2+\delta$, $|\zeta| < 1$, the result is $\frac{1}{2}D_5 = -4e^4-pe^3+qe^2+re+s$, where

$$p = 16-9\theta-5\delta-4\zeta > 2+2\cdot 3^{1/2} > 5,$$
$$q = 24\theta-20+11\delta+8\zeta-3\theta^2-\delta^2-\zeta^2-\delta\zeta-2\delta\theta-2\zeta\theta < 3,$$
$$r = 19\theta-8+5\delta+2\zeta-11\theta^2-6\delta\theta-4\zeta\theta+2\delta\zeta+\delta^2 < 0,$$
$$s = 1.$$

Hence $\frac{1}{2}D_5 < -4e^4-5e^3+3e^2+1 < -91$ for $e > 2$.

REFERENCES

[1] L. M. Blumenthal, *A budget of curiosa metrica*, American Mathematical Monthly 66 (1959), p. 453-460.

[2] F. Herzog, *Completely tetrahedral sextuples*, ibidem 66 (1959), p. 460-464.

[3] H. Steinhaus, *Problem 309*, The New Scottish Book, Wrocław 1946-1958, p. 33.

Reçu par la Rédaction le 5. 6. 1961

COLLOQUIUM MATHEMATICUM

VOL. IX 1962 FASC. 2

TRANSFORMATIONS OF COMPLEX SERIES

BY

B. JASEK (WROCŁAW)

Even the most recent editions of the book "Theorie und Anwendung der unendlichen Reihen" by K. Knopp do not mention certain problems concerning rearrangements and some other transformations of complex series, in particular, problems that have been solved since the first edition of the book had appeared. The intention of the present paper is to give a review of recent research in this domain and of its bibliography.

I. INTRODUCTION

1. Notation. Let S denote a series $S \equiv z_1+z_2+\ldots$ We assume that the sequence $\{z_n\}$ ($z_n = x_n+iy_n$ for $n = 1, 2, \ldots$) contains infinitely many terms different from 0, and that its limit equals 0.

A sequence differing from the natural sequence $1, 2, 3, \ldots$ at most in the order of terms will be called a *permutation* and denoted by $N \equiv \{N_n\}$.

By t we denote a sequence $t \equiv \{t_n\}$, each term of this sequence being a number of a fixed set of complex numbers T.

By St we mean the series $St \equiv t_1z_1+t_2z_2+\ldots$

$S(N)$ denotes a series which arises from the series S by rearrangement of its terms according to the permutation N.

Complex numbers will sometimes be treated as vectors and vice versa; the double notation, however, will not be introduced.

The "ordinary" complex plane will be denoted by H.

A space obtained from the plane H by joining to it a point at infinity will be denoted by H'.

H^* will denote a plane H with joined elements of the form (∞, φ), for which we assume $|(\infty, \varphi)| = \infty$, and $\arg(\infty, \varphi) = \varphi$, where $0 \leqslant \varphi < 2\pi$. By the *neighbourhood* of a point $z_0 \in H^*$ we shall mean the interior of every circle with positive radius and centre at point z_0 if $|z_0| < \infty$. In the opposite case, by the neighbourhood of a point $z_0 \in H^*$ we shall mean the set $A(z_0, \varepsilon)$ of elements satisfying (for $\varepsilon > 0$) the condition $1/\varepsilon < |z| \leqslant \infty$ and $|\arg z - \arg z_0| < \varepsilon$. It is easy to check that in the topology generated by the fundamental system of neighbourhoods which we have defined H^* is a topological compact space.

Let the initial point of the vector z_1 be placed at the origin of the system of coordinates, the initial point of the vector z_2 at the end point of the vector z_1 and the initial point of the vector z_3 at the point of the vector z_2. We proceed analogously with the remaining terms of the series S. In this way we obtain an infinite polygonal line, and we denote it by $L \equiv L(S)$. $L_n \equiv L_n(S)$ denotes the end point of the vector z_n in polygonal line $L(S)$.

