2° It follows from Grothendieck's result quoted above that there is no linear continuous mapping of m onto c_0. Moreover, every linear mapping of m into c_0 is compact.

P 351. Let $\aleph_0 \leqslant \aleph_\tau \leqslant \overline{\overline{S}}$. Does there exist a linear mapping of $m(S)$ onto its subspace $m(S|\aleph_\tau)$? (We recall that $m(S|\aleph_\tau)$ denotes the space of all functions $f(\cdot)$ in $m(S)$ such that $\overline{\overline{\{s \in S: |f(s)| > \varepsilon\}}} < \aleph_\tau$, for any $\varepsilon > 0$.)

We note that there is a mapping T of the space $m(S)$ into $m(S|\aleph_1)$ such that

$$\overline{\overline{\bigcup_{f \in m(S)} \{s \in S: (Tf)(s) > \tfrac{1}{2}\}}} > \aleph_0.$$

This may easily be deduced from the following result comunicated to us by prof. C. Ryll-Nardzewski:

Let $\overline{\overline{S}} = \aleph_1$. Then, under the assumption of the continuum hypothesis there exists a family $(\nu_a)_{a \in \mathfrak{a}}$ ($\overline{\overline{\mathfrak{a}}} = \aleph_1$) of finite-additive set functions with, bounded variation defined on the field of all subsets of the set S such that $\overline{\overline{\{a \in \mathfrak{a}: \nu_a(A) \neq 0\}}} \leqslant \aleph_0$ for any $A \subset S$.

3° It is interesting to compare our result with the following generalization — due to Grünbaum [4] — of the theorem of Sobczyk [8] concerning projections onto c_0.

Let $\overline{\overline{S}} \geqslant \aleph_\tau \geqslant \aleph_0$, let the space $m(S|\aleph_\tau)$ be isomorphically embedded into a B-space X and let the quotient space $X/m(S|\aleph_\tau)$ contain a dense set of cardinality $\leqslant \aleph_\tau$. Then there exists a projection of X onto $m(S|\aleph_\tau)$.

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ON BOUNDED SETS IN F-SPACES

BY

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A subset Z of a metric linear space X is called *bounded* if $\limsup_{t \to 0} \sup_{x \in Z} \varrho(0, tx) = 0$. If every bounded subset of X is compact (i. e. its closure is compact), then X is called a *metrisable Montel space*.

J. Dieudonné [2] proved that every locally convex metrisable Montel space is separable. Using the continuum hypothesis Dieudonné showed in his paper [3] that there is a non-complete locally convex linear metric space which is not separable but every bounded set in this space is separable.

In this paper we prove that Dieudonné's theorem on the separability of Montel spaces is valid in the case of arbitrary metrisable Montel spaces; we also give an example of a non-separable complete linear metric space in which every bounded set is separable. The construction of this example is also based on continuum hypothesis. The problem whether there exist B_0-spaces (i. e., according to [5], locally convex complete metric linear spaces) having this property is still open.

1. Theorem. *Every metrisable Montel space is separable.*

Proof. Let X be non-separable linear metric space and Z an arbitrary uncountable set in X, such that

(1) $$\varrho(z, z') \geqslant \delta > 0 \quad \text{for} \quad z, z' \in Z, \quad z \neq z'.$$

Let us define the sequence of quasi-norms (Hyers [4], see also Bourgin [1] and Rolewicz [6]) by

$$[x]_n = \inf\{t > 0: \varrho(0, tx) \geqslant 1/n\}, \quad x \in X.$$

It is obvious that a set A, $A \subset X$, is bounded if and only if

(2) $$\sup_{x \in A} [x]_n < \infty \quad (n = 1, 2, \ldots).$$

Since $\overline{\overline{Z}} > \aleph_0$, we can find such $M_1 > 0$ that $Z_1 = Z \cap \{x \in X: [x]_1 < M_1\}$ is uncountable. Further we can define by induction a sequence of sets $Z \supset Z_1 \supset Z_2 \dots$ and a sequence (M_n) of positive numbers in such a way that

$$\overline{\overline{Z}}_n > \aleph_0, \quad \sup\{[x]_i : x \in Z_n,\ i = 1, 2, \dots, n\} < M_n \quad (n = 1, 2, \dots).$$

