identity operation, then so is the other, for it should be noted that $e_i(a_1, \ldots, a_n) = a_i \notin S$ while the values attained by operations of the form (1) are always in S (as shown above). Certainly $g = e_i$ and $h = e_j$ implies $i = j$ and then we have $g = h$. In all other cases we may assume that h is given by (1) and similarly

$$g(x_1, \ldots, x_n) \equiv f_r(x_{j_1}, \ldots, x_{j_r}), \tag{3}$$

where $r \leqslant \infty$, $j_1, \ldots, j_r \epsilon \{1, \ldots, n\}$.

It is easily seen that if an operation h of the form (1) satisfies $h(a_1, \ldots, a_n) = 0$, where $\{a_1, \ldots, a_n\} \epsilon \boldsymbol{J}$, then either $l = \infty$ or some number occurs at least twice in the sequence $i_1, \ldots, i_l$. In both cases we have identically $h(x_1, \ldots, x_n) \equiv 0$. The same being true for g, the appearance of 0 in (2) implies $g(x_1, \ldots, x_n) \equiv 0 \equiv h(x_1, \ldots, x_n)$.

If $g(a_1, \ldots, a_n) = h(a_1, \ldots, a_n) \neq 0$, then we must have $l, r \leqslant n$ (cf. (1), (3)) and, by $n \leqslant m$,

$$\begin{aligned} f_l(a_{i_1}, \ldots, a_{i_l}) &= q\{a_{i_1}, \ldots, a_{i_l}\}, \\ f_r(a_{j_1}, \ldots, a_{j_r}) &= q\{a_{j_1}, \ldots, a_{j_r}\}. \end{aligned} \tag{4}$$

It follows now from the one-to-one property of the mapping q, by (1), (2), (3) and (4) that $\{a_{i_1}, \ldots, a_{i_l}\} = \{a_{j_1}, \ldots, a_{j_r}\}$. Hence $l = r$, $\{i_1, \ldots, i_l\} = \{j_1, \ldots, j_r\}$ and, again by (1) and (3), $g = h$. This completes our proof.

MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 10. 2. 1961

COLLOQUIUM MATHEMATICUM

VOL. IX 1962 FASC. 1

REMARKS ON THE CARTESIAN PRODUCT OF TWO GRAPHS

BY

L. SZAMKOŁOWICZ (WROCŁAW)

1. In paper [4] H. S. Shapiro introduced a notion of the *Cartesian product* $G_1 \times G_2$ of two graphs. F. Harary in his paper [2] (see also [3]) introduced a notion of the *composition* $G_1 * G_2$ (we write $G_1 * G_2$ instead of Harary's notation $G_1[G_2]$, according to the associativity of this operation) of two graphs. These notions for connected graphs are special cases of a more general notion of the Cartesian product of two graphs with metrics. In the present note we shall study this product under some natural assumptions concerning these *metrics*, namely those of [1] (p. 630). We shall prove that under these assumptions our product coincides with $G_1 \times G_2$ or $G_1 * G_2$.

2. Definitions. A pair $\langle N, \varrho\rangle$, where N is a finite or infinite set, is said to be an *NS-space* if $\varrho(x, y)$ is a function defined on the whole N whose values are non-negative integers such that

1° $\varrho(x, y) = 0$ if and only if $x = y$,

2° $\varrho(x, y) = \varrho(y, x)$,

3° $\varrho(x, y) + \varrho(y, z) \geqslant \varrho(x, z)$,

4° If $\varrho(x, y) = n (n \geqslant 1)$, then there exists an element $z \epsilon N$ such that $\varrho(x, z) = 1$ and $\varrho(z, y) = n-1$.

The *Cartesian product* of two *NS*-spaces $\langle N_1, \varrho_1\rangle$ and $\langle N_2, \varrho_2\rangle$ we define as an *NS*-space $\langle N_1 \times N_2, \varrho\rangle$, where $N_1 \times N_2$ is the set of ordered pairs (x, y), $x \epsilon N_1$, $y \epsilon N_2$, with the metric ϱ defined by

$$\varrho[(x_1, y_1), (x_2, y_2)] = f[\varrho_1(x_1, x_2), \varrho_2(y_1, y_2)] = f(k, m),$$

$k = \varrho_1(x_1, x_2)$, $m = \varrho_2(y_1, y_2)$, $x_1, x_2 \epsilon N_1$, $y_1, y_2 \epsilon N_2$, where f is a function whose values are non-negative integers and satisfies the following conditions (see [1], p. 630):

(1) $f(k, m) = f(m, k)$ for all non-negative integers m, k,

(2) $f(0,1) = 1$,

(3) $f(k,m) \leqslant f(k',m')$ for $m' \geqslant m$ and $k' \geqslant k$,

(4) $f(s\cdot k, s\cdot m) = s\cdot f(k,m)$ for all non-negative integers s, k, m,

(5) $f(k, f(m,s)) = f(f(k,m), s)$ for all non-negative integers k, m, s.