A set of limit points of the sequence of points $\{L_n\}$ (i. e., of the sequence of partial sums of the series S) in the space H^* will be denoted by $L^* \equiv L^*(S)$.

A set of limit points of the sequence of points $\{L_n\}$ in the space H' will be denoted by $L' \equiv L'(S)$.

2. Classification of the series S. In present-day investigations a division of all series S into four classes is very often used. A formulation of the principle of this division will be preceded by some definitions and theorems.

Definition 1. A unit vector v with initial point at the origin will be called a *direction of convergence* of series S if the series $\sum_{n=1}^{\infty}|(v, z_n)|$ is convergent ((v, z_n) denotes the inner product of vectors v and z_n).

Definition 2. If a unit vector v with initial point at the origin is not a direction of convergence of the series S it is called a *direction of divergence* of this series.

Definition 3. A unit vector v with initial point at the origin is called a *principal direction* of series S if each open angular region with vertex at the origin and containing vector v is such that the sum of absolute values of the terms of series S which belong to this region is equal to ∞.

The notions given in the above definitions were introduced by Steinitz [35]. We must underline that there are also other definitions for these notions, and there is no agreement as to the terminology. E. g., principal direction is called direction of condensation in [29], strong direction in [25], and direction of divergence in [13].

THEOREM 1. *A series S has no principal direction if and only if it is absolutely convergent.*

A proof of this simple theorem can be found in [30] (chapter III, problem 52).

THEOREM 2. *If a series S is not absolutely convergent, it has at most two directions of convergence.*

Proof. Suppose a series S has more than two directions of convergence. Thus among these directions there are two directions linearly independent, say v_1 and v_2. Then each unit vector can be written in the form $v = a_1 v_1 + a_2 v_2$, where a_1 and a_2 are certain real numbers. Then we have

$$|(v, z_n)| = |a_1(v_1, z_n) + a_2(v_2, z_n)| \leqslant |a_1|\,|(v_1, z_n)| + |a_2|\,|(v_2, z_n)|.$$

From the above inequality and from definition 1 it follows that for each vector v the series $\sum_{n=1}^{\infty}|(v, z_n)|$ is convergent. Assuming at first $v = 1$, and then $v = i$ we should obtain the convergence of both series, $\sum_{n=1}^{\infty}|x_n|$ and $\sum_{n=1}^{\infty}|y_n|$, i. e., an absolute convergence of the series S which contradicts the assumption.

THEOREM 3. *If v is a principal direction of a series S and v' is a unit vector perpendicular to v, then for each $\varepsilon > 0$ there exists a sequence of natural numbers $i_1 < i_2 < \ldots$ such that*

$$\sum_{k=1}^{\infty}|(v, z_{i_k})| = \infty \quad \textit{and} \quad \sum_{k=1}^{\infty}|(v', z_{i_k})| < \varepsilon.$$

A simple proof of this theorem can be found in [26].

There are series S for which every unit vector is a principal direction, e. g., a series $\sum_{n=1}^{\infty} e^{in\varphi}/n$ for each φ not commensurable with π [26].

For each series S the set of its principal directions is closed. It is also easy to construct a series S for which the set of principal directions is equal to an arbitrarily given closed set of unit vectors with initial points at the origin of coordinates.

Definition 4. Let us denote by R_1 the class of absolutely convergent series S, by R_2 the class of series S with exactly one axis of convergence (i. e., two opposite directions of convergence in the sense of definition 1), by R_3 the class of series S with exactly one principal direction or exactly two opposite principal directions and without an axis of convergence, finally by R_4 the class of series with at least two principal directions linearly independent.

It is easy to check that none of the sets R_1, R_2, R_3 and R_4 is empty. They are disjoint and each series S belongs to one of them. As far as we know, no definite names for the classes R_1, R_2, R_3 and R_4 have been introduced.

Series with terms

$$z_n = \frac{1}{n} + \frac{1}{2^n} i, \quad \text{and} \quad z_n = \frac{i^n}{n}$$

are examples of the series $S \in R_2$ and $S \in R_4$ respectively.

