COLLOQUIUM MATHEMATICUM

VOL. IX 1962 FASC. 2

ON DETERMINING BOUNDED SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS BY THE SMALL PARAMETER METHOD

BY

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In a previous note [1] we have considered the differential equation of the form

$$y^{(n)}+\psi_{n-1}(t)y^{(n-1)}+\ldots+\psi_0(t)y=f(t). \tag{1}$$

We have assumed that the functions $\psi_i(t)$, $i=0,1,\ldots,n-1$, and $f(t)$ are defined and bounded for $t\epsilon(-\infty,\infty)$. We have shown that under certain assumptions there exists a solution of (1) defined and bounded for $t\epsilon(-\infty,\infty)$. This solution has been expressed in the form of a uniformly convergent series the terms of which are bounded solutions of a sequence of differential equations with constant coefficients:

$$y_k^{(n)}+A_{n-1}y_k^{(n-1)}+\ldots+A_0y_k=f_k(t), \qquad k=0,1,2,\ldots \tag{2}$$

We have given in [1] sufficient conditions for the series formed from the bounded solutions of the differential equations (2) to converge to the bounded solution of the differential equation (1).

Namely we have assumed that the differential equation (1) can be written in the form

$$y^{(n)}+[A_{n-1}+\varphi_{n-1}(t)]y^{(n-1)}+\ldots+[A_0+\varphi_0(t)]y=f(t), \tag{3}$$

so that the characteristic equation

$$r^n+A_{n-1}r^{n-1}+\ldots+A_1r+A_0=0 \tag{4}$$

has only single roots and, moreover, that the condition $S<1$ is satisfied (see formula (21) in [1]).

It is evident that if the functions $\psi_i(t)$, $i=0,1,\ldots,n-1$, in equation (1) are bounded, then they can be written in the form

$$\psi_i(t)=A_i+\varphi_i(t), \qquad i=0,1,\ldots,n-1, \tag{5}$$

in such a way that the characteristic equation (4) has single roots. We cannot, however, state beforehand whether in a decomposition of the function $\psi_i(t)$, $i = 0, 1, \ldots, n-1$, in which the characteristic equation (4) has single roots, the condition $S < 1$ is satisfied. Roughly speaking, it follows from the form of the formula defining S that the condition $S < 1$ is satisfied when the numbers $\overline{\varphi}_i = \sup_t |\varphi_i(t)|$, $i = 0, 1, \ldots, n-1$, are sufficiently small as compared with the distance of the roots of the characteristic equation (4).

If we accept the rule that in decomposition (5) numbers $\overline{\varphi}_i$ must be as small as possible, it can occur that in such a decomposition the characteristic equation (4) has multiple roots, and the theorem given in [1] does not include such a case.

If, however, while decomposing $\psi_i(t)$, $i = 0, 1, \ldots, n-1$, we accept the rule that the characteristic equation (4) must have single roots, it may happen that $\overline{\varphi}_i$, $i = 0, 1, \ldots, n-1$, will be so great that the condition $S < 1$ may fail to be fulfilled. Thus it is necessary to consider also the case in which the characteristic equation (4) has multiple roots. There is one more reason to consider this case. Namely, in technical problems, in which we often deal with equations of the form (1), the coefficients (which are interpreted as physical quantities) are given at once in form (5); therefore another decomposition is not recommended. Thus, having functions $\psi_i(t)$, $i = 0, 1, \ldots, n-1$, given a priori in form (5), one cannot exclude the case of the characteristic equation having multiple roots.

In the present paper we give a generalization of the theorem published in [1]. Namely, we shall free ourselves from the rather inconvenient assumption that the characteristic equation (4) has only single roots. However, we still assume that the real parts of all roots of the characteristic equation (4) are non-zero.

§1. Let us consider a differential equation with constant coefficients,

$$y^{(n)} + A_{n-1}y^{(n-1)} + \ldots + A_0 y = f(t). \tag{6}$$

Let us assume that characteristic equation

$$r^n + A_{n-1}r^{n-1} + \ldots + A_1 r + A_0 = 0 \tag{7}$$

has m different roots r_i, $i = 1, 2, \ldots, m$, and that k_i is the multiplicity order of the root r_i $\left(\sum_{i=1}^{m} k_i = n\right)$.

