COLLOQUIUM MATHEMATICUM

VOL. IX 1962 FASC. 2

ON GROUPS OF n INDEPENDENT RANDOM VARIABLES WHOSE PRODUCT FOLLOWS THE BETA DISTRIBUTION

BY

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1. Introduction. Let us have n independent real random variables $X_1, X_2, \ldots, X_n$ having non-degenerate distributions. If for real non-vanishing numbers $c_1, c_2, \ldots, c_n$ the linear combination $c_1X_1+c_2X_2+\ldots+c_nX_n$ is normal, then every X_k is also normal. This suggests the general problem of finding out when the distribution of the function $f(X_1, X_2, \ldots, X_n)$ determines the distribution of the arguments uniquely. In the terms of statistics it is the problem of finding when the distribution of a sampling statistics determines the distribution of the population.

In this paper we shall consider a group of n independent non-degenerate real random variables $X_1, X_2, \ldots, X_n$ having beta distributions. We denote by $X_{p,q}$ a random variable having the density

$$(1.1)\qquad f_{p,q}(x) = \begin{cases} 0 & \text{for} \quad x \leqslant 0,\, x \geqslant 1, \\ \dfrac{1}{\mathrm{B}(p,q)} x^{p-1}(1-x)^{q-1} & \text{for} \quad 0 < x < 1. \end{cases}$$

It is well known that if there are n independent random variables $X_{p_0,p_1-p_0}, X_{p_1,p_2-p_1}, \ldots, X_{p_{n-1},p_n-p_{n-1}}$ having beta distributions where all the numbers $p_0, p_1, \ldots, p_n$ satisfy the condition

$$(1.2)\qquad 0 < p_0 < p_1 < p_2 < \ldots < p_n = p,$$

then their product

$$(1.3)\qquad U = X_{p_0,p_1-p_0}\cdot X_{p_1,p_2-p_1}\cdot\ldots\cdot X_{p_{n-1},p_n-p_{n-1}}$$

has a beta distribution with density $f_{p_0,p-p_0}(x)$ (see [2]).

The following question arises: if the product

$$(1.4)\qquad U = X_1\cdot X_2\cdot\ldots\cdot X_n$$

of n independent random variables has a beta distribution with density $f_{p_0,p-p_0}(x)$, must the factors X_k also have beta distributions? The answer is in this case negative: there exist groups of n independent random variables which do not have beta distributions though their product has a beta distribution (see section 5 of this paper). Thus we are interested in the enumeration of the set of groups $(X_1, X_2, \ldots, X_n)$ of independent random variables whose product follows the beta distribution $X_{p_0,p}$. We shall denote this set by $\mathscr{X}_{p_0,p}$.

The problem of enumeration of the set $\mathscr{X}_{p_0,p}$ will be dealt with below (see section 4) by using the Mellin transforms of positive random variables (sections 2 and 3).

2. The Mellin transforms of positive random variables. We define the Mellin transform of a positive random variable V having the distribution function $P\{V \leqslant x\} = F(x)$ $(F(0) = 0)$ by the formula

$$h(s) = \mathrm{E}[V^s] = \int_0^\infty x^s\, dF(x) \tag{2.1}$$

(see [1]) where s is a complex variable. The function $h(s)$ is always defined in some strip S: $c_1 < \operatorname{Re} s < c_2$ $(c_1 < 0 < c_2)$ containing the imaginary axis and parallel to it. The Mellin transform $h(s)$ defines the law of the corresponding positive random variable V uniquely.

If the distribution function $F(x)$ has density $F'(x)$, then it is given at every point of its continuity by the formula

$$F'(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-1} h(s)\, ds, \tag{2.2}$$

where the path of integration is the imaginary axis or any line parallel to it and lying within S.

