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Schoenflies problems

by

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Dedicated to the Fundamenta Mathematicae on the occasion of the publication of its 50th volume, with grateful appreciation of what this journal of mathematics has meant to the world of mathematics during the last fifty years.

§ 1. Introduction. The theorem that the union of a Jordan curve and its interior in a 2-plane is a closed 2-cell is commonly called a Schoenflies theorem. The problems which arise in attempting to generalize this theorem in euclidean spaces of higher dimension are called *Schoenflies problems*.

The generalization which suggests itself first is false. Let M be a topological $(n-1)$-sphere in an euclidean n-space E with $n>1$. Let $\mathring{J}M$ be the open interior of M and JM the closure of $\mathring{J}M$. It is not always true that JM is a closed n-cell for $n>2$. See Ref. [0].

A major advance in formulating a valid Schoenflies extension theorem when $n>2$ was made by Barry Mazur in Ref. [2]. Mazur concerned himself with a topological $(n-1)$-sphere on an euclidean n-sphere. We shall present a theorem which is essentially that of Mazur, but in which Mazur's n-sphere is replaced by the euclidean n-space E. This use of an euclidean n-space E in place of an eucliden n-sphere accords with subsequent developments which we shall present.

Mazur's theorem. Mazur made two assumptions, the second of which, as we shall see, is unnecessary. Let S be an $(n-1)$-sphere in E with center at the origin and radius 1. In our formulation of Mazur's theorem these hypotheses are as follows.

I. Let φ be a homeomorphism of an open neighborhood N of S into E under which points interior (exterior) to S go into points interior (exterior) to the $(n-1)$-manifold $\varphi(S)=M$.

II. Suppose that there exists in N a neighborhood of a point P of S in the form of a star of euclidean cells incident with P such that on each cell of the star φ is linear.

Conclusion (α). *Then $\varphi|S$ admits an extension as a homeomorphism λ_φ of JS onto JM such that λ_φ agrees with φ at all points of some neighborhood $W \subset N$ of S interior to S.*

It is trivial that λ_φ can be extended as a homeomorphism to agree with φ at points of N exterior to S. In general W cannot include all points of N interior to S.

The extension problem (φ, N). *Topological case.* Given φ and N as in Hypothesis I, and omitting Hypothesis II, the problem of finding λ_φ so as to satisfy (α) will be called the *extension problem* (φ, N) in the *topological case.*

The topological case is to be distinguished from the *differentiable case* in which φ is a C^m-diffeomorphism of S into E, $m > 0$, and the *analytic case* in which φ is an analytic diffeomorphism of S into E.

In Ref. [3] we have established the following theorem, reducing Mazur's hypotheses to I.

THEOREM 1.1 (i). *Given the topological extension problem* (φ, N), *there exists a homeomorphic mapping f of E into E such that $f\varphi$ reduces to the identity on a neighborhood of some point of S.*

(ii). *A necessary and sufficient condition that there exist a solution of the extension problem (φ, N) is that there exist a solution of problem $(f\varphi, N)$.*

Since the homeomorphism $f\varphi$ satisfies Mazur's Hypothesis II, Mazur's theorem and the above theorem have the following corollary.

COROLLARY 1.1. *Each Schoenflies extension problem (φ, N) in the topological case admits a solution λ_φ satisfying Conclusion* (α).

Theorem 1.1 was first discovered by the author in a form appropriate to the differentiable case. (Cf. Lemma 3.1 of Ref. [4].) In Ref. [4] the first solution of the extension problem in the differentiable case was announced. This solution was a diffeomorphism with one differential singularity; it was a homeomorphism. The complete exposition in the differentiable case was given in Ref. [5]. With the discovery of Theorem 1.1 and its analogue in the differentiable case it was recognized that Part II of Ref. [5], with appropriate verbal changes, gave a solution of the extension problem (φ, N) *both* in the topological and the differentiable case. A simplified exposition with explicit formulas that made this clear has been presented by Huebsch and Morse in Ref. [6].

Three proofs of Corollary 1.1 appeared at approximately the same time.

