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ON COMPACTIFICATIONS OF SOME SUBSETS OF EUCLIDEAN SPACES

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Let S_n be the unit sphere, i. e. the sphere with centre 0 and radius 1 in the $(n+1)$-dimensional Euclidean space E^{n+1}. I say that a set $X \subset S_n$ is *densely connected* in S_n if the set $R \cap X$ is connected for every connected open subset R of S_n. Obviously, each set in S_n (where $n = 0, 1, \ldots$) that is non-degenerate (i. e. containing at least two distinct points) and densely connected in S_n is dense in S_n, but not inversely.

THEOREM. *If a non-degenerate set $X \subset S_n$ is densely connected in S_n $(n = 0, 1, \ldots)$, Y is a compact metric space and $h \colon X \to Y$ is a homeomorphism such that* $\dim[Y - h(X)] \leqslant 0$, *then* $n \leqslant \dim Y$.

Proof. Let $p, q \in X$ and $p \neq q$. Since the sphere S_n is topologically homogeneous, we can assume that p, q are the poles p_N (north) and p_S (south) of S_n, respectively. The set $Y - h(X)$ being empty or 0-dimensional, there exists (see [3], p. 164) an open neighbourhood G of $h(p)$ in Y such that

$$\mathrm{Fr}(G) \subset h(X) \tag{1}$$

and $h(q) \in Y - \overline{G}$ (1). Then neither $h(p)$ nor $h(q)$ belongs to $\mathrm{Fr}(G)$ and so there are such sufficiently small open neighbourhoods P and Q of p and q in S_n, respectively, that

$$\mathrm{Fr}(G) \subset Y - [\overline{h(P \cap X)} \cup \overline{h(Q \cap X)}]. \tag{2}$$

The theorem being evidently true for $n = 0$, let us assume that $n > 0$ and denote by r the projection of $S_n - \{p_N, p_S\}$ onto the equator S_{n-1} of S_n along the meridians of S_n. Since r is a continuous mapping

(1) $\overline{G}$ and $\mathrm{Fr}(G)$ denote the closure and the boundary of G in Y, respectively. The notation from [3] and [4] is used throughout in this proof.

and $S_n-(P\cup Q)$ is a compact set, there exists a number $\varepsilon>0$ such that

$$(3)\qquad \begin{gathered} x,x'\in S_n-(P\cup Q) \text{ and } |x-x'|<\varepsilon \text{ imply} \\ |r(x)-r(x')|<\sqrt{2(n+1)/n}. \end{gathered}$$

But $\operatorname{Fr}(G)$ is a compact set and h^{-1} is a continuous mapping. It follows from (1) and (2) that there exist open sets $H_1,\ldots,H_j$ in Y such that

$$(4)\qquad \operatorname{Fr}(G)\subset H_1\cup\ldots\cup H_j\subset Y-[\overline{h(P\cap X)}\cup\overline{h(Q\cap X)}],$$

$$(5)\qquad \delta[h^{-1}(H_i)]<\varepsilon \quad \text{for} \quad i=1,\ldots,j.$$

Now let us suppose on the contrary that $\dim Y\leqslant n-1$. Then (4) and the compactness of $\operatorname{Fr}(G)$ imply the existence of open sets $I_1,\ldots,I_k$ in Y, satisfying

$$(6)\qquad \operatorname{Fr}(G)\subset I_1\cup\ldots\cup I_k\subset \bar{I}_1\cup\ldots\cup\bar{I}_k\subset H_1\cup\ldots\cup H_j$$

and $\dim\operatorname{Fr}(I_i)\leqslant n-2$ for $i=1,\ldots,k$. Hence the union $\operatorname{Fr}(I_1)\cup\ldots\cup\operatorname{Fr}(I_k)$ is an at most $(n-2)$-dimensional set (see [3], p. 176) contained in the open set $H_1\cup\ldots\cup H_j$, according to (6). Therefore there exist (see [3], p. 182 and 184) open sets $J_1,\ldots,J_j$ in Y such that

$$(7)\qquad \operatorname{Fr}(I_1)\cup\ldots\cup\operatorname{Fr}(I_k)\subset J_1\cup\ldots\cup J_j,$$

$$(8)\qquad J_i\subset H_i \quad \text{for} \quad i=1,\ldots,j,$$

$$(9)\qquad J_{i_0}\cap\ldots\cap J_{i_{n-1}}=0 \quad \text{for} \quad 1\leqslant i_0<\ldots<i_{n-1}\leqslant j.$$