It is known that if $y_1(t)$ is a particular solution of a homogeneous differential equation corresponding to the differential equation (6) which satisfies the initial conditions

$$y_1^{(i)}(0) = \begin{cases} 0 & \text{for} \quad i = 0, 1, \ldots, n-2, \\ 1 & \text{for} \quad i = n-1, \end{cases} \tag{8}$$

then the particular solution $y_2(t)$ of the differential equation (6) can be written in the form

$$y_2(t) = \int_0^t y_1(t-\tau) f(\tau)\, d\tau. \tag{9}$$

The validity of this fact may easily be verified. The particular solution of a homogeneous differential equation corresponding to the differential equation (6) which satisfies the initial conditions (8) (cf. [2], p. 5-9) has the form

$$y_1(t) = \sum_{i=1}^{m} \frac{1}{(k_i-1)!} \frac{\partial^{k_i-1}}{\partial r^{k_i-1}} \left\{\frac{e^{rt}}{\Phi_i(r)}\right\}_{r=r_i}, \tag{10}$$

where $\Phi_i(r) = 1$ when $m = 1$, and

$$\Phi_i(r) = \prod_{\substack{j=1 \\ i \neq j}}^{m} (r - r_j)^{k_j} \quad \text{if} \quad m \geqslant 2. \tag{11}$$

Since

$$\begin{aligned} y_1^{(p)}(t) &= \sum_{i=1}^{m} \frac{1}{(k_i-1)!} \frac{\partial^{k_i-1+p}}{\partial r^{k_i-1}\partial t^p} \left\{\frac{e^{rt}}{\Phi_i(r)}\right\}_{r=r_i} \\ &= \sum_{i=1}^{m} \frac{1}{(k_i-1)!} \frac{\partial^{k_i-1}}{\partial r^{k_i-1}} \left\{\frac{r^p e^{rt}}{\Phi_i(r)}\right\}_{r=r_i} \end{aligned} \tag{12}$$

in virtue of (8) the following relations hold:

$$\sum_{i=1}^{m} \frac{1}{(k_i-1)!} \frac{\partial^{k_i-1}}{\partial r^{k_i-1}} \left\{\frac{r^p}{\Phi_i(r)}\right\}_{r=r_i} = \begin{cases} 0 & \text{for} \quad p = 0, 1, \ldots, n-2, \\ 1 & \text{for} \quad p = n-1. \end{cases} \tag{13}$$

From (9) and (10) it follows that the particular solution of the differential equation (6) can be written in the form

$$y_2(t) = \sum_{i=1}^{m} \frac{1}{(k_i-1)!} \int_0^t f(\tau) \frac{\partial^{k_i-1}}{\partial r^{k_i-1}} \left\{\frac{e^{r(t-\tau)}}{\Phi_i(r)}\right\}_{r=r_i} d\tau. \tag{14}$$

LEMMA. *If $g(t)$ is a function defined and bounded for $t \in (-\infty, \infty)$, i. e. $|g(t)| \leqslant \overline{g}$, then the function*

$$f(t) = \int_t g(\tau)(t-\tau)^n e^{r(t-\tau)} d\tau \quad \textit{for} \quad n = 0, 1, 2, \ldots, \tag{15}$$

where $\int\limits_t$ *stands for* $\int\limits_t^\infty$ *when* $\alpha = \mathrm{Re}(r) > 0$ *and for* $\int\limits_{-\infty}^t$ *when* $\alpha = \mathrm{Re}(r) < 0$, *is also a bounded function and the following inequality holds*:

$$|f(t)| \leqslant \bar{g}\frac{n!}{|\alpha|^{n+1}}.$$

The elementary proof is here omitted.

§ 2. THEOREM 1. *If the characteristic equation* (7) *of the differential equation* (6) *has* m *different roots* r_i *such that* $\mathrm{Re}(r_i) = \alpha_i \neq 0$, *and* k_i *is the multiplicity of the root* r_i $\left(\sum\limits_{i=1}^m k_i = n\right)$ *and* $f(t)$ *is a function defined and bounded for* $t\in(-\infty,\infty)$, *then there exists a solution* $y = y(t)$ *of equation* (6) *which satisfies the following conditions*:

$$(16)\qquad |y^{(j)}(t)| \leqslant \bar{f}\sum_{i=1}^m \sum_{p=1}^{k_i} \frac{1}{(p-1)!\,|\alpha_i|^{k_i-p+1}}\left|\frac{d^{p-1}}{dr^{p-1}}\left\{\frac{r^j}{\Phi_i(r)}\right\}_{r=r_i}\right|$$

$$\text{for}\quad j = 0, 1, \ldots, n-1,$$

where $\bar{f} = \sup\limits_t |f(t)|$.

