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REMARKS ON FIXED POINT THEOREM FOR INVERSE LIMIT SPACES

BY

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1. Introduction. A topological space X has the *fixed point property* (FPP) if for every continuous (single-valued) function $f\colon X \to X$ there exists a point $x^* \epsilon X$ such that $f(x^*) = x^*$. Let us consider an inverse system $\{X_n, \pi_n^m, M\}$ of spaces and functions (see [2]), where $\pi_n^m\colon X_m \to X_n$, $m \geqslant n$, are continuous and onto (π_n^n is the identity), and $m, n \epsilon M$, where M is a directed set. The inverse limit $X = \varprojlim \{X_n, \pi_n^m, M\}$ consists of all points $x = \{x_m\}$, $m \epsilon M$, for which $\pi_n^m(x_m) = x_n$. Let $\pi_n\colon X \to X_n$ be projections, i. e. functions defined by $\pi_n(x) = x_n$. We suppose that X_n are polyhedra (¹) and we consider the following question: under what conditions concerning the inverse system the inverse limit space has FPP. In this paper some sufficient conditions will be given.

It is known that the snake-like continua, i. e. continua which are inverse limit spaces of arcs, M being the sequence of natural numbers, have FPP (see [4]; in [7] a more general result is given, namely for some class of multi-valued functions). We investigate the FPP in a more general situation and the fixed point theorem for snake-like continua is a special case of our theorem. But we do not know whether it is possible to obtain in that way fixed point theorems (if they are true) for so called *tree-like continua* (i. e. continua which are inverse limit spaces of finite dendrites, M being the sequence of natural numbers) and for continua which do not separate the plane.

We say that the inverse system $\{X_n, \pi_n^m, M\}$ has the *special incidence point property* (SIPP) if for every continuous (single-valued) function $f\colon X_m \to X_n$, $m \geqslant n$, there exists a point x_m^* such that $f(x_m^*) = \pi_n^m(x_m^*)$.

(¹) This makes no restriction for the investigation of FPP on compact metric spaces, as every space of this kind is an inverse limit of polyhedra with projections onto [3].

We shall use the following property of inverse systems of polyhedra (Theorem X.11.9 of [2] with Lemmas X.3.7 and X.3.8):

(1) *Let Y be a polyhedron. Then for every continuous function $f\colon X \to Y$ and every $m_0 \in M$ there exist an m, $m \geqslant m_0$, and a continuous function $f_m\colon X_m \to Y$ such that for every $x \in X$ the values $f(x)$ and $f_m\pi_m(x)$ lie in the same simplex of Y, in other words, such that the diagram*

$$\begin{tikzcd} & Y & \\ X_m \arrow[ur, "f_m"] & & X \arrow[ul, "f"'] \arrow[ll, "\pi_m"] \end{tikzcd}$$

is "nearly commutative".

Let M be a directed set. A *net* in X with respect to M, in symbols $\{x_m\}_M$, is a function $\alpha\colon M \to X$, $\alpha(m) = x_m$. A point $x \in X$ is said to be the *limit* of $\{x_m\}_M$, in symbols $x = \lim\limits_M x_m$, if for every neighbourhood U of x there is an $m_0 \in M$ such that if $m \geqslant m_0$, then $x_m \in U$. A *subnet* of $\{x_m\}_M$ is a net $\{x'_{m'}\}_{M'}$, with a function $\varphi\colon M' \to M$ having the following properties: 1° $x_{\varphi(m')} = x'_{m'}$, 2° for every $m \in M$ there exists an $m_0 \in M'$ such that if $p' \in M'$ and $p' \geqslant m_0$, then $\varphi(p') \geqslant m$. It is known (see [5], p. 136) that if X is a compact Hausdorff space, then for every net there exist a subnet and a point of X which is the limit of that subnet.

We shall consider in the sequel nets $\{C_m\}_M$ whose values C_m are compact subsets of X. Suppose that

(i) *for every finite open covering $\{U_i\}$, $i = 1, 2, \ldots, k$, of X there is $m_0 \in M$ such that if $m \geqslant m_0$ then C_m lies in some U_i.*

Choose in every C_m a point x_m and consider a point $x \in X$ and a subnet $\{x'_{m'}\}_{M'}$ of $\{x_m\}_M$ with $x = \lim\limits_{M'} x'_{m'}$. We prove that

(ii) *the limit does not depend on the choice of $x'_{m'}$ in $C'_{m'} = C_{\varphi(m')}$.*

