D'une part, on a en vertu de l'inégalité $|\sin x| < |x|$ et de (B) pour $0 \leqslant h \leqslant \pi/4n$

$$I_1 \leqslant \frac{\pi}{4n}\Big(\sum_{k=1}^{n}[k(|a_k|+|b_k|)]^p\Big)^{1/p}$$

$$= \frac{\pi}{4n}\Big\{\sum_{1}^{n}[k^{1+\varepsilon_1}(|a_k|+|b_k|)]^p\frac{1}{k^{p\varepsilon_1}}\Big\}^{1/p}$$

$$\leqslant \frac{\pi}{4n}n^{1+\varepsilon_1}(|a_n|+|b_n|)\Big\{\sum_{1}^{n}\frac{1}{k^{p\varepsilon_1}}\Big\}^{1/p}.$$

A cause de $p\varepsilon_1 < 1$, le dernier facteur est d'ordre $O(n^{-p\varepsilon_1+1})^{1/p} = O(n^{-\varepsilon_1+1/p})$ et il vient

$$(7)\qquad I_1 \leqslant C_1' n^{\varepsilon_1}(|a_n|+|b_n|)n^{-\varepsilon_1+1/p} = C_1'(|a_n|+|b_n|)n^{1/p}.$$

D'autre part, on a pour tout $h$

$$I_2 = \Big\{\sum_{n+1}^{\infty}(|a_k|+|b_k|)^p|\sin kh|^p\Big\}^{1/p} \leqslant \Big\{\sum_{n+1}^{\infty}(|a_k|+|b_k|)^p\Big\}^{1/p}$$

$$= \Big\{\sum_{n+1}^{\infty}k^{p-p\varepsilon}(|a_k|+|b_k|)^p\frac{1}{k^{p-p\varepsilon}}\Big\}^{1/p}.$$

En vertu de (A), la dernière somme peut être majorée par le produit

$$n^{1-\varepsilon}(|a_n|+|b_n|)\Big\{\sum_{n+1}^{\infty}\frac{1}{k^{p-p\varepsilon}}\Big\}^{1/p},$$

dont le dernier facteur est convergent d'ordre $O(n^{1/p-1+\varepsilon})$ à cause de $p-p\varepsilon > 1$, c'est-à-dire de $\varepsilon < 1-1/p$. On a donc

$$(8)\qquad I_2 \leqslant C_1''(|a_n|+|b_n|)n^{1/p}.$$

Les formules (6), (7) et (8) entraînent directement la première partie de (*).

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*Reçu par la Rédaction le 22. 7. 1961*

COLLOQUIUM MATHEMATICUM

VOL. IX 1962 FASC. 2

# ON CHANGE OF VARIABLE IN THE DENJOY-PERRON INTEGRAL (II)

BY

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This paper continues the investigations concerning change of variable in the Denjoy-Perron integral contained in [2]. The notation and terminology used in this paper are the same as in [2]. We begin with the following definitions:

A function $F$ defined on an interval $I$ will be said to be *non-decreasing* (resp. *non-increasing*) *in the restricted sense on a set* $E \subset I$ if for every pair of points $x_1, x_2, x_1 < x_2$, belonging to $[\inf E, \sup E]$, $F(x_1) \leqslant F(x_2)$ (resp. $F(x_1) \geqslant F(x_2)$), provided that at least one of the points $x_1$, $x_2$ belongs to $E$. A function which is either non-decreasing or non-increasing in the restricted sense on a set $E$ will be termed *monotone in the restricted sense*, or $M_*$ *on* $E$. A function $F$ defined on an interval $I$ will be termed $MG_*$ *on a set* $E \subset I$ if $E$ is expressible as the sum of a finite or enumerable sequence of sets on each of which $F$ is $M_*$.

Let us denote by $N(F; I)$ the set of the values assumed an infinity of times by a function $F$ on an interval $I$.

A function $F$ will be said to *fulfil the condition* $(\mathrm{T}_0)$ on an interval $I$ if (i) the set $N(F; I)$ is at most enumerable; (ii) for each $y$ belonging to $N(F; I)$ the set $F^{-1}(y) - \operatorname{int}(F^{-1}(y))$ is at most enumerable.