Proof. Let us consider the function

$$(17)\qquad y(t) = y_2(t)+y_3(t),$$

where $y_2(t)$ is defined by formula (14), and y_3 has the form

$$(18)\qquad y_3(t) = \sum_{i=1}^m \frac{\mathrm{sgn}(-\alpha_i)}{(k_i-1)!}\int\limits_0 f(\tau)\frac{\partial^{k_i-1}}{\partial r^{k_i-1}}\left\{\frac{e^{r(t-\tau)}}{\Phi_i(r)}\right\}_{r=r_i} d\tau.$$

It is easy to see that $y_3(t)$ given by formula (18) is a solution of a homogeneous differential equation corresponding to the differential equation (6). Since $\int\limits_0^t + \mathrm{sgn}(-\alpha)\int\limits_0 = \mathrm{sgn}(-\alpha)\int\limits_t$, the function $y(t)$ defined by (17), being a particular solution of the differential equation (6), can be written in the form

$$(19)\qquad y(t) = \sum_{i=1}^m \frac{\mathrm{sgn}(-\alpha_i)}{(k_i-1)!}\int\limits_t f(\tau)\frac{\partial^{k_i-1}}{\partial r^{k_i-1}}\left\{\frac{e^{r(t-\tau)}}{\Phi_i(r)}\right\}_{r=r_i} d\tau.$$

Because of relations (13) we have

$$(20)\qquad y^{(p)}(t) = \sum_{i=1}^m \frac{\mathrm{sgn}(-\alpha_i)}{(k_i-1)!}\int\limits_t f(\tau)\frac{\partial^{k_i-1}}{\partial r^{k_i-1}}\left\{\frac{r^p e^{r(t-\tau)}}{\Phi_i(r)}\right\}_{r=r_i} d\tau$$

$$\text{for}\quad p = 0, 1, \ldots, n-1.$$

Having transformed (20), we can write

$$(21)\qquad y^{(p)}(t) =$$

$$\sum_{i=1}^m \sum_{j=1}^{k_i} \frac{\mathrm{sgn}(-\alpha_i)}{(j-1)!(k_i-j)!}\,\frac{d^{j-1}}{dr^{j-1}}\left\{\frac{r^p}{\Phi_i(r)}\right\}_{r=r_i}\int\limits_t f(\tau)\,(t-\tau)^{k_i-j}e^{r_i(t-\tau)}d\tau.$$

Thus, by the Lemma, we have

$$(22)\qquad |y^{(p)}(t)| \leqslant \sum_{i=1}^m \sum_{j=1}^{k_i} \frac{1}{(j-1)!(k_i-1)!}\left|\frac{d^{j-1}}{dr^{j-1}}\left\{\frac{r^p}{\Phi_i(r)}\right\}_{r=r_i}\right|\times$$

$$\times\int\limits_t |f(\tau)|\,|t-\tau|^{k_i-j}e^{\alpha_i(t-\tau)}d\tau \leqslant \bar{f}\sum_{i=1}^m \sum_{j=1}^{k_i} \frac{1}{(j-1)!\,|\alpha_i|^{k_i-j+1}}\left|\frac{d^{j-1}}{dr^{j-1}}\left\{\frac{r^p}{\Phi_i(r)}\right\}_{r=r_i}\right|,$$

q. e. d.

THEOREM 2. *We are given the differential equation*

$$(23)\qquad y^{(n)} + [A_{n-1}+\varphi_{n-1}(t)]y^{(n-1)} + \ldots + [A_0+\varphi_0(t)]y = f(t),$$

where $\varphi_i(t)$ *for* $i = 0, 1, \ldots, n-1$ *and* $f(t)$ *are functions defined and bounded for* $t\in(-\infty,\infty)$.

Let the characteristic equation (7) *have* m *different roots* r_i, $i = 1, 2, \ldots, m$, *and let* k_i *be the multiplicity of the root* r_i $\left(\sum\limits_{i=1}^m k_i = n\right)$.