Morton Brown in Ref. [7] gave a direct solution of the extension problem (φ, N) in the topological case. Mazur's result and Theorem 1.1, as combined by the author in Ref. [3], gave another proof. These two proofs apply only to the topological case. The methods of Part II of Ref. [5] (later simplified in Ref. [6]) afford a proof both in the topological and the differentiable case.

The explicit solution of the problem (φ, N), in Ref. [6], is the basis of what follows concerning *cell problems.* Cell problems are distinguished in § 3 from extension problems. The results of Ref. [6] lead to the theorem that the interior of M (in the differentiable case) is an open analytic n-ball, that is the image of an open euclidean n-ball under a real analytic diffeomorphism. The methods of Ref. [6] also serve as a model for a theory of families of Schoenflies extensions. See Ref. [10] and Ref. [11].

Γ-problems and simple extension problems. The data in the problem of obtaining continuous families of Schoenflies extensions includes a topological space Γ on which a parametric point p rests. Such a problem will be called a Γ-problem to distinguish it from an extension problem (φ, N). We term the latter problem a *simple* extension problem. Both in Γ-problems and in simple extension problems one distinguishes between the *topological*, the *differentiable* and the *analytic case*, first in the data, and, subsequently, in the conclusions. See §§ 4 and 5.

Extension problems and cell problems.

(i) Theorem 1.1 and its corollary concern an extension problem.

(ii) Given a C^1-diffeomorphism f of S onto $f(S) = M$, the corresponding *open cell problem* is to find an analytic diffeomorphism F of an open n-ball onto $\mathring{J}M$. It is solved by Huebsch and Morse in Ref. [12]. In general F is in no sense an extension of f.

(iii) In § 3 we define a *closed cell problem* (with differentiability index $m = \infty$). We show how this closed cell problem is related to the general extension problem. In this connection we introduce "differentiable isotopies". These isotopies differ from those introduced by Milnor in Ref. [13]. For $n = 2$, or 3, a solution of this closed cell problem exists, but the existence of a solution in the general case is an open question.

The reader familiar with the theory of extensions of continuous mappings, as developed by Borsuk (Ref. [17]), or Hurewicz, will note a formal similarity between certain of the results presented here and earlier extension theorems. In particular, there exist in the present theory strong relations between differentiable isotopies and the existence of differentiable extensions (§ 2). However the similarity is no more than formal. In fact, the Borsuk theory aims at continuous extensions over sets of great generality, whereas the theory of Schoenflies extensions starts with domains of extreme simplicity, such as S, and finds its principal difficulty in showing that extensions exist as homeomorphisms or diffeomorphisms.

In the next section we shall present some of the principal theorems concerning simple Schoenflies problems in the differentiable case.

§ 2. The extension problem (φ, N). Differentiable case. We begin with notation and conventions.

Notation. As previously E is an euclidean n-space, S an $(n-1)$-sphere in E with center at the origin 0, and M a topological $(n-1)$-sphere in E. If 0 is in $\mathring{J}M$ we set

$$JM-0 = J_0M .$$

Conventions. A real analytic or C^m-mapping F into E of an open subset X of E, is defined in the usual way, as is an analytic or C^m-diffeomorphism of X into E. We suppose that $m > 0$. We shall extend these definitions to any subset X of E such that $X \subset \mathrm{Cl}\,\mathring{X}$. For such a set X we say that F is an *analytic* or C^m-*mapping* into E, or an *analytic* or C^m-*diffeomorphism* of X into E, if F admits an extension F^e over an open subset $Y \supset X$ of E with the respective properties of F, when Y replaces X.

A_0 *or* C_0^m-*diffeomorphisms*, $m > 0$. Let X be a subset of E such that $\mathring{X}$ contains the origin and $X \subset \mathrm{Cl}\,\mathring{X}$. Among such sets are JS and $\mathring{J}S$. Set $X-0 = X_0$. We understand that an A_0 *or* C_0^m-*mapping* F of X into E is a continuous mapping of X into E whose restriction to X_0 is an analytic or C^m-mapping, respectively, of X_0 into E. Such a mapping will be called an A_0 or C_0^m-diffeomorphism of X into E, if it is a homeomorphism of X into E, and if $F|X_0$ is an analytic or C^m-diffeomorphism, respectively, of X_0 into E.