It should be noted that the Mellin transform $h(s)$ of a positive random variable V is for $s = it$ the characteristic function $\varphi(t)$ of the variable $\ln V$ because of

$$h_V(it) = \mathrm{E}[V^{it}] = \mathrm{E}[e^{it\ln V}] = \varphi_{\ln V}(t). \tag{2.3}$$

Thus $h(s)$ shows for $s = it$ all the properties of characteristic functions: for t real

$$\begin{cases} h(it) \text{ is continuous along the whole axis } t; \\ h(0) = 1,\ |h(it)| \leqslant 1,\ h(-it) = \overline{h(it)}; \\ h(it) \text{ is a positive definite function.} \end{cases} \tag{2.4}$$

If we have n independent positive random variables $X_1, X_2, \ldots, X_n$ with the corresponding Mellin transforms $h_1(s), h_2(s), \ldots, h_n(s)$, then the Mellin transform of their product (1.4) is

$$h_U(s) = h_1(s)\cdot h_2(s)\cdot\ldots\cdot h_n(s). \tag{1.4'}$$

From the theory of Cramér and Lévy it follows that the necessary and sufficient condition of weak convergence of the sequence of distribution functions $F_n(x)$ $(F(0) = 0)$ to a distribution function $F_0(x)$ is the convergence of the sequence of corresponding Mellin transforms $h_n(s)$ to a continuous function $h_0(s)$ for every $s = it$ (t real); $F_0(x)$ corresponds in this case to $h_0(s)$. Hence we see that the integer n in the formulae (1.4), (1.4') may be changed into indefinite.

More directly, the Mellin transform of the product

$$U = \prod_{k=1}^{\infty} X_k \tag{2.5}$$

is

$$h_U(s) = \prod_{k=1}^{\infty} h_k(s) \tag{2.5'}$$

(the limes in (2.5) is in the sense of weak convergence, the limes in (2.5') is for every $s = it$ and $h_U(it)$ should be continuous).

Hence we see that the relation between the multiplication of independent positive random variables and the multiplication of their Mellin transforms is exactly the same as the relation between the addition of independent random variables and the multiplication of their characteristic functions.

3. The functional equation for Mellin transforms of the group $(X_1, X_2, \ldots, X_n)$ belonging to $\mathscr{X}_{p_0,p}$. The Mellin transform of the variable (1.1) is

$$h_{p,q}(s) = \frac{\mathrm{B}(p+s, q)}{\mathrm{B}(p, q)} \quad (\operatorname{Re} s > -p). \tag{1.1'}$$

The Mellin transforms of both sides of formula (1.3) are

$$h_U(s) = h_{p_0,p_1-p_0}(s)\cdot h_{p_1,p_2-p_1}(s)\cdot\ldots\cdot h_{p_{n-1},p_n-p_{n-1}}(s) \tag{1.3'}$$

or

$$\frac{\mathrm{B}(p_0+s, p_n-p_0)}{\mathrm{B}(p_0, p_n-p_0)} \tag{1.3''}$$

$$= \frac{\mathrm{B}(p_0+s, p_1-p_0)}{\mathrm{B}(p_0, p_1-p_0)}\cdot\frac{\mathrm{B}(p_1+s, p_2-p_1)}{\mathrm{B}(p_1, p_2-p_1)}\cdot\ldots\cdot\frac{\mathrm{B}(p_{n-1}+s, p_n-p_{n-1})}{\mathrm{B}(p_{n-1}, p_n-p_{n-1})}.$$

Hence we see that for the enumeration of the set $\mathscr{X}_{p_0,p_n-p_0}$ it is enough to solve the equation

$$h_1(s)\cdot h_2(s)\cdot\ldots\cdot h_n(s) = \frac{\mathrm{B}(p_0+s,\, p_n-p_0)}{\mathrm{B}(p_0,\, p_n-p_0)} \qquad (\mathrm{Re}\, s > -p_0) \tag{3.1}$$

in terms of Mellin transforms of positive random variables.

4. Solving equation (3.1). Let us take arbitrary positive numbers $p_1, p_2, \ldots, p_{n-1}$ satisfying condition (1.2). From formula (1.3″) we see that the group of functions

$$h_k(s) = h_{p_{k-1},p_k-p_{k-1}}(s) = \frac{\mathrm{B}(p_{k-1}+s,\, p_k-p_{k-1})}{\mathrm{B}(p_{k-1},\, p_k-p_{k-1})} \qquad (k = 1, 2, \ldots, n) \tag{4.1}$$

are for every group $p_1, p_2, \ldots, p_{k-1}$ the Mellin transforms of a group $(X_1, X_2, \ldots, X_n)$ belonging to $\mathscr{X}_{p_0,p_n-p_0}$.