Applying $\dim[Y-h(X)]\leqslant 0$, let us take a finite cover $K_1,\ldots,K_l$ of the compact set $\operatorname{Fr}(I_1)\cup\ldots\cup\operatorname{Fr}(I_k)$, where all the sets K_i are open in Y, have the boundaries contained in $h(X)$ and the diameters less than the Lebesgue number of the cover $J_1,\ldots,J_j$, according to (7). Then we have

$$(10)\qquad \operatorname{Fr}(I_1)\cup\ldots\cup\operatorname{Fr}(I_k)\subset K_1\cup\ldots\cup K_l,$$

$\operatorname{Fr}(K_i)\subset h(X)$ and for every $i=1,\ldots,l$ a number $\varphi(i)=1,\ldots,j$ exists such that

$$(11)\qquad \bar{K}_i\subset J_{\varphi(i)} \quad \text{for} \quad i=1,\ldots,l.$$

Putting

$$C_i=\bigcup_{\substack{\varphi(m)=i\\ m=1,\ldots,l}}\operatorname{Fr}(K_m)$$

for $i=1,\ldots,j$, we thus get

$$(12)\qquad C_1\cup\ldots\cup C_j=\operatorname{Fr}(K_1)\cup\ldots\cup\operatorname{Fr}(K_l)\subset h(X),$$

$$(13)\qquad C_i\subset J_i\subset H_i\subset Y-[h(P\cap X)\cup h(Q\cap X)] \quad \text{for} \quad i=1,\ldots,j,$$

by (4), (8) and (11). Moreover, each set C_i is compact and (12) implies that the union

$$(14)\qquad B=h^{-1}(C_1)\cup\ldots\cup h^{-1}(C_j)$$

is a compact subset of $X\subset S_n$. According to (13), the sets $G_i=B\cap h^{-1}(J_i)$ $(i=1,\ldots,j)$ are open in B and their union is B. It follows from (8) that $G_i\subset h^{-1}(H_i)$, which gives $\delta(G_i)<\varepsilon$, by (5). We also have

$$G_{i_0}\cap\ldots\cap G_{i_{n-1}}\subset h^{-1}(J_{i_0}\cap\ldots\cap J_{i_{n-1}})=0$$

for $1\leqslant i_0<\ldots<i_{n-1}\leqslant j$, according to (9), whence the inequality $d_{n-1}(B)<\varepsilon$ follows (see [4], p. 60). Thus there is such a continuous mapping f of B that

$$(15)\qquad \dim f(B)\leqslant n-2,$$

$$(16)\qquad \delta[f^{-1}(y)]<\varepsilon \quad \text{for} \quad y\in f(B)$$

(see [4], p. 64). Since $C_i\cap[h(P\cap X)\cup h(Q\cap X)]=0$, by (13), we obtain $h^{-1}(C_i)\cap(P\cup Q)\cap X=0$ for $i=1,\ldots,j$ and so $B\subset S_n-(P\cup Q)$, by (14) and the inclusion $B\subset X$. Therefore the projection r is determined on B. If we had $r|B$ *non* $\simeq 1$, then, by (3) and (16), there would exist an essential mapping of $f(B)$ onto S_{n-1} (see [4], p. 284), contrary to (15) (see [2], p. 88). Thus we have the homotopy $r|B\simeq 1$, which means that the set B does not separate the points p and q in S_n (see [4], p. 187 and 345). Let R be the component of S_n-B, containing p and q. Hence R is a connected open set in S_n (see [4], p. 163) and so the set

$$(17)\qquad R\cap X\subset S_n-B$$

is connected, X being densely connected.