The following theorem was established by Morse in Ref. [5]. A C^m-diffeomorphism of S into E, $m > 0$, is understood in the usual sense.

THEOREM 2.1. *A C^m-diffeomorphism f of S into E, $m > 0$, can be extended by a C_0^m-diffeomorphism F of JS onto $Jf(S)$.*

Extensions over JS without differential singularity. The question arises, when can the extension F of f in Theorem 2.1 be chosen so as to be a C^m-diffeomorphism of JS into E.

In Theorem 2.2 and its corollaries we shall suppose that all diffeomorphisms are of class C^∞. This is to avoid the detail which is necessarily associated with a treatment under weaker hypotheses.

A definition is needed.

Differentiable isotopies. Let R be the axis of reals. Two C^∞-diffeomorphisms $f, g\colon S \to E$ will be said to be *differentiably isotopic* if there exists a C^∞-diffeomorphism

$$h\colon S\times R \to E\times R; \qquad (x, t) \to \big(h(x, t), t\big)$$

such that $h(x, t) = f(x)$ for $t \leqslant 0$, and $h(x, t) = g(x)$ for $t \geqslant 1$. We term h a *differentiable isotopy of f into g* or simply a differentiable isotopy of f. A differentiable isotopy of two C^∞-diffeomorphisms $F, G\colon JS \to E$ is similarly defined.

This definition should be compared with that of Milnor, Ref. [13]. Under Milnor's definition a differentiable isotopy of a diffeomorphism f_0 of S onto S is a deformation of f_0 through a family f_t, $-\infty < t < \infty$, of diffeomorphisms each in S^S, but under our definition is through a family of diffeomorphisms each in E^S.

Theorem 2.2 is the fundamental theorem of this section. The necessity of the condition is relatively easy to establish. The sufficiency of the condition may be established by means of methods introduced in §§ 1-6 of Ref. [14].

THEOREM 2.2. *A necessary and sufficient condition that a C^∞-diffeomorphism f_0 of S into E be extendable as an orientation-preserving C^∞-diffeomorphism of JS into E, is that there exist a differentiable isotopy of $f_0(f_t \in E^S)$ into the identity map of S onto S.*

According to Theorem 5 of Milnor, *loc. cit.*, there exists a C^∞-diffeomorphism f_0 of degree 1 of a 6-sphere S onto S such that f_0 is not differentiably isotopic to the identity in Milnor's sense, that is with $f_t \in S^S$. This result of Milnor has an extension in the form of a corollary of Theorem 2.2. See Ref. [21].

COROLLARY 2.1. *When $n = 7$ there exists a C^∞-diffeomorphism f_0 of a 6-sphere S onto S of degree $+1$, which is not differentiably isotopic to the identity with $f_t \in E^S$.*

In establishing this corollary use is made of Milnor's manifold M_3^7 as well as of Theorem 2.2.

The following theorem gives an "extension" of a differentiable isotopy.

THEOREM 2.3. *Corresponding to a differentiable isotopy f_t, $-\infty < t < \infty$, of a C^∞-diffeomorphism f_0 into the identity f_1, with $f_t \in E^S$, there exist C^∞-diffeomorphisms F_t, $-\infty < t < \infty$, of JS into E extending the respective mappings f_t, and defining a differentiable isotopy of F_0 into the identity map of JS onto JS.*

The sufficiency of the condition in Theorem 2.2 is a corollary of Theorem 2.3.

Theorem 2.3 would be false if one added that F_0 could be taken arbitrarily among the C^∞-diffeomorphisms of JS into E which extend f_0. An example will show this.

We distinguish three differentiable extension problems.

(a) A C_0^m-extension problem, given $f \in E^S$.

(b) A C^m-extension problem, given $f \in E^S$.

(c) A C^m-extension problem, given $f \in S^S$.

Problem (a) is to find a C_0^m-diffeomorphism F of JS into E extending the given C^m-diffeomorphism of f into E. It is solvable. Problems (b) and (c) are similar, except that F is required to be a C^m-diffeomorphism of JS into E and in Problem (c), $f(S) = S$. As Milnor has shown, Problem (c) may fail to have a solution for certain values of n. Since a Problem (c) is a Problem (b), a Problem (b) may likewise fail to have a solution.

