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A LOCAL PROPERTY OF POINTWISE PERIODIC HOMEOMORPHISMS

BY

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A function $T: S \to S$ is said to be *pointwise periodic* if for each $x \in S$ there is a positive integer $n(x)$ such that

$$T^{n(x)}(x) = x.$$

An account of such functions may be found in Gordon Whyburn's book [5]. Our purpose is to localize a result of Hall and Schweigert [2]. After this paper had been written, Walter Gottschalk kindly informed me that a somewhat better result could be obtained using some results in his joint book with Hedlund [1]. However, the proof is rather short and seems worth making public in view of the direct methods employed.

THEOREM. *Let $f: X \to X$ be a pointwise periodic homeomorphism on the totally disconnected locally compact Hausdorff space X, let $a \in X$ and let A be an open set about a. If $f^m(a) = a$, then there is a compact open set V with*

$$a \in V = f^m(V) \subset A.$$

Proof. Suppose first that $m = 1$ and let $\boldsymbol{V}$ be the collection of all compact open sets which contain a and are contained in A. If $V \in \boldsymbol{V}$, then

$$V \subset f(V) \cup \ldots \cup f^n(V) \cup \ldots$$

and, since V is compact and open and f is a homeomorphism,

$$V \subset f(V) \cup \ldots \cup f^n(V),$$

where we may suppose that n is the least positive integer for which such an inclusion is valid. If we let

$$V' = V \cup f(V) \cup \ldots \cup f^{n-1}(V),$$

then it follows at once that $V' \subset f(V')$ and, from the fact that f is pointwise periodic, we easily see that $f(V') = V'$. It should be observed that V'

is compact open and if $p \geqslant n-1$, where n is the integer used in the definition of V', then $f^p(V) \subset V'$. From this it follows that for $V_1, V_2 \in \boldsymbol{V}$ and $V_1 \subset V_2$ we have $V_1' \subset V_2'$ and thus that the collection of all sets V' with $V \in \boldsymbol{V}$ has the finite intersection property.

Suppose that no V' with $V \in \boldsymbol{V}$ is a subset of A so that every set in the collection

$$\{V' \setminus A \mid V \in \boldsymbol{V}\}$$

is non-void. The collection has the finite intersection property so let b be an element of the intersection. The orbit, B, of b is finite and does not contain a, since $f(a) = a$. Thus there is an element V in $\boldsymbol{V}$ which does not intersect B. It follows at once that V' does not intersect B, contrary to the fact that $b \in B \cap V'$.

We conclude that there is an open and closed set V' such that

$$a \in V' = f(V') \subset A.$$

The function f^m being pointwise periodic, the result follows from what has just been proved.

Corollary 1. *If $f\colon X \to X$ is a pointwise periodic homeomorphism on the locally compact totally disconnected Hausdorff space X and if A is a compact open set in X, then there is an integer n such that*

$$f^n(A) = A.$$

Proof. For any $x \in A$ there is a positive integer $n(x)$ such that $f^{n(x)} = x$ and hence a compact open set U_x such that

$$x \in U_x = f^{n(x)}(U_x) \subset A.$$

Thus there is a finite set I and for each $i \in I$ a positive integer n_i and an open set U_i such that

$$A = \bigcup \{U_i | i \in I\} \quad \text{and} \quad f^{n_i}(U_i) = U_i,\ i \in I.$$

If n is the product of all the integers n_i we have

$$f^n(A) = A.$$

If X is a Stone space (compact totally disconnected Hausdorff) then the compact open subsets of X form a Boolean algebra B, see Sikorski [3], p. 23. Moreover, a homeomorphism f taking X onto X determines an automorphism g taking B onto B. From the corollary we clearly have

Corollary 2. *If X is a Stone space, then any pointwise periodic homeomorphism on X determines a pointwise periodic automorphism on the associated Boolean algebra, that is on the 0-dimensional cohomology group of X* [4].

It seemed plausible that a pointwise periodic automorphism on a Boolean algebra might induce a pointwise periodic homeomorphism on the associated Stone space. Professor F. B. Wright informs me that such need not be the case. Suppose that g is a Boolean homeomorphism on B such that for each $b \in B$ there is a positive integer n such that $g^n(b) = 0$. How does one describe the induced transformation on the associated Stone space [3], p. 30 **(P 349)**?

Let X be compact Hausdorff and define $R \subset X \times X$ by $(x, y) \in R$ if and only if x and y are in the same component of X. Then R is clearly an equivalence on X and is a closed set. If X/R denotes the set of equivalence classes (the component space of X) we define $h\colon X \to X/R$ by letting $h(x)$ be the component of X containing x. The topology of X/R is the usual one the set W in X/R being open if $h^{-1}(W)$ is open in X. Then for any continuous function $f\colon X \to X$ there is a continuous function $g\colon X/R \to X/R$ such that

$$gh = hf.$$

Now it is clear that if f is pointwise periodic then so also is g. Since X/R is a compact totally disconnected Hausdorff space and since $A = h^{-1}h(A)$ for any compact open set $A \subset X$ we have, from Corollary 1,

Corollary 3 (Hall and Schweigert [2]). *If f is a pointwise periodic homeomorphism on a compact Hausdorff space and if A is open and closed, then there is an integer n such that $f^n(A) = A$.*

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Added in proof. Two papers by K. H. Hofmann and F. B. Wright, *The automorphism group of certain function spaces and pointwise periodic groups*, extend the observations of this note and resolve some of the problems raised herein.

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REMARKS ON FIXED POINT THEOREM FOR INVERSE LIMIT SPACES

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1. Introduction. A topological space X has the *fixed point property* (FPP) if for every continuous (single-valued) function $f\colon X \to X$ there exists a point $x^* \in X$ such that $f(x^*) = x^*$. Let us consider an inverse system $\{X_n, \pi_n^m, M\}$ of spaces and functions (see [2]), where $\pi_n^m\colon X_m \to X_n$, $m \geqslant n$, are continuous and onto (π_n^n is the identity), and $m, n \in M$, where M is a directed set. The inverse limit $X = \varprojlim \{X_n, \pi_n^m, M\}$ consists of all points $x = \{x_m\}$, $m \in M$, for which $\pi_n^m(x_m) = x_n$. Let $\pi_n\colon X \to X_n$ be projections, i. e. functions defined by $\pi_n(x) = x_n$. We suppose that X_n are polyhedra ([1]) and we consider the following question: under what conditions concerning the inverse system the inverse limit space has FPP. In this paper some sufficient conditions will be given.

It is known that the snake-like continua, i. e. continua which are inverse limit spaces of arcs, M being the sequence of natural numbers, have FPP (see [4]; in [7] a more general result is given, namely for some class of multi-valued functions). We investigate the FPP in a more general situation and the fixed point theorem for snake-like continua is a special case of our theorem. But we do not know whether it is possible to obtain in that way fixed point theorems (if they are true) for so called *tree-like continua* (i. e. continua which are inverse limit spaces of finite dendrites, M being the sequence of natural numbers) and for continua which do not separate the plane.

We say that the inverse system $\{X_n, \pi_n^m, M\}$ has the *special incidence point property* (SIPP) if for every continuous (single-valued) function $f\colon X_m \to X_n$, $m \geqslant n$, there exists a point x_m^* such that $f(x_m^*) = \pi_n^m(x_m^*)$.

([1]) This makes no restriction for the investigation of FPP on compact metric spaces, as every space of this kind is an inverse limit of polyhedra with projections onto [3].