Since the neighbourhood G of $h(p)$ does not contain the point $h(q)$, the connected set $h(R\cap X)$ containing these points must intersect $\operatorname{Fr}(G)$ (see [4], p. 80). But $h(p)\in h(P)$, whence the point $h(p)$ lies outside any of the sets $I_1,\ldots,I_k$, according to (4) and (6), as well as of the sets $K_1,\ldots,K_l$, according to (4), (8) and (11). Thus, for the same reason as above, (6) implies that $h(R\cap X)$ must intersect at least one of the sets $\operatorname{Fr}(I_1),\ldots,\operatorname{Fr}(I_k)$ and (10) implies that $h(R\cap X)$ must intersect at least one of the sets $\operatorname{Fr}(K_1),\ldots,\operatorname{Fr}(K_l)$. It follows from (12) that $h(R\cap X)$ intersects some C_{i_0} $(i_0=1,\ldots,j)$. We infer, by virtue of (14), that

$$0\neq R\cap X\cap h^{-1}(C_{i_0})\subset R\cap X\cap B,$$

which contradicts (17).

COROLLARY. *If $\tilde{X}$ is a metrizable compactification of a set $X \subset E^n$ $(n = 1, 2, \ldots)$ such that*

$$\dim(\tilde{X}-X) \leqslant 0 \quad \textit{and} \quad \dim(E^n-X) \leqslant n-2,$$

then $n \leqslant \dim\tilde{X} \leqslant \dim X+1$. *Hence if* $X \subset E^n$ *and*

$$\dim X \leqslant n-2 \geqslant \dim(E^n-X),$$

then X has no metrizable compactification $\tilde{X}$ satisfying the inequality $\dim(\tilde{X}-X) \leqslant 0$, *i. e. X is not peripherically compact* (see [1], p. 58).

For we evidently have $\dim\tilde{X} \leqslant \dim X+1$ (see [3], p. 175) and the inequality $\dim(E^n-X) \leqslant n-2$ implies, by virtue of the Mazurkiewicz Theorem (see [4], p. 343), that the set $R-(E^n-X) = R \cap X$ is a semi-continuum for every connected open subset R of E^n. Thus the set X is densely connected in the compactification S_n of E^n and the inequality $n \leqslant \dim\tilde{X}$ follows, according to the theorem.

EXAMPLE. *There exists for every $n = 1, 2, \ldots$ a separable metric space A_n such that* 1° $\dim A_n = n$, 2° *A_n is topologically complete and peripherically compact and* 3° *if C is a metrizable compactification of A_n, satisfying* $\dim(C-A_n) \leqslant 0$, *then* $n+1 \leqslant \dim C$.

For denote by M_n^m the set of points in E^n at most m of whose coordinates are rational and put $A_n = M_{n+1}^n$ for $n = 1, 2, \ldots$ Hence 1° follows (see [2], p. 29). Obviously, $E^{n+1}-A_n$ is an F_σ-set and the completeness of A_n follows by the Aleksandrov Theorem (see [3], p. 316). Each open cell in E^{n+1}, bounded by hyperplanes with the equations of type $x_i = a$ $(1 \leqslant i \leqslant n+1)$, where a is an irrational number, has a boundary contained in A_n; thus we get 2°. But $\dim(E^{n+1}-A_n) = \dim L_{n+1}^{n+1} = 0 \leqslant n-1$ (²) and so 3° follows, according to the corollary.

Remarks. The above example answers in the negative a question of Aleksandrov (see [1], p. 59) for the case of metrizable compactifications. It is answered in the general case by Sklyarenko (see [5], p. 41), who indicates the set $I^2 \cap L_2^1$ (²). The possibility of such an answer had been suggested to me by R. Engelking before Sklyarenko's paper [5] appeared.

It seems very probable that applying a theorem announced by Sklyarenko (see [5], p. 40, Theorem 3), one could generalize our theorem to the case where the set X is peripherically compact and densely connected in E^n, the space Y is compact, completely regular and not necessarily metrizable, and the set $Y-h(X)$ is punctiform (i. e. it contains only degenerate continua) instead of being 0-dimensional.

(²) In the notation from [2] (see p. 162-163).

Finally, the following question arises:

P 350. Is it true that if a non-degenerate set $X \subset S_n$ is densely connected in S_n $(n = 0, 1, \ldots)$, Y is a compact metric space and $h\colon X \to Y$ is a homeomorphism such that $\dim[Y-h(X)] \leqslant 0$, then every closed separator C of $\overline{h(X)}$ satisfies $n-1 \leqslant \dim C$, that is the inequality $n \leqslant \mathrm{dc}\,\overline{h(X)}$ holds (see [4], p. 105)?

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