Now we shall expand the unknown functions $h_k(s)$ into products

$$h_k(s) = \frac{\mathrm{B}(p_{k-1}+s,\, p_k-p_{k-1})}{\mathrm{B}(p_{k-1},\, p_k-p_{k-1})}\, e^{\gamma_k(s)} \qquad (k = 1, 2, \ldots, n), \tag{4.2}$$

where $\gamma_k(s)$ are new unknown functions. Substituting functions (4.2) in equation (3.1) we see that the functions $\gamma_k(s)$ should satisfy the equation

$$\sum_{k=1}^{n} \gamma_k(s) \equiv 0 \tag{4.3}$$

and they should be taken in such a way as to make all the functions (4.2) Mellin transforms of positive random variables. Since $h(it)$ should be a characteristic function, it should satisfy conditions (2.4). Thus substituting the unknown complex functions $\gamma_k(it)$ in their real and imaginary parts

$$\gamma_k(it) = \alpha_k(t) + i\beta_k(t) \tag{4.4}$$

we obtain the following conditions:

$$\left\{\begin{array}{l} \alpha_k(t),\ \beta_k(t) \text{ are real and continuous along the axis } t, \\ \sum_{k=1}^{n} \alpha_k(t) = 0, \qquad \sum_{k=1}^{n} \beta_k(t) = 0, \\ \alpha_k(0) = 0, \qquad \beta_k(0) = 0, \\ \alpha_k(-t) = \alpha_k(t), \qquad \beta_k(-t) = -\beta_k(t), \\ \alpha_k(t) \leqslant \ln \left| \dfrac{\mathrm{B}(p_{k-1},\, p_k-p_{k-1})}{\mathrm{B}(p_{k-1}+it,\, p_k-p_{k-1})} \right|. \end{array}\right. \tag{4.5}$$

The result of these cosiderations may be formulated as the following theorem:

THEOREM. *For a group of n positive independent random variables $(X_1, X_2, \ldots, X_n)$ to belong to the set $\mathscr{X}_{p_0,p_n-p_0}$ it is necessary and sufficient that their Mellin transforms* (2.1) *are given for $s = it$ (t real) in the form*

$$h_k(it) = \frac{\mathrm{B}(p_{k-1}+it,\, p_k-p_{k-1})}{\mathrm{B}(p_{k-1},\, p_k-p_{k-1})}\, e^{\alpha_k(t)+i\beta_k(t)}, \tag{4.6}$$

where $p_1, p_2, \ldots, p_{n-1}$ are arbitrary positive numbers satisfying conditions (1.2), *the functions $\alpha_k(t)$, $\beta_k(t)$ are real and continuous on the whole axis t satisfying conditions* (4.5) *and functions* (4.6) *are positive definite functions.*

It would be interesting to give an enumeration of the functions $\alpha_k(t)$, $\beta_k(t)$ making functions (4.6) positive definite, but this problem seems to be difficult. We shall give instead some examples of groups of random variables belonging to $\mathscr{X}_{p_0,p-p_0}$.

5. Examples of groups $(X_1, X_2, \ldots, X_n)$ belonging to $\mathscr{X}_{p_0,p-p_0}$.

5.1. Using the known formula for beta and gamma functions (see [3], p. 341)

$$\mathrm{B}(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad (\mathrm{Re}\, p > 0,\ \mathrm{Re}\, q > 0) \tag{5.1}$$

and the following formula for gamma function (see [3], p. 333):

$$\Gamma(mw) = \frac{m^{mw-1/2}}{(2\pi)^{(m-1)/2}} \prod_{r=1}^{m} \Gamma\left(w + \frac{r-1}{m}\right) \qquad (\mathrm{Re}\, w > 0), \tag{5.2}$$

we may write equation (3.1) in the form

$$h_1(s)\cdot h_2(s)\cdot\ldots\cdot h_n(s) = \prod_{r=1}^{m} \frac{\mathrm{B}\left(\dfrac{p_0+r-1+s}{m},\, \dfrac{p_n-p_0}{m}\right)}{\mathrm{B}\left(\dfrac{p_0+r-1}{m},\, \dfrac{p_n-p_0}{m}\right)}. \tag{5.3}$$