To prove this consider an open finite covering $\{U_i\}$, $i = 1, 2, \ldots, k$, of X such that $x \in U_1$ and such that there is a neighbourhood V of x having no point in common with U_i for $i \neq 1$. Since $x = \lim\limits_{M'} x'_{m'}$, then there exists an $m_0 \in M'$ such that if $m' \geqslant m_0$ then $x'_{m'} \in V$. By (i), there exists an $m_1 \in M'$ such that if $m' \geqslant m_1$ then $C'_{m'}$ lies in some U_i. Let $m_2 \in M'$ be greater than m_0 and m_1. Hence for $m' \geqslant m_2$ we have $C'_{m'} \subset U_1$, because $C_{m'} \cap V \neq 0$ and $V \subset U_1$. Since the neighbourhood U_1 of x may be arbitrarily chosen, (ii) is proved.

2. The multi-valued functions F_{mn}. Let $X = \varprojlim\{X_n, \pi_n^m, M\}$ and $Y = \varprojlim\{Y_n, \sigma_n^m, N\}$. Let $f\colon X \to Y$ be single-valued function. Consider

the multi-valued functions [1] $F_{mn}\colon X_m \to Y_n$ defined by

(2) $$F_{mn}(x_m) = \sigma_n f \pi_m^{-1}(x_m) \quad \text{for} \quad x_m \in X_m.$$

We have for every $m', m'' \in M$ and $n', n'' \in N$,

$$F_{m'n'}(x_{m'}) = \sigma_{n'}^{n''} F_{m''n''}(\pi_{m'}^{m''})^{-1}(x_{m'}) \quad \text{for} \quad x_{m'} \in X_{m'}.$$

Note that if π_n^m are continuous, then $(\pi_n^m)^{-1}$ are upper semicontinuous (see [6], II). Therefore, if f is continuous, then F_{mn} are upper semicontinuous.

Let $U_1, U_2, \ldots, U_k$ be a finite open covering of X. Then

(3) *There exists an $m_0 \in M$ such that if $m \geqslant m_0$, then every set $\pi_m^{-1}(x_m)$, $x_m \in X_m$, lies in some U_i, $i = 1, 2, \ldots, k$.*

This is a consequence of lemma X.3.7 of [2].

From (1) and (3) it follows that

(4) *For every $n \in N$ and $m_0 \in M$ there exist an $m \geqslant m_0$ and a continuous (single-valued) function $f_{mn}\colon X_m \to Y_n$ such that the values $f_{mn}(x_m)$ and $F_{mn}(x_m)$ lie, for every $x_m \in X_m$, in the star of the same vertex of Y_n.*

Proof. Consider the finite open covering of Y_n consisting of stars of vertices of Y_n. Let $U_1, U_2, \ldots, U_k$ be a finite open covering of X such that every $\sigma_n f(U_i)$, $i = 1, 2, \ldots, k$, lies in the star of some vertex of Y_n. Let $m_1 \in M$ be such that $m_1 \geqslant m_0$ and every $\pi_{m'}^{-1}(x_{m'})$ lies in some U_i, $i = 1, 2, \ldots, k$, if $m' \geqslant m_1$. Such an m_1 exists according to (3). By (1), for $f = \sigma_n f$, there exist an m, $m \geqslant m_1$, and a continuous function $f_{mn}\colon X_m \to Y_n$ such that $\sigma_n f(x)$ and $f_{mn}\pi_m(x)$ lie in the same simplex of Y_n for every $x \in X$. We see that if $x_m \in X_m$, then $f_{mn}\pi_m\pi_m^{-1}(x_m)$ and $\sigma_n f\pi_m^{-1}(x_m)$ lie in the star of the same vertex of Y_n. But we have $f_{mn}(x_m) = f_{mn}\pi_m\pi_m^{-1}(x_m)$ and $F_{mn}(x_m) = \sigma_n f\pi_m^{-1}(x_m)$. Thus (4) is proved.

3. The fixed point theorem. Let $\{X_n, \pi_n^m, M\}$ be an inverse system, where X_n are compact polyhedra and π_n^m are continuous and onto.

THEOREM. *If $\{X_n, \pi_n^m, M\}$ has SIPP, then $X = \varprojlim\{X_n, \pi_n^m, M\}$ has FPP.*

Proof. Let $f\colon X \to X$ be a continuous (single-valued) function. Let $F_{mn}\colon X_m \to X_n$ be multi-valued functions defined for f by the formula (2) for $X = Y$. Consider an element $n \in M$ and the element $m \in M$, $m \geqslant n$, for which (4) holds (for $X = Y$, $M = N$ and $n = m_0$). According to SIPP, there exists a point $x_m^* \in X_m$ such that

(5) $$\pi_n^m(x_m^*) = f_{mn}(x_m^*),$$

where $f_{mn}\colon X_m \to X_n$ is the function whose existence was shown in (4).