If $\mathrm{Re}(r_i) = \alpha_i \neq 0$ *for* $i = 1, 2, \ldots, m$, *and*

$$S = \sum_{\nu=0}^{n-1}\sum_{i=1}^m \sum_{j=1}^{k_i} \frac{\bar{\varphi}_\nu}{(j-1)!\,|\alpha_i|^{k_i-j+1}}\left|\frac{d^{j-1}}{dr^{j-1}}\left\{\frac{r^\nu}{\Phi_i(r)}\right\}_{r=r_i}\right| < 1,$$

where $\bar{\varphi}_i = \sup\limits_t|\varphi_i(t)|$ *for* $i = 0, 1, \ldots, n-1$, *then equation* (23) *has a solution bounded for* $t\in(-\infty,\infty)$, *and defined by the series*

$$(24)\qquad y(t) = \sum_{k=0}^\infty y_k(t)$$

converging uniformly in the interval $-\infty < t < \infty$, *where* $y_k(t)$ *are the bounded solutions of differential equations*

$$(25)\qquad \begin{cases} y_0^{(n)} + A_{n-1}y_0^{(n-1)} + \ldots + A_0y_0 = f(t), \\ y_k^{(n)} + A_{n-1}y_k^{(n-1)} + \ldots + A_0y_k = -\varphi_{n-1}(t)y_{k-1}^{(n-1)} - \ldots - \varphi_0(t)y_{k-1}. \end{cases}$$

Moreover, the remainder

$$(26)\qquad R_k = y_{k+1}+y_{k+2}+\ldots$$

of series (24) *can be estimated either by*

$$(27)\qquad |R_k| \leqslant \frac{1}{1-S}\sum_{i=1}^{m}\sum_{j=1}^{k_i}\frac{1}{(j-1)!\,|a_i|^{k_i-j+1}}\left|\frac{d^{j-1}}{dr^{j-1}}\left\{\frac{1}{\Phi_i(r)}\right\}\right|_{r=r_i}\times$$
$$\times\sup_t|\varphi_{n-1}(t)y_k^{(n-1)}+\ldots+\varphi_0(t)y_k|$$

or by

$$(28)\qquad |R_k| \leqslant \bar{f}_0\frac{S^{k+1}}{1-S}\sum_{i=1}^{m}\sum_{j=1}^{k_i}\frac{1}{(j-1)!\,|a_i|^{k_i-j+1}}\left|\frac{d^{j-1}}{dr^{j-1}}\left\{\frac{1}{\Phi_i(r)}\right\}_{r=r_i}\right|,$$

where $\bar{f}_0 = \bar{f}$.

Proof. Let us notice that if $y_k(t)$ are bounded solutions of differential equations (25), then we have

$$(29)\qquad \bar{f}_k \leqslant S\bar{f}_{k-1},$$

where

$$\bar{f}_k = \sup_t|y_k^{(n)}+A_{n-1}y_k^{(n-1)}+\ldots+A_0y_k|$$
$$= \sup_t|-\varphi_{n-1}(t)y_{k-1}^{(n-1)}-\ldots-\varphi_0(t)y_{k-1}|\quad\text{for}\quad k=1,2,\ldots$$

In fact, by theorem 1 we have

$$\bar{f}_k \leqslant \bar{f}_{k-1}\bar{\varphi}_{n-1}\sum_{i=1}^{m}\sum_{j=1}^{k_i}\frac{1}{(j-1)!\,|a_i|^{k_i-j+1}}\left|\frac{d^{j-1}}{dr^{j-1}}\left\{\frac{r^{n-1}}{\Phi_i(r)}\right\}_{r=r_i}\right|+\ldots+$$
$$+\bar{f}_{k-1}\bar{\varphi}_0\sum_{i=1}^{m}\sum_{j=1}^{k_i}\frac{1}{(j-1)!\,|a_i|^{k_i-j+1}}\left|\frac{d^{j-1}}{dr^{j-1}}\left\{\frac{r^0}{\Phi_i(r)}\right\}_{r=r_i}\right|$$
$$=\bar{f}_{k-1}\sum_{\nu=0}^{n-1}\sum_{i=1}^{m}\sum_{j=1}^{k_i}\frac{\bar{\varphi}_\nu}{(j-1)!\,|a_i|^{k_i-j+1}}\left|\frac{d^{j-1}}{dr^{j-1}}\left\{\frac{r^\nu}{\Phi_i(r)}\right\}_{r=r_i}\right| = S\bar{f}_{k-1}.$$