We shall say that a function $F$ is *non-decreasing* (resp. *non-increasing*) *at a point* $x_0$ if there exists a neighbourhood of $x_0$ such that for $x$ belonging to this neighbourhood

$$(x-x_0)\big(F(x)-F(x_0)\big) \geqslant 0 \qquad (\text{resp. } (x-x_0)\big(F(x)-F(x_0)\big) \leqslant 0).$$

A function which is either non-decreasing or non-increasing at $x_0$ will be termed *monotone at* $x_0$. We shall now prove the following

THEOREM 1. *Let $F$ be a continuous function defined on an interval $[a, b]$. Then the following conditions are equivalent:*

(i) *$F$ is $MG_*$ on $[a, b]$,*

(ii) *every perfect subset of $[a, b]$ contains a portion on which the function $F$ is $M_*$,*

(iii) *$F$ fulfils the condition* $(\mathrm{T}_0)$ *on* $[a, b]$,

(iv) *at each point of* $(a, b)$, *except perhaps those of an enumerable set, $F$ is monotone.*

Proof. In order to prove that (i) implies (ii) it is enough to use Baire's theorem. Therefore, we shall now show that (ii) implies (iii). For this purpose, let $\boldsymbol{K}$ be the class of all closed subintervals $I$ of $[a, b]$ such that $F$ fulfils the condition $(\mathrm{T}_0)$ on $I$. We shall show that the class $\boldsymbol{K}$ satisfies the following conditions:

(a) if $[a_0, b_0]$ and $[b_0, c_0]$ belong to $\boldsymbol{K}$, then $[a_0, c_0]$ also belongs to $\boldsymbol{K}$,

(b) if $I_0 \epsilon \boldsymbol{K}$, then every interval $I \subset I_0$ also belongs to $\boldsymbol{K}$,

(c) if every interval $I \subset \mathrm{int}(I_0)$ belongs to $\boldsymbol{K}$, then the interval $I_0$ belongs to $\boldsymbol{K}$,

(d) if each interval contiguous to a perfect set $E$ belongs to $\boldsymbol{K}$, then there exists an interval $I_0$ such that $I_0 \epsilon \boldsymbol{K}$ and $E \cdot \mathrm{int}(I_0) \neq 0$.

We see at once that (a) and (b) are satisfied. In order to prove (c) let every interval $I \subset \mathrm{int}(I_0)$, where $I_0 = [a_0, b_0]$, belong to $K$. Then, for sufficiently large positive integer $n$, the interval $I_n = [a_0 + 1/n, b_0 - 1/n]$ belongs to $\boldsymbol{K}$. Further, since

$$N(F; I_0) \subset \sum_n N(F; I_n) + \{F(a_0), F(b_0)\},$$

the set $N(F; I_0)$ is at most enumerable. It is easy to see that for $y$ belonging to $N(F; I_0)$ the set $F^{-1}(y) - \mathrm{int}\big(F^{-1}(y)\big)$ is at most enumerable. Thus $F$ fulfils the condition $(\mathrm{T}_0)$ on $I_0$, and this completes the proof of (c). Now we shall show that $\boldsymbol{K}$ satisfies (d). Let $E$ be a perfect set, and let each interval contiguous to $E$ belong to $\boldsymbol{K}$. Since (ii) is satisfied, there exists a portion $P$ of $E$ such that $F$ is $\mathrm{M}_*$ on $\overline{P}$. Let $I_0$ be the smallest closed interval containing $\overline{P}$. We shall show that the interval $I_0$ is the required one. In fact, let $\{I_n\}$ denote the sequence of the intervals contiguous to $\overline{P}$, and let $A$ be the set of all numbers $y$ such that the set $F^{-1}(y)\overline{P}$ contains at least two points. We find that

$$N(F; I_0) \subset A + \sum_n N(F; I_n);$$

hence it follows that the set $N(F; I_0)$ it at most enumerable. Further, let $y$ belong to $N(F; I_0)$. Since $F$ is $\mathrm{M}_*$ on $\overline{P}$ and each $I_n$ belongs to $\boldsymbol{K}$, it easily follows that the set $F^{-1}(y) - \mathrm{int}\big(F^{-1}(y)\big)$ is at most enumerable. Thus, since also $E \cdot \mathrm{int}(I_0) \neq 0$, the proof of (d) is completed. We have shown that the class $\boldsymbol{K}$ satisfies the conditions (a)-(d). Hence, by Romanowski's lemma (p. 39 in [4]), it follows that the interval $[a, b]$ belongs to $\boldsymbol{K}$ and therefore $F$ fulfils the condition $(\mathrm{T}_0)$ on $[a, b]$.