We shall see in the theorems given in chapter III that the division of the series into the four classes R_1, R_2, R_3, R_4 agrees with many other properties of these series.

II. PROBLEMS

A. For a given series S we shall mean by $\Lambda_0(S)$ the set of sums of all series $S(N)$ which are convergent in H. What are the properties of the set $\Lambda_0(S)$?

B. Let K denote a set of all permutations N such that series $S(N)$ is convergent for all convergent series S. A set of permutations K will be called the *class of permutations preserving convergence*. What condition is necessary and sufficient that the permutation N belong to the class K?

C. What are the properties of sets $L'(S)$ and $L^*(S)$ (I. 1)?

D. A series S is fixed. To each permutation N there correspond sets $L'[S(N)]$ and $L^*[S(N)]$. Hence we have two families of sets, denoted by $\Lambda'(S)$ and $\Lambda^*(S)$ respectively. What subsets of the space $H'(H^*)$ belong to the family $\Lambda'(S)$ $(\Lambda^*(S))$?

E. A series S and a set T are fixed. We denote by $\Gamma_0(S, T)$ the set of sums of all series St convergent in H. What are the properties of the set $\Gamma_0(S, T)$?

F. A set of complex numbers M will be called a *set of convergence factors* if for every series S a sequence $m = \{m_j\}$ ($m_j \in M$ for $j = 1, 2, \ldots$) can be chosen so that series Sm is convergent (in H). What is the class of sets that are sets of convergence factors?

G. A set Q of complex numbers will be called a *set of sum factors* if, for every series S with an infinite sum of absolute values of terms and for every $z \in H$, there exists a sequence $q = \{q_n\}$ ($q_n \in Q$ for $n = 1, 2, \ldots$) such that series Sq is convergent and has the sum equal to z. What is the class of sets that are sets of sum factors?

H. A series S and a set T are fixed. Let $\Gamma^*(S, T)$ denote a family of all sets of the form $L^*(St)$. What is the family $\Gamma^*(S, T)$?

III. RESULTS

The successive sections of this chapter are devoted to the discussion of the results concerning the problems A-H.

A. Cauchy was the first to notice that the convergence of some series (with real terms) depends upon the order of their terms. Dirichlet explained this fact partially, proving that the convergence and the sum of an absolutely convergent series do not depend on the order of terms. At

the same time he gave several examples in which a rearrangement of terms of not absolutely convergent series yielded a change of its sum. Riemann [31] showed that unconditional convergence, i. e., convergence for any order of terms, implies absolute convergence, and that any preassigned sum can be obtained by an adequate rearrangement of terms of a conditionally convergent series. This theorem stimulated further research. More detailed studies of this subject were conducted by M. Ohm, O. Schlömilch, E. Borel (cf. § 44 in Knopp's book quoted at the beginning of this paper) and Pringsheim [29].

Riemann's theorem was generalized by Sierpiński [34], who proved that in order to get a preassigned sum of conditionally convergent series it is sufficient to rearrange its terms of one sign only (the sign depending on the sum wanted).

The first theorem on the structure of the set $\Lambda_0(S)$ for series S with complex terms was formulated by Lévy [23] in 1905.

The set $\Lambda_0(S)$ is a point, a straight line or a plane H. The paper of Lévy, however, is not quite clear and has some gaps. A complete proof of this theorem was given by Steinitz [35], in 1913-1916. His studies concerned the structure of the set $\Lambda_0(S)$ in arbitrary Euclidean spaces of finite dimension. His work contains about 150 pages. Its volume is due to the fact (which came out later) that the author had introduced an artificial connection of the problem with the properties of convex sets.

It is not surprising that studies concerning this subject were continued by many authors. A paper of Gross [10] (written in 1917) referred to the same problem. The methods which he used are similar to those of Steinitz, but his paper is much shorter and clearer. In 1926 Threfall [37] published a short proof of the following theorem: *if series S is not absolutely convergent, then by rearranging its terms we can obtain a convergent series with another sum.*

The same subject was studied by: Bergström [5] in 1931, Ness [26] in 1937, Šklyarskiĭ [36] in 1944 and Kadec [20] in 1953.