Dividing the set of positive integers $R = \{1, 2, \ldots, m\}$ $(m > n)$ into n mutually exclusive and exhaustive subsets $R_1, R_2, \ldots, R_n$, we may write

$$h_k(s) = \prod_{r \in R_k} \frac{\mathrm{B}\left(\dfrac{p_0+r-1+s}{m},\, \dfrac{p-p_0}{m}\right)}{\mathrm{B}\left(\dfrac{p_0+r-1}{m},\, \dfrac{p-p_0}{m}\right)} \qquad (k = 1, 2, \ldots, n). \tag{5.4}$$

The variables X_k corresponding to the Mellin transforms (5.4) are

$$X_k = \prod_{r\in R_k} Y_r \qquad (k = 1, 2, \ldots, n), \tag{5.4'}$$

where Y_r are independent positive random variables having densities

$$g_r(x) = \begin{cases} 0 & \text{for} \quad x \leqslant 0, x \geqslant 1, \\ \dfrac{m}{B\left(\dfrac{p_0+r-1}{m}, \dfrac{p-p_0}{m}\right)} x^{p_0+r-1}(1-x^m)^{\frac{p-p_0}{m}-1} & \text{for} \quad 0 < x < 1. \end{cases} \tag{5.5}$$

5.2. Using formula (5.1) and the known formula for gamma functions (see [3], p. 332)

$$\Gamma(1+w) = 4^w \prod_{k=1}^{\infty} \frac{\Gamma\left(\frac{1}{2}+\frac{w}{2^k}\right)}{\Gamma(\frac{1}{2})} \qquad (\operatorname{Re} w > -\tfrac{1}{2}), \tag{5.6}$$

we may write equation (3.1) in the form

$$h_1(s)\cdot h_2(s)\cdot\ldots\cdot h_n(s) = \prod_{r=1}^{\infty} \frac{B\left(\dfrac{2^{r-1}+p_0-1+s}{2^r}, \dfrac{p-p_0}{2^r}\right)}{B\left(\dfrac{2^{r-1}+p_0-1}{2^r}, \dfrac{p-p_0}{2^r}\right)}. \tag{5.7}$$

Dividing the set of positive integers $N = \{1, 2, \ldots\}$ into n mutually exclusive and exhaustive subsets $N_1, N_2, \ldots, N_n$, we obtain

$$h_k(s) = \prod_{r\in N_k} \frac{B\left(\dfrac{2^{r-1}+p_0-1+s}{2^r}, \dfrac{p-p_0}{2^r}\right)}{B\left(\dfrac{2^{r-1}+p_0-1}{2^r}, \dfrac{p-p_0}{2^r}\right)} \qquad (k = 1, 2, \ldots, n). \tag{5.8}$$

The variables X_k corresponding to the Mellin transforms (5.8) are

$$X_k = \prod_{r\in N_k} Y_r \qquad (k = 1, 2, \ldots, n), \tag{5.8'}$$

where Y_k are positive independent random variables having densities

$$g_r(x) = \begin{cases} 0 & \text{for} \quad x \leqslant 0, x \geqslant 1, \\ \dfrac{2^r}{B\left(\dfrac{2^{r-1}+p_0-1}{2^r}, \dfrac{p-p_0}{2^r}\right)} x^{2^{r-1}-p_0-1}(1-x^{2^r})^{\frac{p-p_0}{2^r}-1} & \text{for} \quad 0 < x < 1. \end{cases} \tag{5.9}$$

5.3. Using formula (5.1) and the known formula (see [3], p. 331)

$$\Gamma(w) = e^{-Cw}\frac{1}{w} \prod_{r=1}^{\infty} \frac{e^{w/r}}{1+\dfrac{w}{r}} \qquad (\operatorname{Re} w > 0), \tag{5.10}$$

we may write equation (3.1) in the form

$$h_1(s)\cdot h_2(s)\cdot\ldots\cdot h_n(s) = \prod_{r=0}^{\infty} \frac{p_0+r}{p+r}\cdot\frac{p+r+s}{p_0+r+s}. \tag{5.11}$$