From (29) we immediately obtain

$$(30)\qquad \sum_{k=0}^{\infty}\bar{f}_k \leqslant \bar{f}_0\frac{1}{1-S}.$$

From the convergence of series (30) and from theorem 1 it follows the uniform convergence of the series

$$(31)\qquad \sum_{k=0}^{\infty}y_k^{(p)}(t)\quad\text{for}\quad p=0,1,\ldots,n-1.$$

In fact

$$\sum_{k=0}^{\infty}|y_k^{(p)}(t)| \leqslant \sum_{k=0}^{\infty}\left(\bar{f}_k\sum_{i=1}^{m}\sum_{j=1}^{k_i}\frac{1}{(j-1)!\,|a_i|^{k_i-j+1}}\left|\frac{d^{j-1}}{dr^{j-1}}\left\{\frac{r^p}{\Phi_i(r)}\right\}_{r=r_i}\right|\right)$$
$$\leqslant\sum_{i=1}^{m}\sum_{j=1}^{k_i}\frac{1}{(j-1)!\,|a_i|^{k_i-j+1}}\left|\frac{d^{j-1}}{dr^{j-1}}\left\{\frac{r^p}{\Phi_i(r)}\right\}_{r=r_i}\right|\cdot\sum_{k=0}^{\infty}\bar{f}_k$$
$$\leqslant\frac{\bar{f}_0}{1-S}\sum_{i=1}^{m}\sum_{j=1}^{k_i}\frac{1}{(j-1)!\,|a_i|^{k_i-j+1}}\left|\frac{d^{j-1}}{dr^{j-1}}\left\{\frac{r^p}{\Phi_i(r)}\right\}_{r=r_i}\right|.$$

From relations (25) and the uniform convergence of series (29) and (31) follows the uniform convergence of the series

$$(32)\qquad \sum_{k=0}^{\infty}y_k^{(n)}(t).$$

We shall now show that (24) is a bounded solution of the differential equation (23).

From equations (25) it follows that

$$\sum_{k=0}^{\infty}[y_k^{(n)}+A_{n-1}y_k^{(n-1)}+\ldots+A_0y_k] = f(t)-\sum_{k=1}^{\infty}[\varphi_{n-1}(t)y_{k-1}^{(n-1)}+\ldots+\varphi_0(t)y_{k-1}],$$

whence, by the uniform convergence of series (31) and (32), we have

$$\Big(\sum_{k=0}^{\infty}y_k\Big)^{(n)}+[A_{n-1}+\varphi_{n-1}(t)]\Big(\sum_{k=0}^{\infty}y_k\Big)^{(n-1)}+\ldots+[A_0+\varphi_0(t)]\sum_{k=0}^{\infty}y_k = f(t).$$

Thus the function defined by formula (24) is the solution of the differential equation (23).

Since $y_k(t)$ are bounded functions and, moreover, series (24) is uniformly convergent, the function defined by this series is bounded. The remainder R_k fulfils the inequality

$$|R_k| \leqslant |y_{k+1}|+|y_{k+2}|+\ldots$$
$$\leqslant\sum_{i=1}^{m}\sum_{j=1}^{k_i}\frac{1}{(j-1)!\,|a_i|^{k_i-j+1}}\left|\frac{d^{j-1}}{dr^{j-1}}\left\{\frac{1}{\Phi_i(r)}\right\}_{r=r_i}\right|\cdot(\bar{f}_{k+1}+\bar{f}_{k+2}+\ldots)$$
$$=\bar{f}_{k+1}\sum_{i=1}^{m}\sum_{j=1}^{k_i}\frac{1}{(j-1)!\,|a_i|^{k_i-j+1}}\left|\frac{d^{j-1}}{dr^{j-1}}\left\{\frac{1}{\Phi_i(r)}\right\}_{r=r_i}\right|(1+S+S^2+\ldots)$$
$$=\frac{\bar{f}_{k+1}}{1-S}\sum_{i=1}^{m}\sum_{j=1}^{k_i}\frac{1}{(j-1)!\,|a_i|^{k_i-j+1}}\left|\frac{d^{j-1}}{dr^{j-1}}\left\{\frac{1}{\Phi_i(r)}\right\}_{r=r_i}\right|.$$