The structure of a set $\Lambda_0(S)$ can be characterized more precisely as follows:

THEOREM 4. *The set $\Lambda_0(S)$ is empty (in H) if and only if at least one of the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ is unconditionally divergent, i. e., preserves the divergence for every order of its terms.*

THEOREM 5. *The set $\Lambda_0(S)$ contains exactly one point if and only if $S \in R_1$.*

THEOREM 6. *If the set $\Lambda_0(S)$ is not empty, then it is a straight line if and only if $S \in R_2$.*

THEOREM 7. *If the set $\Lambda_0(S)$ is not empty, then it is the plane H if and only if $S \in R_3 \cup R_4$.*

The following theorems on vectors in a plane are essential for all papers devoted to the structure of the set $\Lambda_0(S)$ (except the paper of Ness [26]).

THEOREM 8. *If $c_1+c_2+\ldots+c_p = 0$, and $\max|c_i| = 1$, then there exists an arrangement $c_{N_1}, \ldots, c_{N_p}$ such that $|c_{N_1}+c_{N_2}+\ldots+c_{N_k}| \leqslant a$ for $k = 1, 2, \ldots, p$, where a does not depend on p.*

A simple proof of this theorem for an arbitrary Euclidean space is given in Kadec [20].

Ness applies essentially the following theorem of Sierpiński [34]:

THEOREM 9. *If $p_n \to 0$, $p_n > 0$ and $\sum_{n=1}^{\infty} p_n = \infty$, then for every $d > 0$ there exists a permutation N such that*

$$\sum_{k=1}^{n} p_k - \sum_{k=1}^{n} p_{N_k} \to d \quad \textit{as} \quad n \to \infty.$$

B. Trivial examples of permutation of the class K are permutations $\{N_n\}$ satisfying the condition $|N_n - n| < c$, where $c \geqslant 0$ and does not depend on n.

We denote by B a finite set of natural numbers. Let $\delta(B)$ denote the smallest closed interval containing B. If B is identical with the set of all natural numbers from the interval $\delta(B)$, then this set will be called a *segment* of the sequence of natural numbers. In particular, single natural numbers are segments of the sequence of natural numbers. Every set of natural numbers can be represented as a sum of segments. If a set contains p numbers, then it may be represented as a sum of at most p segments.

For a given permutation N and fixed n let us consider natural numbers $x \leqslant n$ for which the condition $N_x > n$ is fulfilled. Let $p_1, p_2, \ldots, p_r$ be those numbers. Analogously let us denote by $q_1, q_2, \ldots, q_s$ natural numbers $y > n$ for which $N_y \leqslant n$. Let us represent the set $p_1, \ldots, p_r$ as the sum of the least possible number of segments of a sequence of natural numbers, and let $\underline{l}(n)$ denote the number of those segments. By $\bar{l}(N)$ we shall denote the number of analogical segments for the set $q_1, q_2, \ldots, q_s$. We assume $l(n) = \underline{l}(n) + \bar{l}(n)$.

THEOREM 10. *A permutation $N \in K$ if and only if the sequence $l(n)$ is bounded. Permutations of the class K preserve also the sums of all convergent series.*

A simple proof of this theorem is given by Kronrod [21].

The theorem remains true if, instead of the usual summability, we consider summability by Toeplitz's methods [3].

Some theorems on permutations, preserving convergence and sum of series in particular cases, can be found in [7], [8], [19], and [29].