Dividing the set of non-negative integers $N = \{0, 1, 2, \ldots\}$ into n mutually exclusive and exhaustive subsets $N_1, N_2, \ldots, N_n$, we obtain

$$h_k(s) = \prod_{r\in N_k} \frac{p_0+r}{p+r}\cdot\frac{p+r+s}{p_0+r+s} \qquad (k = 1, 2, \ldots, n). \tag{5.12}$$

The variables X_k corresponding to the Mellin transforms (5.12) are

$$X_k = \prod_{r\in N_k} V_r \qquad (k = 1, 2, \ldots, n), \tag{5.12'}$$

where the Mellin transforms of V_r are

$$H_r(s) = \frac{p_0+r}{p+r} + \frac{p-p_0}{p+r}\cdot\frac{p_0+r}{p_0+r+s} \qquad (r = 0, 1, 2, \ldots). \tag{5.13}$$

Thus the distributions of V_r are

$$G_r(x) = \begin{cases} 0 & \text{for} \quad x < 0, \\ \dfrac{p-p_0}{p+r} x^{p_0+r} & \text{for} \quad 0 \leqslant x < 1, \\ 1 & \text{for} \quad x \geqslant 1. \end{cases} \tag{5.13'}$$

Every random variable V_r is here a mixture of two random variables V'_r, V''_r, of which the first, V'_r, is degenerate,

$$P\{V'_r = 1\} = 1, \tag{5.15}$$

and the second is continuous, its density being

$$g_r(x) = \begin{cases} 0 & \text{for} \quad x \leqslant 0,\ x \geqslant 1, \\ (p_0+r)x^{p_0+r-1} & \text{for} \quad 0 < x < 1. \end{cases} \tag{5.16}$$

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Reçu par la Rédaction le 4. 3. 1961

COLLOQUIUM MATHEMATICUM

VOL. IX 1962 FASC. 2

ERGODICITÉ ET STATIONNARITÉ DES CHAÎNES DE MARKOFF VARIABLES À UN NOMBRE FINI D'ÉTATS POSSIBLES

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Introduction. Les problèmes liés aux chaînes de Markoff constantes ont été largement discutés par de nombreux auteurs. Les chaînes de Markoff variables ont été traitées beaucoup moins fréquemment et l'ergodicité de ces chaînes, qui avait déjà attiré l'attention de Markoff (cf. [6]), quoique le terme „ergodicité" n'ait été introduit que plus tard, soulève encore des questions non résolues. Parmi les mathématiciens qui s'en occupent, il y a d'une part l'école soviétique: Bernstein [1], Kolmogoroff [4], Siragedinoff [11], Sarimsakoff [9, 10] et Moustafin [10], dont les recherches n'ont abouti qu'à des résultats partiels. D'autre part, Hajnal [2, 3], Mott [7, 8] et Schneider [8] ont dernièrement repris le sujet.

La présente communication fait suite à celle [5], publiée il y a quelques années, sur l'ergodicité des chaînes de Markoff variables à deux états possibles. Son but est de généraliser au cas d'un nombre fini d'états possibles les résultats obtenus précédemment. Au cours des recherches, il a paru juste d'étudier en même temps la stationnarité des chaînes, vu que cette propriété joue un aussi grand rôle que l'ergodicité dans l'étude des chaînes.

§ 1. Notations. Nous écrirons $D = (d_1, \ldots, d_r)$ pour mettre en évidence que D est un vecteur aux composantes d_i $(i = 1, \ldots, r)$; $0 = (0, \ldots, 0)$ est le vecteur nul.

Nous écrirons $P = \{p_{ab}\}$ pour exprimer que P est une matrice dont les éléments sont p_{ab} $(a, b = 1, \ldots, r)$; $0 = \{0\}$ est la matrice nulle dont tous les éléments sont nuls; $I = \{\delta_{ab}\}$ est la matrice-unité, où $\delta_{aa} = 1$ et $\delta_{ab} = 0$ pour $a \neq b$.

E ou E_D est une matrice dite *ergodique* dont toutes les lignes sont identiques. On la désigne par $E = \{e_a\}$ $(a = 1, \ldots, r)$ ou par $E = E_D$ suivant qu'on veut mettre en évidence ses éléments e_a ou ses lignes D.

§ 2. Définition et propriétés fondamentales d'une chaîne. Les chaînes qui seront étudiées sont celles de Markoff discontinues à un