From (4) and (2) it follows that

(6) $$f(0,m) = m\cdot f(0,1) = m.$$

Hence, according to (1), we get the equality

(7) $$f(k,0) = k.$$

The inequality

(8) $$f(k,m) \leqslant k+m$$

follows from the triangle inequality 3° for the points (x_1, y_1), (x_2, y_1) and (x_2, y_2), and from (6) and (7).

Further, from (3), (2) and (8) we obtain

(9) $$1 \leqslant \varrho(1,1) \leqslant 2.$$

3. Bohnenblust proved in [1] that if values of $f(\xi,\eta)$ and ξ,η are non-negative real numbers, then any function satisfying the five conditions (1)-(5) is necessarily of the form

$$f(\xi,\eta) = (\xi^p + \eta^p)^{1/p}$$

for some real number p $(0 < p \leqslant \infty)$, and if, in addition, (8) is assumed, then $1 \leqslant p \leqslant \infty$. We are going to prove the following.

THEOREM. *Any function f, whose values are non-negative integers, satisfying the conditions (1)-(5) and (8) is of one of the forms*

$$f(k,m) = \max(k,m) \quad \text{or} \quad f(k,m) = k+m.$$

Proof. According to (9), we consider two cases according as $f(1,1) = 1$ or $f(1,1) = 2$.

A. $f(1,1) = 1$. From (6), (7) and (3) it follows that $f(k,m) \geqslant \max(k,m)$. On the other hand, from (4) and (3) we obtain $f(k,m) \leqslant \max(k,m)$.

Consequently,

$$f(k,m) = \max(k,m).$$

B. $f(1,1) = 2$. Any function f satisfying conditions (1)-(5) generates a sequence $\{a_n\}$, $n = 1, 2, \ldots$, defined inductively by

(*) $$a_1 = 1; \quad a_n = f(1, a_{n-1}) \quad \text{for } n > 1.$$

This sequence is evidently non-decreasing.

Furthermore, in virtue of (1), (5) and definition (*), we have the equality

(**) $$a_{n+m} = f(a_n, a_m)$$

for any integers $n, m > 0$. This assertion was proved in paper [1] (Lemma 4.1).

We now prove that

(***) $$a_k = k \quad \text{for any} \quad k > 0.$$

By (*) and (8), we have

(i) $$a_{2k-1} = f(1, a_{2k-2}) \leqslant a_{2k-2}+1.$$

From (*), (**) and (4) we obtain

(ii) $$a_{2k} = f(1, a_{2k-1}) = f(a_k, a_k) = a_k\cdot f(1,1) = 2\cdot a_k,$$

hence, by (8),

(iii) $$a_{2k-1} \geqslant 2\cdot a_k - 1.$$

To prove (***), we proceed by induction. We have, evidently, $a_1 = 1$, $a_2 = f(1,1) = 2$. Assume that

$$a_s = s \quad \text{for} \quad s \leqslant 2k-2.$$

Then from (i) it follows that $a_{2k-1} \leqslant 2k-1$, and from (iii) $a_{2k-1} \geqslant 2k-1$. Hence $a_{2k-1} = 2k-1$.

The equality $a_{2k} = 2k$ follows at once from (ii) and the induction hypothesis. Thus (***) is proved.

From (**) and (***) we obtain $f(k,m) = k+m$. Thus if $f(1,1) = 2$, then $f(k,m) = k+m$.

3. Applications to connected graphs. A *graph G* is a pair $\langle N, W\rangle$, where N is a finite or infinite set of elements (vertices) and W a relation for which the following conditions hold:

$\langle 1\rangle$ $$xWy \to yWx,$$

$\langle 2\rangle$ $$\sim xWx.$$

Let $x, y \in N$. A *path in G from x to y* is a finite sequence $\{x_k\}$, $x_k \in N$, $k = 0, 1, \ldots, n$, such that

(a) $x_k \neq x_j$ for $k \neq j$, $k, j = 0, 1, \ldots, n$,

(aa) $x_k W x_{k+1}$ for $k = 0, 1, \ldots, n$,

(aaa) $x_0 = x$, $x_n = y$.

The number n is the *length of the path*.

The graph G is *connected* if for every point x and y of that graph there exists a path joining these points in G.