[1] See [2], where F_{mn} are defined for multi-valued functions $F\colon X \to Y$.

Consider the compact sets $C_{mn} = \pi_m^{-1}(x_m^*)$. Here m depends on n and $\{C_{mn}\}_M$ is a net whose values are compact subsets of X. According to (3), condition (i) holds. Hence let us consider, according to (ii), a subnet $\{C'_{m'n'}\}_{M'}$ of $\{C_{mn}\}_M$ and a point $x^* \in X$ such that $x^* = \lim_{M'} x'_{m'n'}$ for every net $\{x'_{m'n'}\}_{M'}$, where $x'_{m'n'} \in C'_{m'n'}$. We prove that $f(x^*) = x^*$. In fact, we have

$$\pi_n f(C_{mn}) = \pi_n f \pi_m^{-1}(x_m^*) = F_{mn}(x_m^*), \tag{6}$$
$$\pi_n(C_{mn}) = \pi_n \pi_m^{-1}(x_m^*) = \pi_n^m(x_m^*).$$

By (4) and (5), the values $F_{mn}(x_m^*)$ and $\pi_n^m(x_m^*)$ lie in the star V_{mn} of some vertex of X_n. By (6), the same is true for $\pi_n f(C_{mn})$ and $\pi_n(C_{mn})$, i. e.

$$f(C_{mn}) \cup C_{mn} \subset \pi_n^{-1}(V_{mn}). \tag{7}$$

Assume that the triangulations of X_n are such that $\{\pi_n^{-1}(V_{mn})\}_M$ form a net having property (i) of § 1 (see Lemma X. 3.7 of [2]). Consider the subnet $\{\pi_{n'}^{-1}(V'_{m'n'})\}_{M'}$. Let $\{x'_{m'n'}\}_{M'}$ be a net such that $x'_{m'n'} \in C'_{m'n'}$. Hence, by definition of M', $\lim_{M'} x'_{m'n'} = x^*$. Since $\{\pi_{n'}^{-1}(V'_{m'n'})\}_{M'}$ has property (i), then, by (7) and (ii), we have $\lim_{M'}\{\xi_{m'n'}\}_{M'} = x^*$ for every net $\{\xi_{m'n'}\}_{M'}$ with $\xi_{m'n'} \in \pi_{n'}^{-1}(V'_{m'n'})$. In particular this is true, according to (7), for $\{f(x'_{m'n'})\}_{M'}$. Hence, by the continuity of f, we obtain (see [5], p. 86) $f(x^*) = f(\lim_{M'} x'_{m'n'}) = \lim_{M'} f(x'_{m'n'}) = x^*$.

4. Remarks. We do not know whether the theorem is true under the hypothesis that every X_n has FPP. If π_n^m are not onto, it is not true because there exists a continuum (see [1]) which is the common part of a decreasing sequence $\{Q_n\}$ of solid spheres in E^3, which does not have FPP. Such a continuum may be considered as the inverse limit of $\{Q_n, \pi_n^m\}$, where π_n^m are inclusions.

However, we have the following

COROLLARY. *Let $\{X_n, \pi_n^m, M\}$ be an inverse system such that*

1° *$\{X_n\}$ is an increasing sequence of compact polyhedra with FPP,*

2° *π_n^m are retractions, i. e. $\pi_n^m(x_m) = x_m$ if $x_m \in X_n \subset X_m$.*

Then $X = \varprojlim\{X_n, \pi_n^m, M\}$ has FPP.

Proof. It is sufficient to show that $\{X_n, \pi_n^m, M\}$ has SIPP. Let $f\colon X_m \to X_n$, $m \geqslant n$, be a (single-valued) continuous function. Consider $f' = f|X_n$ and $\pi'^m_n = \pi_n^m|X_n$. We have $f'(X_n) \subset X_n$ and $\pi'^m_n(x_m) = x_m$ for $x_m \in X_n \subset X_m$. By FPP for X_n, there exists a point $x_m^* \in X_n$ such that $f'(x_m^*) = x_m^*$. Hence we have $f(x_m^*) = \pi_n^m(x_m^*)$.

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