In order to prove that (iii) implies (iv), let $M$ denote the set of the points at which $F$ assumes a strict extremum. Since $F$ fulfils the condition $(\mathrm{T}_0)$ and the set $M$ is at most enumerable, it is enough to show that $F$ is monotone at each point $x$ belonging to $(a, b) - \big(M + F^{-1}[N(F; [a, b])]\big)$. For this purpose, let us remark that since the value $F(x_0)$ is assumed only finite numbers of times, there exists a neighbourhood of $x_0$ such that for $x$ belonging to this neighbourhood $F(x) \neq F(x_0)$, provided that $x \neq x_0$. Further, since $F$ does not assume a strict extremum at $x_0$ and is continuous, it is monotone at $x_0$.

In order to complete the proof of the theorem, it is enough to show that (iv) implies (i). For this purpose, let $E_n^1$ (resp. $E_n^2$), $n = 1, 2, \ldots$, be the set of all $x$ belonging to $(a, b)$ such that $|s - x| \leqslant 1/n$, $s \epsilon [a, b]$ implies

$$(s-x)\big(F(s) - F(x)\big) \geqslant 0 \qquad (\text{resp. } (s-x)\big(F(s)-F(x)\big) \leqslant 0).$$

Further, let $E_{n,k}^1$ (resp. $E_{n,k}^2$), $k = 0, 1, \ldots, n-1$, denote the intersection $E_n^1$ (resp. $E_n^2$) with $[a + k(b-a)/n, a + (k+1)(b-a)/n]$. We see at once that $F$ is non-decreasing (resp. non-increasing) in the restricted sense on each $E_{n,k}^1$ (resp. $E_{n,k}^2$). Further, since the set $[a, b] - \sum_{i=1}^{2} \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} E_{n,k}^i$ is at most enumerable, $F$ is $\mathrm{MG}_*$ on $[a, b]$. Thus the theorem is proved.

For simplicity of wording, every continuous function which is $\mathrm{MG}_*$ and fulfils condition (N) on an interval will be called $\mathrm{ACMG}_*$ *on that interval.* By [5], Theorem 8.8, p. 233, and Theorem 6.8, p. 228, it follows that every function which is $\mathrm{ACMG}_*$ on an interval is $\mathrm{ACG}_*$ on that interval.

THEOREM 2. *Let $\varphi$ be a function $ACMG_*$ on an interval $[c, d]$, and let $f$ be a finite function defined on the interval $[a, b] = \varphi[[c, d]]$. Then the following conditions are equivalent:*

(i) *the function $f$ is $D_*$-integrable on* $[a, b]$,

(ii) *the function $f(\varphi)\varphi'$ is $D_*$-integrable on* $[c, d]$.

*Moreover, if one of these conditions is satisfied, then*

$$(D_*) \int_{\varphi(c)}^{\varphi(d)} f(x)\,dx = (D_*) \int_c^d f\big(\varphi(t)\big) \cdot \varphi'(t)\,dt. \tag{1}$$

Proof. First let, (i) be satisfied, and let $F$ be an indefinite $D_*$-integral of $f$. We shall show that the function $F_1 = F(\varphi)$ is $\mathrm{ACG}_*$ on $[c, d]$. For this purpose, on account of [5], Theorem 8.8 (p. 233) and Theorem 6.8 (p. 228), it is enough to prove that $F_1$ is $\mathrm{VBG}_*$ on $[c, d]$. The function $F$ is $\mathrm{ACG}_*$ on $[a, b]$, and thus it is $\mathrm{VBG}_*$; therefore $[a, b]$ is the sum of a sequence of sets $E_n$ on each of which $F$ is $\mathrm{VB}_*$. Let us put $T_n = \varphi^{-1}[E_n]$. Since $\varphi$ is $\mathrm{MG}_*$, we can express each $T_n$ as the sum of a sequence of sets $T_{n,k}$ on each of which $\varphi$ is $\mathrm{M}_*$. Now it is enough to show that $F_1$ is $\mathrm{VB}_*$ on each $T_{n,k}$. For this purpose, let $\{I_p\}$ be any finite sequence of non-overlapping intervals whose end-points belong to fixed $T_{n,k}$. Since $\varphi$ is monotone in the restricted sense on this $T_{n,k}$, it easily follows that $O(F_1; I_p) = O(F; \varphi[I_p])$. Now, since the intervals $\varphi[I_p]$ are non-overlapping and have end-points belonging to fixed $E_n$ on which $F$ is $\mathrm{VB}_*$, this completes the proof that $F_1$ is $\mathrm{VB}_*$ on each $T_{n,k}$. We have thus shown that $F_1$ is $\mathrm{ACG}_*$ on $[c, d]$. By Theorem 2 in [2] it follows that the function $f(\varphi)\varphi'$ is $D_*$-integrable on $[c, d]$ and (1) holds.