Let us choose a sequence (z_n) with $z_n \in Z_n$, $z_i \neq z_k$ for $i \neq k$. We have $\sup [z_n]_k \leqslant \sup\{M_k, [z_1]_k, \dots, [z_k]_k\} < \infty$ $(k = 1, 2, \dots)$; therefore, according to (2), the sequence (z_n) is bounded. We conclude from (1) that the set $\{z_n\}$ is not compact. Hence X is not a Montel space, q. e. d.

2. Example of a non-separable complete metric linear space in which every bounded set is separable. We shall need the following

LEMMA. *Under the assumption of the continuum hypothesis there exists such an uncountable family $\{f_\lambda\}$ (λ runs over an abstract set Λ of indices) of real non-negative concave continuous functions vanishing at $t = 0$ that no uncountable subfamily $\{f_\varkappa\}$, $\varkappa \in K \subset \Lambda$ is equicontinuous at the point* 0.

Proof. Let E be the space given in Dieudonne's [3] example, let $(|\cdot|_n)$ be the system of homogeneous pseudonorms determining the topology of E and let $\varrho(x, y) = \sum_{n=1}^{\infty} |x-y|_n / [2^n(1+|x-y|_n)]$. Obviously for every $x \in E$ the function $g_x(t) = \varrho(0, tx)$ is continuous, concave and vanishing at $t = 0$.

Since E is not separable, there exist $\delta > 0$ and an uncountable set $\{e_\lambda\}$, $\lambda \in \Lambda$, such that $\varrho(e_\lambda, e_{\lambda'}) > \delta$ for $\lambda \neq \lambda'$. Put

$$f_\lambda(t) = \varrho(0, te).$$

The family $\{f_\lambda\}$, $\lambda \in \Lambda$, has the required properties.

In fact, if a subfamily $\{f_\varkappa\}$, $\varkappa \in K \subset \Lambda$, $\overline{\overline{K}} > \aleph_0$ were equicontinuous, then the set $Z = \{e_\varkappa : \varkappa \in K\}$ would have to be bounded, which would contradict the properties of the space E.

Example. Let $\{f_\lambda\}$ be the family of functions mentioned in the lemma. Let X be the set of all real functions $x = x(\lambda)$ such that

$$\overline{\overline{\{\lambda \in \Lambda : x(\lambda) \neq 0\}}} < \aleph_0, \quad \sum_{\lambda \in \Lambda} f_\lambda(|x(\lambda)|) < \infty,$$

considered as a linear metric space with the metric

$$\varrho(x, y) = \sum_{\lambda \in \Lambda} f(|x(\lambda) - y(\lambda)|).$$

It is immediately seen that X is a complete and non-separable space. We shall prove that every bounded set in X is separable. Let Z be a non-separable set in X. Let

$$\Lambda_z = \bigcup_{x \in Z} \{\lambda \in \Lambda : x(\lambda) \neq 0\}.$$

Since Z is non-separable, $\overline{\overline{\Lambda}}_z > \aleph_0$ and for some $\varepsilon > 0$ the set

$$\Lambda_Z^\varepsilon = \bigcup_{x \in Z} \{\lambda \in \Lambda : |x(\lambda)| > \varepsilon\}$$

is uncountable. Hence the functions $f_\lambda(t)$ $(\lambda \in \Lambda_Z^\varepsilon)$ are not equicontinuous at zero, therefore there exists a $\delta > 0$ such that

$$0 < \delta < \limsup_{t \to 0} \sup_{\lambda \in \Lambda_Z^\varepsilon} f_\lambda(t\varepsilon) \leqslant \limsup_{t \to 0} \sup_{x \in Z} \sup_{\lambda \in \Lambda_Z^\varepsilon} f_\lambda(t|x(\lambda)|) < \limsup_{t \to 0} \sup_{x \in Z} \varrho(0, tx).$$

That means that the set Z is unbounded, q. e. d.

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