Schoenflies integers. We shall call an integer $n > 1$, a Schoenflies integer, if every C^∞-extension Problem (b) has a solution when $E = E_n$.

Milnor integers. We shall call an integer $n > 1$, a Milnor integer, if when $S = S_{n-1}$ every C^∞-extension Problem (c) has a solution.

It is obvious that each Schoenflies integer is a Milnor integer. We are ignorant as to the converse. According to Munkres in 6.4 of Ref. [15], $n = 2$ and $n = 3$ are Milnor integers (our notation). See also Smale, Ref. [19]. The author has proved that $n = 2$ and $n = 3$ are also Schoenflies integers, confirming and extending the above results of Munkres and Smale.

The integer 7 is not a Milnor integer, as one sees with the aid of Milnor's manifold M_8^7. Hence 7 is not a Schoenflies integer.

§ 3. Schoenflies cell problems. The image in E of an open n-ball in E under an analytic diffeomorphism will be called an *open analytic n-ball*. The fundamental theorem of this section is the following.

THEOREM 3.1. *If M is the image of S in E under a C^1-diffeomorphism f, the interior of M is an open analytic n-ball.*

The open-cell problem. Given M the problem of finding the analytic diffeomorphism F of an open n-ball onto $\mathring{J}M$ will be called the Schoenflies *open cell problem*. This problem is not an extension problem. The manifold M may be given as the C^1-diffeomorphic image in E of S under diffeomorphisms other than f, but the open-cell problem remains the same. The diffeomorphism F may not extend any diffeomorphism of S into E which represents M. The proof of Theorem 3.1 is given in Ref. [12], by showing that there exists a C^1-diffeomorphism G of $\mathring{J}S$ onto $\mathring{J}M$, and making use of a general theorem in Ref. [12] that any C^1-diffeomorphism of an open subset X of E onto an open subset Y of E can be approximated by a real analytic diffeomorphism of X onto Y.

The purely topological version of Theorem 3.1 involves topological $(n-1)$-spheres Σ in E, termed *elementary*. Cf. Ref. [6], § 4. By definition, Σ is elementary if it is the image of S in E under a homeomorphism F of S onto Σ which is extendable as a homeomorphism into E over an open neighborhood of S. Not every topological $(n-1)$-sphere in E is elementary. See Ref. [1]. If Σ is a topological $(n-1)$-sphere which is not elementary the interior of Σ is not always an open topological n-ball. See Ref. [0]. However if Σ is elementary, not only is $\mathring{J}\Sigma$ an open topological n-ball but $J\Sigma$ is a closed topological n-ball. This is a consequence of Corollary 1.1.

To Theorem 3.1 we add the following conjecture.

CONJECTURE. *The interior of an elementary topological $(n-1)$-sphere Σ in E is an open analytic n-ball.*

This conjecture is correct when $n = 2$ or 3. It follows from the conformal mapping theorem when $n = 2$. When $n = 3$ the proof runs as follows. By Corollary 1.1, $\mathring{J}S$ is the homeomorph of $\mathring{J}\Sigma$. Both $\mathring{J}S$ and $\mathring{J}\Sigma$ are differentiable 3-manifolds. Hence they are C^1-diffeomorphic. See Theorem 6.3, Munkres, Ref. [15]. See also Thom, Ref. [20]. By a general theorem of Huebsch and Morse in Ref. [12], $\mathring{J}S$ and $\mathring{J}\Sigma$ are then analytically diffeomorphic.

If the Hauptvermutung (cf. Munkres, loc. cit., p. 544) is here valid our conjecture for general n is provable by a similar argument. However a more likely mode of proof would be by a limiting process, applying Theorem 3.1 to C^1-topological $(n-1)$-spheres in $\mathring{J}\Sigma$ which approximate Σ.

The closed cell problem. Let M be the image of S in E under a C^∞-diffeomorphism f of S into E. A C^∞-diffeomorphism of S onto M which is extendable as an orientation-preserving C^∞-diffeomorphism of JS onto JM will be called a *preferred representation* of M. Given a C^∞-representation of M, the problem of finding a preferred representation of M will be called the *closed cell problem* (of index $m = \infty$).