Since

$$\bar{f}_{k+1} = \sup_t |\varphi_{n-1}(t) y_k^{(n-1)} + \ldots + \varphi_0(t) y_k|,$$

we have

$$|R_k| \leqslant \frac{1}{1-S} \sum_{i=1}^{m} \sum_{j=1}^{k_i} \frac{1}{(j-1)!\,|a_i|^{k_i-j+1}} \left| \frac{d^{j-1}}{dr^{j-1}} \left\{ \frac{1}{\Phi_i(r)} \right\}_{r=r_i} \right| \times$$
$$\times \sup_t |\varphi_{n-1}(t) y_k^{(n-1)} + \ldots + \varphi_0(t) y_k|.$$

Thus inequality (27) has been proved. Inequality (28) results from (27) and (29).

§ 3. As an example let us consider the differential equation

$$y'' + 7y' + 2y = A, \qquad A = \text{const}. \tag{33}$$

If we assume $\varphi_1(t) = 8$ and $\varphi_0(t) = 4$ and express the functions $\psi_1(t) = 7$ and $\psi_0(t) = 2$ in form (5), then the condition $S < 1$ will not be fulfilled, and the series formed from the bounded solutions of the corresponding sequence of differential equations will be divergent. However, if we express $\psi_1(t) = 7$ and $\psi_0(t) = 2$ in form (5), assuming $\varphi_1(t) = 10$ and $\varphi_0(t) = 0$, then the condition $S < 1$ will not be satisfied, and nevertheless the series formed from the bounded solutions of the corresponding sequence of differential equations will be convergent to the bounded solution of equation (33).

This shows that the conditions given in theorem 2 are only sufficient for the existence of a bounded solution of the differential equation (6).

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Reçu par la Rédaction le 4. 6. 1961

COLLOQUIUM MATHEMATICUM

VOL. IX 1962 FASC. 2

SUR L'ORDRE DE GRANDEUR DES COEFFICIENTS DE FOURIER D'UNE CLASSE SPÉCIALE DES FONCTIONS L^p

PAR

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Soit

$$\mathfrak{S}[f] = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) \tag{1}$$

la série de Fourier d'une fonction L-intégrable. Posons

$$\Delta f = f(x+h) - f(x-h), \quad h > 0, \quad 0 < a \leqslant 1 \quad \text{et} \quad p \geqslant 1.$$

Alors $f(x)$ appartient à $\mathrm{Lip}(a, p)$ si

$$\left\{ \int_{-\pi}^{\pi} |\Delta f|^p dx \right\}^{1/p} = O(h^a) \quad \text{où} \quad h \to 0 \tag{2}$$

et à $\mathrm{Lip}^*(a, p)$ si on a (2) avec o au lieu de O. Une fonction de la classe $\mathrm{Lip}(a, p)$ appartient nécessairement à L^p (voir [1]). Il est aussi connu que $(f(x) \in \mathrm{Lip}(a, p)$ entraîne $a_\nu, b_\nu = O(\nu^{-a})$ et que $f(x) \in \mathrm{Lip}(a, p)$ entraîne $o(\nu^{-a})$. Cet ordre ne peut pas être amélioré.

Nous allons donner dans cette communication des bornes inférieures et supérieures plus précises pour les coefficients de Fourier des fonctions L^p où $p \geqslant 2$, en supposant en outre que ces coefficients sont monotones par valeur absolue.

Soit $\mathfrak{S}[f]$ une série de Fourier de la forme (1). Posons

$$\frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p \leqslant 2, \quad 2 \leqslant q < \infty \quad \text{et} \quad f(x) \in L^q.$$

Désignons par $\omega_q(\delta)$ le module de continuité intégrale d'ordre q de $f(x)$, c'est-à-dire que

$$\omega_q(\delta) = \omega_q(\delta, f) = \sup_{0 \leqslant h \leqslant \delta} \left\{ \int_{-\pi}^{\pi} |f(x+h) - f(x-h)|^q dx \right\}^{1/q}.$$