C. In the case of $S \in R_1$ the sets $L'(S)$ and $L^*(S)$ reduce to single points. If all terms of series S are real, and the series is unconditionally divergent, then the limit point of partial sums is either $+\infty$ or $-\infty$. Let μ and ν denote, respectively, the upper and the lower limit of the sequence of partial sums of the series S. The set of limit points of this sequence is identical with the closed interval $[\mu, \nu]$ (which can be infinite). In both cases mentioned above, sets $L'(S)$ and $L^*(S)$ are closed and connected. These cases are not exceptional; on the contrary, the following general theorem is true:

THEOREM 11. *For each series S the sets $L'(S)$ and $L^*(S)$ are closed and connected. For each closed and connected set $C \subset H^*$ there exists a series S such that $C = L^*(S)$.*

The theorem remains true if we replace H^* by H', and L^* by L'.

The proof of theorem 11 can be found in [6].

D. The first theorem on the structure of the family $\Lambda'(S)$ was proved by Hadwiger [11].

THEOREM 12. *If a set $\Lambda_0(S)$ is a straight line, then for each closed and connected subset C on this line, there exists a permutation N such that $L'[S(N)] = C$. If $\Lambda_0(S) = H$, then for each closed and convex set $D \subset H'$ there exists a permutation N such that $L'[S(N)] = D$.*

The following theorem, proved by Lapčik [24], is a generalization of this theorem:

THEOREM 13. *If $\Lambda_0(S) = H$, then the family $\Lambda'(S)$ is identical with the family of all continua of the space H'.*

An analogical theorem for the space $\Lambda^*(S)$ can be proved without any essential difficulty.

E. The problem of the structure of the set $\Gamma_0(S, T)$ has been formulated too generally. We shall discuss particular sets T.

THEOREM 14. *If $S \in R_1$ and T_0 consists of two numbers, 0 and 1, then $\Gamma_0(S, T_0)$ is a perfect subset of the rectangle $|x| \leqslant \alpha$ and $|y| \leqslant \beta$, where $\alpha = \sum_{n=1}^{\infty} |x_n|$ and $\beta = \sum_{n=1}^{\infty} |y_n|$.*

In [17], [25], and [30] we can find proofs of this theorem in the case where all terms of the series S are real. The proof for the series with complex terms does not differ essentially. The same theorem is valid for unconditionally convergent series in Banach spaces.

More detailed studies of the set $\Gamma_0(S, T_0)$ for $S \in R_1$ can be found in [13].

THEOREM 15. *If $S \in R_1$ and T_1 denotes the set composed of two numbers, $+1$ and -1, then the set $\Gamma_0(S, T_1)$ arises by linear transformation of the set $\Gamma_0(S, T_0)$* [17].

THEOREM 16. *If $S \in R_1$ and a set T is compact in H, then $\Gamma_0(S, T)$ is perfect.*

The proof of this theorem differs slightly from the proof of theorem 14.

The question of the structure of the set $\Gamma_0(S, T_0)$ when $S \notin R_1$ is much more difficult. Studies devoted to this case can be found in [13], [15], and [27].

The set $\Gamma_0(S, T_1)$ has an interesting structure in the case of $S \notin R_1$. It was studied by Hanani [14]. The formulations of his theorems will be preceded by the following notices:

For $S \in R_2$ we shall denote by $R(S, T_1)$ the set of numbers $\sum_{n=1}^{\infty} \pm(v', z_n)$, where v' is a direction of convergence of series S. According to theorem 15 the set $R(S, T_1)$ is perfect. By $P(S, T_1)$ we denote the Cartesian product of the set $R(S, T_1)$ and a straight line perpendicular to vector v' (the rectangular system of coordinates is determined by vectors v' and $v \perp v'$).

THEOREM 17. *If $S \in R_2$, then $\Gamma_0(S, T_1)$ is dense in the set $p \cap P(S, T_1)$ for every straight line p non-parallel to v. In the case when p is parallel to v and passes through a point of the set $P(S, T_1)$ the set $p \cap \Gamma_0(S, T_1)$ can be empty.*

THEOREM 18. *If $S \in R_3$, then $\Gamma_0(S, T_1)$ is dense on every straight line non-parallel to the principal direction of series S. If p is parallel to the principal direction of series S, then the set $p \cap \Gamma_0(S, T_1)$ can be empty.*

THEOREM 19. *There are series $S \in R_3$ for which the equality $\Gamma_0(S, T_1) = H$ occurs.*

THEOREM 20. *If $S \in R_4$, then $\Gamma_0(S, T_1) = H$.*

The case of more general sets T will be discussed in subsequent paragraphs.