Let us suppose that (ii) is satisfied. We shall prove that $f$ is $D_*$-integrable on $[a, b]$. For this purpose, let us denote by $\boldsymbol{K}$ the class of all closed subintervals $I$ of $[c, d]$ such that $f$ is $D_*$-integrable on $\varphi[I]$. We shall show that the class $\boldsymbol{K}$ satisfies the conditions (a)-(d) from the proof of Theorem 1. The conditions (a) and (b) are obvious. In order to prove (c), let every interval $I \subset (c_0, d_0)$ belong to $\boldsymbol{K}$, and let us write $m_0 = \inf\limits_{c_0 \leqslant t \leqslant d_0} \varphi(t)$, $M_0 = \sup\limits_{c_0 \leqslant t \leqslant d_0} \varphi(t)$. We may clearly suppose that $\varphi$ does not assume the values $m_0$, $M_0$ at $t \epsilon (c_0, d_0)$ and that $m_0 = \varphi(c_0)$, $M_0 = \varphi(d_0)$. Let $\{a_n\}$, $\{b_n\}$ be any sequences such that $\lim\limits_n a_n = m_0$, $\lim\limits_n b_n = M_0$ and $m_0 < a_n < b_n < M_0$ for $n = 1, 2, \ldots$ Further, let

$$c_n = \inf\{t: a_n = \varphi(t), c_0 \leqslant t \leqslant d_0\} \quad \text{and} \quad d_n = \sup\{t: b_n = \varphi(t), c_0 \leqslant t \leqslant d_0\}.$$

We see at once that $\lim\limits_n c_n = c_0$ and $\lim\limits_n d_n = d_0$; moreover, $c_n \neq c_0$ and $d_n \neq d_0$ for $n = 1, 2, \ldots$ Hence, in view of our hypothesis, it follows that $f$ is $D_*$-integrable on each $[a_n, b_n]$. Therefore, on account of the part which has already been proved, we obtain

$$(D_*)\int_{a_n}^{b_n} f(x)\,dx = (D_*)\int_{c_n}^{d_n} f(\varphi(t))\cdot\varphi'(t)\,dt.$$