The *distance* $\mathrm{d}(x, y)$ of points x and y of graph G is the minimum of the lengths of all paths joining x and y in G. It is easy to verify that d is a metric satisfying conditions 1°-4°. Thus every connected graph may be considered as an NS-space.

On the other hand, for every NS-space $\langle N, \varrho\rangle$ there exists exactly one connected graph $\langle N, W\rangle$, where W is given by

$$xWy \quad \text{if and only if} \quad \varrho(x, y) = 1.$$

The distance $\mathrm{d}(x, y)$ in $G = \langle N, W\rangle$ is evidently $\varrho(x, y)$.

The *Cartesian product* of two connected graphs G_1, G_2 is the Cartesian product of these graphs considered as NS-spaces $G_1 = \langle N_1, W_1\rangle \sim \sim \langle N_1, \varrho_1\rangle$ and $G_2 = \langle N_2, W_2\rangle \sim \langle N_2, \varrho_2\rangle$. From our theorem it follows that the Cartesian product of G_1 and G_2 is a graph $G = \langle N_1 \times N_2, W\rangle \sim \sim \langle N_1 \times N_2, \varrho\rangle$, where

$$\varrho[(x_1, y_1), (x_2, y_2)] = \begin{cases} \max[\varrho_1(x_1, x_2), \varrho_2(y_1, y_2)] \\ \text{or } \varrho_1(x_1, x_2) + \varrho_2(y_1, y_2). \end{cases}$$

In the first case the relation W in G is given by

$W[(x_1, y_1), (x_2, y_2)]$ *if and only if* $x_1 = x_2$ *and* $W_2(y_1, y_2)$, *or* $W_1(x_1, x_2)$ *and* $y_1 = y_2$, *or* $W_1(x_1, x_2)$ *and* $W_2(y_1, y_2)$.

We see that$G$ is in this case the comopsition $G_1 * G_2$ in the sense of Harary.

In the second case the relation W in G is given by

$W[(x_1, y_1), (x_2, y_2)]$ *if and only if* $x_1 = x_2$ *and* $W_2(y_1, y_2)$, *or* $W_1(x_1, x_2)$ *and* $y_1 = y_2$.

G is the Cartesian product $G_1 \times G_2$ in the sense of Shapiro.

4. Remarks and problems. It is obvious that both products $G_1 \times G_2$ and $G_1 * G_2$ are associative. Therefore such products as $G_1 \times \ldots \times G_n$ or $G_1 * \ldots * G_n$ can be written without brackets. Such products can be defined also for infinite systems of graphs $G_t = \langle N_t, W_t\rangle$ $(t \in T)$:

$$\prod_{t\in T}^{x} G_t = \langle \mathop{\mathrm{P}}_{t\in T} N_t \{\{f, g\}:\ f, g \in \mathop{\mathrm{P}}_{t\in T} N_t,\ f \neq g;\ f(t_0) \neq g(t_0) \text{ for exactly one } t_0 \in T \text{ and } \{f(t_0), g(t_0)\} \in W_{t_0}\}\rangle,$$

$$\prod_{t\in T}^{*} G_t = \langle \mathop{\mathrm{P}}_{t\in T} N_t,\ \{\{f, g\}:\ f, g \in \mathop{\mathrm{P}}_{t\in T} N_t\ f \neq g;\ f(t) \neq g(t) \text{ implies } \{f(t), g(t)\} \in W_t\}\rangle,$$

where $\mathop{\mathrm{P}}_{t\in T} N_t$ denotes the Cartesian product of the system of sets N_t $(t \in T)$, i. e., the set of all functions over T satisfying $f(t) \in N_t$ for all $t \in T$. Clearly Π^x is a generalization of x and Π^* a generalization of $*$.

The definitions of Π^x and Π^* and the following problem are due to Jan Mycielski.

P 348. Is the decomposition of a graph into a Π^x or Π^* product of indecomposable non-one-point graphs unique (disregarding the order of its terms)?

REFERENCES

[1] F. Bohnenblust, *An axiomatic characterization of L_p-spaces*, Duke Mathematical Journal 6 (1940), p. 627-640.

[2] F. Harary, *On the group of the composition of two graphs*, ibidem 26 (1959), p. 29-34.

[3] G. Sabidussi, *The composition of graphs*, ibidem 26 (1959), p. 693-696.

[4] H. S. Shapiro, *The embedding of graphs in cubes and the design of sequential relay circuits*, unpublished Bell Telephone Laboratories Memorandum, July 1953.

Reçu par la Rédaction le 14. 2. 1961