F. It is evident that T_0 is a set of convergence factors.

THEOREM 21. *A set T_1 is a set of convergence factors.*

A proof of this theorem can be found in [9]. The most important part there of is the proof of a lemma which we shall formulate as

THEOREM 22. *If complex numbers $c_1, c_2, \ldots, c_p$ satisfy condition $|c_j| \leqslant 1$ for $j = 1, 2, \ldots, p$, then there exists a sequence $t_j = \pm 1$ for $j = 1, 2, \ldots, p$ such that*

$$|t_1 c_1 + t_2 c_2 + \ldots + t_k c_k| < \sqrt{3} \quad \textit{for} \quad k = 1, 2, \ldots, p.$$

A definite solution of the problem of characterization of sets of convergence factors was given by Calabi and Dvoretzky [6], who proved

THEOREM 23. *A bounded set of complex numbers M is a set of convergence factors if and only if 0 belongs to the smallest closed and convex set containing M.*

G. The first non-trivial example of a set of sum factors was given by Hornich [18]: a set of numbers of the form $\sqrt[k]{v}$, where $v \neq 0$ and k is a natural number $\geqslant 3$, is a set of sum factors.

The definite solution of problem G can be found in [6]. The result can be formulated as

THEOREM 24. *A given set of complex numbers Q is a set of sum factors if and only if 0 is an interior point of the smallest convex set containing Q.*

Thus any three complex numbers which are vertices of a triangle containing in its interior the origin of coordinates form a set of sum factors.

H. As far as we know, the family $\Gamma^*(S, T_0)$ has not yet been studied. The family $\Gamma^*(S, Q)$, when Q is a set of sum factors is characterized by

THEOREM 25. *If $S \notin R_1$ and Q is a set of sum factors, then the family $\Gamma^*(S, Q)$ is identical with the family of all continua of the space H^*.*

A proof can be found in [6].

Theorem 24 implies that there exist series $S \in R_2 \cup R_3 \cup R_4$ for which the family $\Gamma^*(S, T_1)$ is not identical with the family of all continua of the space H^*.

I am preparing a paper about the structure of the family $\Gamma^*(S, T_1)$.

BIBLIOGRAPHY

[1] P. R. Agnew, *Subseries of series which are not absolutely convergent*, Bulletin of the American Mathematical Society 53 (1947), p. 118-120.

[2] — *On rearrangements of series*, ibidem 46 (1940), p. 797-799.

[3] — *Permutations preserving convergence of series*, Proceedings of the American Mathematical Society 6 (4) (1955), p. 563-564.

[4] F. Bagemill and P. Erdös, *Rearrangements of C_1-summable series*, Acta Mathematica 92 (1954), p. 35-53.

[5] V. Bergström, *Ein neuer Beweis eines Satzes von E. Steinitz*, Abhandlungen aus dem mathematischen Seminar der Universität Hamburg 8 (1931), p. 148-152.

[6] E. Calabi and A. Dvoretzky, *Convergence- and sum-factors for series of complex numbers*, Transactions of the American Mathematical Society 70 (1951), p. 177-194.

[7] B. C. Das, *Effect of rearrangement of terms on some semi-convergent series*, The Mathematics Student 5 (1937), p. 12-18.

[8] P. H. Diananda, *On rearrangements of series II*, Colloquium Mathematicum 9 (1961), p. 277-279.

[9] A. Dvoretzky and C. Hanani, *Sur les changements des signes des termes d'une série à termes complexes*, Comptes rendus de l'Académie des Sciences (Paris) 225 (1947), p. 516-518.

[10] W. Gross, *Bedingt konvergente Reihen*, Monatshefte für Mathematik und Physik 28 (1917), p. 221-227.