Hence, the definite $D_*$-integrals of $f$ over $[a_n, b_n]$ tend to a finite limit as $n \to \infty$ and therefore $f$ is $D_*$-integrable on $[m_0, M_0]$. This completes the proof of (c). We shall now show that $\boldsymbol{K}$ satisfies (d). Let $E$ be a perfect set, and let each interval contiguous to $E$ belong to $\boldsymbol{K}$. On account of our Theorem 1 and of [5], Theorem 1.4 (p. 244), there exists a portion $P$ of $E$ such that (a) $\varphi$ is $\mathrm{M}_*$ on $\overline{P}$ and (b) $f(\varphi)\varphi'$ is summable on $\overline{P}$ and the series of the oscillations of the indefinite $D_*$-integrals of $f(\varphi)\varphi'$ over the intervals contiguous to $\overline{P}$ is convergent. Let $I_0$ be the smallest interval containing $\overline{P}$, and let $\{I_n\}$ be the sequence of the intervals contiguous to $\overline{P}$. We shall show that $f$ is $D_*$-integrable on $\varphi[I_0]$. In fact, on account of (a), it follows that the intervals $I_n' = \varphi[I_n]$, $n = 1, 2, \ldots$, are contiguous to the closed set $Q = \varphi[P]$ (1). Since each $I_n$, $n = 1, 2, \ldots$, belongs to $\boldsymbol{K}$, it follows that $f$ is $D_*$-integrable on each $I_n'$. Moreover, by (a) and the part of the theorem which has already been proved, we obtain $O(D_*; f; I_n') = O(D_*; f(\varphi)\varphi'; I_n)$ for positive integer $n$. Therefore, by the second part of (b), it follows that $\sum\limits_n O(D_*; f; I_n') < +\infty$. Further, since $\varphi$ is clearly monotone and AC on $\overline{P}$, in view of the first part of (b) and the well-known theorem concerning change of variable in the Lebesgue integral, we infer that $f$ is summable on $Q$. Now, it is enough to use Theorem 5.1 of [5] (p. 257) to prove that $f$ is $D_*$-integrable on $\varphi[I_0]$, and since $E\cdot\mathrm{int}(I_0) \neq 0$, this completes the proof of (d). We have thus shown that the class $\boldsymbol{K}$ satisfies the conditions (a)-(d), p. 318. Hence, by Romanowski's lemma ([4], p. 39) it follows that the interval $[c, d]$ belongs to $\boldsymbol{K}$, and so $f$ is $D_*$-integrable on $[a, b]$. Thus the theorem is proved.

(1) of course some of $I_n'$ can reduce to points.



Theorem 2 generalizes Karták's result ([1], p. 414), and gives more than the result of Mařik ([3], p. 292) applied to the Denjoy-Perron integral. We shall now prove

THEOREM 3. *Let $\varphi$ be a function defined on an interval $[c, d]$. If, for every function $F$ increasing and AC on the set $\varphi[[c, d]]$, the function $G = F(\varphi)$ is $ACG_*$ on $[c, d]$, then $\varphi$ is $ACMG_*$ on $[c, d]$.*

Proof. Suppose that $\varphi$ is not $\mathrm{ACMG}_*$ on $[c, d]$. Then, since $\varphi$ is clearly continuous and fulfils condition (N) on $[c, d]$, it is not $\mathrm{MG}_*$ on $[c, d]$. Therefore, on account of Theorem 1, there exists a perfect set $E \subset [c, d]$ such that $\varphi$ is not $\mathrm{M}_*$ on any portion of $E$. Let $E_1$ be the set of points $t$ such that (i) $t$ is not the end of the interval contiguous to $E$, (ii) every neighbourhood of $t$ contains points $t_1$, $t_2$ such that $\varphi(t) = \varphi(t_1)$ and $\varphi(t) \neq \varphi(t_2)$, provided that either $t_1 > t$ and $t_2 > t$ or $t_1 < t$ and $t_2 < t$. The set $E_1$ is dense in $E$, since otherwise there would exist a portion $P$ of $E$ such that $PE_1 = 0$. We see at once that $\varphi$ is monotone at each point of $P$ at which it does not assume a strict extremum and which is not the end of the interval contiguous to $E$. Therefore, by an argument similar to that used in the proof of Theorem 1, it follows that $\varphi$ is $\mathrm{MG}_*$ on $\overline{P}$, and hence, by Baire's theorem, $\mathrm{M}_*$ on any portion of $\overline{P}$, and so $\mathrm{M}_*$ on any portion of $E$. This contradicts the hypothesis. We have thus proved that $E_1$ is dense in $E$. Let $\{t_n\}_{n=1,2,\ldots}$ be a sequence of points belonging to $E_1$ dense in $E$. We see at once that there exist points $t_{n,k}^i$ ($n, k = 1, 2, \ldots$ and $i = 1, 2, 3, 4$) such that the following conditions are satisfied:

(a) $t_{n,k}^i \epsilon E$ for $i = 1, 2$ and $n, k = 1, 2, \ldots$,

(b) $t_{n,k}^1 \leqslant t_{n,k}^3 < t_{n,k}^4 \leqslant t_{n,k}^2$ for $n, k = 1, 2, \ldots$,

(c) $\lim\limits_k t_{n,k}^i = t_n$ for $i = 1, 2$ and $n = 1, 2, \ldots$,

(d) $\varphi(t_{n,k}^3) = \varphi(t_n)$ for $n, k = 1, 2, \ldots$,

(e) for fixed $n$, either $t^2_{n,k+1} < t^1_{n,k}$ for $k = 1, 2, \ldots$ or $t^1_{n,k+1} > t^2_{n,k}$ for $k = 1, 2, \ldots,$