[11] H. Hadwiger, *Ein Satz über bedingt konvergente Vektorreihen*, Mathematische Zeitschrift 47 (1942), p. 663-668.

[12] — *Über die Umordnungs stärke und eine Erweiterung des Steinitzschen Satzes*, Mathematische Annales 118 (1943), p. 702-717.

[13] H. J. Hamilton, *Some theorems on subsequences*, Bulletin of the American Mathematical Society 44 (1938), p. 298-304.

[14] H. Hanani, *On sums of series of complex numbers*, Pacific Journal of Mathematics 3 (1953), p. 695-709.

[15] J. D. Hill, *Some theorems on subseries*, Bulletin of the American Mathematical Society 48 (1942), p. 103-108.

[16] H. Hornich, *Über Reihen, deren Glieder gegen Null konvergieren*, Monatshefte für Mathematik und Physik 45 (1937), p. 432-434.

[17] — *Über beliebige Teilsummen absolut konvergenter Reihen*, ibidem 49. (1941), p. 316-320.

[18] — *Zu einer geometrischen Theorie der Reihen*, ibidem 46 (1938), p 266-276.

[19] B. Jasek, *Über Umordnung von Reihen*, Colloquium Mathematicum 7 (1960), p. 257-259.

[20] М. И. Кадец, *Об одном свойстве векторных ломаных в n-мерном пространстве*, Успехи математических наук 8 (1) (1953), p. 139-143.

[21] А. С. Кронрод, *О перестановках членов числовых рядов*, Математический сборник 18 (60) (1946), p. 237-276.

[22] F. W. Levi, *Rearrangements of convergent series*, Duke Mathematical Journal 13 (1946), p. 579-585.

[23] P. Lévy, *Sur les séries semi-convergentes*, Nouvelles annales de mathématiques 5 (1905), p. 506-511.

[24] И. П. Лапчик, *О множестве предельных точек рядов с комплексными членами*, Известия Академии Наук СССР 22 (1958), p. 641-666.

[25] P. K. Menon, *On a class of perfect sets*, Bulletin of the American Mathematical Society 54 (1948), p. 706-711.

[26] W. Ness, *Über die Umordnung von bedingt konvergenten Reihen*, Mathematische Zeitschrift 42 (1937), p. 31-50.

[27] H. Pollard, *Subseries of a convergent series*, Bulletin of the American Mathematical Society 49 (1943), p. 730-731.

[28] K. Pracher, *Zur Geometrie der Reihen*, Monatshefte für Mathematik und Physik 52 (1948), p. 255-259.

[29] A. Pringsheim, *Über die Wertveränderungen bedingt convergenter Reihen und Producte*, Mathematische Annalen 22 (1883), p. 455-503.

[30] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, Berlin 1925.

[31] B. Riemann, *Abhandlungen der Gesellschaft zu Göttingen* 12 (1866-1868), p. 97-101.

[32] T. Šalát, *Remarks on Riemann's theorem on divergent series*, Matematicko-Fysikálny Časopis 5 (1955), p. 94-100.

[33] H. M. Sengupta, *Rearrangements of series*, Proceedings of the American Mathematical Society 7 (1956), p. 347-350.

[34] W. Sierpiński, *Sur une propriété des séries qui ne sont pas absolument convergentes*, Bulletin International de l'Académie des Sciences de Cracovie, série A (1911), p. 149-158.

[35] E. Steinitz, *Bedingt konvergente Reihen und konvexe Systeme*, Journal für reine und angewandte Mathematik 143 (1913), p. 128-179; 144 (1914), p. 1-40; 146 (1916), p. 1-52.

[36] Г. О. Шклярсий, *Условно сходящиеся ряды векторов*, Успехи Математических Наук 10 (1944), p. 51-59.

[37] W. Threfall, *Bedingt konvergente Reihen*, Mathematische Zeitschrift 24 (1926), p. 212-215.

Reçu par la Rédaction le 31. 10. 1961