(f) for fixed $n$, either 1° $\varphi(t^4_{n,k}) > \varphi(t^4_{n,k+1}) > \varphi(t_n)$ for $k = 1, 2, \ldots$ or 2° $\varphi(t^4_{n,k}) < \varphi(t^4_{n,k+1}) < \varphi(t_n)$ for $k = 1, 2, \ldots$

Let us define, for every positive integer $n$, the function $F_n$ on the interval $[a, b] = \varphi[[c, d]]$ as follows:

$$F_n(x) = \begin{cases} 0 & \text{at} \quad x = \varphi(t_n), \\ \varepsilon_n/k & \text{at} \quad x = \varphi(t^4_{n,k}) \text{ for } k = 1, 2, \ldots, \\ \text{linear for other } x \text{ so that } F_n \text{ is increasing}, \end{cases} \tag{2}$$

where $\varepsilon_n = +1$ for 1° and $\varepsilon_n = -1$ for 2°. Let us put

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{n^2 M_n}, \tag{3}$$

where $M_n = \sup_{a \leqslant x \leqslant b} |F_n(x)|$. The function $F_n$ is evidently increasing. We shall show that $F$ is also AC on $[a, b]$. For this purpose, let us remark that by Fubini's theorem (p. 117 in [5])

$$F'(x) = \sum_{n=1}^{\infty} \frac{F'_n(x)}{n^2 M_n}$$

almost everywhere on $[a, b]$; hence

$$(L)\int_a^x F'(s)\,ds = \sum_{n=1}^{\infty} \frac{1}{n^2 M_n} (L)\int_a^x F'_n(s)\,ds \tag{4}$$

for every $x$ belonging to $[a, b]$. Since each $F_n$ is evidently AC, it follows that $(L)\int_a^x F'_n(s)\,ds = F_n(x) - F_n(a)$ for $x \in [a, b]$ and every positive integer $n$. Therefore, in view of (3) and (4), we obtain $(L)\int_a^x F'(s)\,ds = F(x) - F(a)$. We have thus shown that $F$ is AC on $[a, b]$. We shall now prove that the function $G = F(\varphi)$ is not ACG$_*$ on $[c, d]$, and this will contradict the hypothesis of the theorem. For this purpose, in view of [5], Theorem 9.1 (p. 233), it suffices to show that $G$ is not VB$_*$ on any portion of $E$. Let $P$ be any portion of $E$. Since the sequence $\{t_n\}$ is dense in $E$, there exists a point $t_{n_0} \in P$. Further, in view of (c), it follows that $t^i_{n_0,k} \in P$ for $i = 1, 2$ and $k \geqslant k_{n_0}$. Now, by (d), (f) and (2) we obtain

$$\varepsilon_{n_0}\big(G(t^4_{n_0,k}) - G(t^3_{n_0,k})\big) > \frac{\varepsilon_{n_0}}{n_0^2 M_{n_0}} \Big(F_{n_0}\big(\varphi(t^4_{n_0,k})\big) - F_{n_0}\big(\varphi(t^3_{n_0,k})\big)\Big) = \frac{1}{k n_0^2 M_{n_0}},$$



whence $O(G; [t^1_{n_0,k}, t^2_{n_0,k}]) > 1/k n_0^2 M_{n_0}$. Since the intervals $[t^1_{n_0,k}, t^2_{n_0,k}]$, $k = k_{n_0}, k_{n_0}+1, \ldots,$ are non-overlapping, it follows that $G$ is not VB$_*$ on $P$. Thus the theorem is proved.

By the preceding theorem and [2], Theorem 2, we obtain

COROLLARY. *Let $\varphi$ be a function which is continuous, derivable almost everywhere and fulfils the condition* (N) *on an interval* $[c, d]$. *If, for every non-negative function $f$, summable on the interval $\varphi[[c, d]]$, the function $f(\varphi)\varphi'$ is $D_*$-integrable on $[c, d]$, then the function $\varphi$ is ACMG$_*$ on* $[c, d]$.

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*Reçu par la Rédaction le 13. 3. 1961*