Let an orientation-preserving C^∞-diffeomorphism of JS into E be termed a $(+)$ C^∞-diffeomorphism of JS into E.

The group Λ_{n-1}. Given $n > 1$ let Λ_{n-1} be the group of C^∞-diffeomorphisms f_0 of S onto S which are preferred representations of S, or equivalently (by Theorem 2.2) the group of all C^∞-diffeomorphisms f_0 of S onto S which are differentiably isotopic ($f_t \in E^S$) to the identity. We understand that the product of two diffeomorphisms in Λ_{n-1} is their composition.

The closed cell problem differs radically from the C^∞-extension problem. Let f be a C^∞-diffeomorphism of S onto M. When $M = S$ the corresponding closed cell problem always has a trivial solution obtained by representing M by the identity map g of S onto $M = S$, and extending g over JS by the identity.

We shall prove a lemma giving a relation between three Schoenflies problems.

(α) *The closed cell problem (of index $m = \infty$).*

(β) *The C^∞-extension problem, given $f \in E^S$.*

(γ) *The C^∞-extension problem, given $f \in S^S$.*

LEMMA 3.1. *Let M be the image in $E = E_n$ of $S = S_{n-1}$ under a C^∞-diffeomorphism f. If M admits a preferred representation g, then f is a preferred representation of M if and only if $g^{-1}f$ is in Λ_{n-1}.*

To verify the lemma note that $f = g(g^{-1}f)$. By hypotheses there exists a $(+)$ C^∞-diffeomorphism G of JS onto JM extending g.

If $g^{-1}f$ is in A_{n-1}, there exists a $(+)$ C^∞-diffeomorphism K of JS onto JS extending $g^{-1}f$. The mapping

$$F = GK \tag{3.1}$$

is a $(+)$ C^∞-diffeomorphism of JS onto JM extending f.

Conversely, the existence of a $(+)$ C^∞-diffeomorphism F of JS onto JM extending f, and the existence of G as above, implies the existence of the $(+)$ C^∞-diffeomorphism

$$K = G^{-1}F$$

of JS onto JS extending $g^{-1}f$, so that $g^{-1}f$ is in A_{n-1}.

We give below a proof that the closed-cell problem has a solution when $n = 2$. The author has had for several years a proof that the closed cell problem for $n = 3$ has a solution. A solution for this case will presently be published. We know of no evidence that the closed cell problem for arbitrary n ever fails to have a solution. It differs in this respect from the C^m-extension problem.

We shall further clarify the relations between the problems (α), (β), (γ) by introducing the following sets of integers.

The set τ. Let τ denote the set of all integers $n > 1$ such that the closed cell problem in $E = E_n$ always has a solution.

The set σ_1. Let σ_1 be the set of Śchoenflies integers.

The set σ_2. Let σ_2 be the set of Milnor integers.

The definitions of τ, σ_1, and σ_2, taken with Lemma 3.1, imply the following.

LEMMA 3.2. *Between the sets τ, σ_1, and σ_2 there exist the relations*

$$\sigma_1 \subset \sigma_2, \qquad \tau \cap \sigma_2 = \sigma_1. \tag{3.2}$$

It is conceivable that τ includes all integers $n > 1$, and in such a case $\sigma_1 = \sigma_2$. Since it is known that $n = 2$ and $n = 3$ are contained in σ_2 and τ, it follows from the relation $\tau \cap \sigma_2 = \sigma_1$ that 2 and 3 are contained in σ_1.

Proof that $2 \in \tau$. There is given a Jordan curve M in E_2 such that M is regular and of class C^∞. There exists a Green's function G on JM with logarithmic pole at a point of $\mathring{J}M$. By definition G reduces to zero on M. Let $z = x+iy$. If e is a positive constant the level curve M_e on which $G(z) = -e$, will be a non-singular, analytic, closed curve in $\mathring{J}M$ and, if e is sufficiently small, will be so near M that M is included in an open "field" of short normals to M_e with one and only one normal through each point of the "field". It is then easy to set up an orientation-preserving C^∞-diffeomorphism T of E_2 onto E_2 such that $T(M_e) = M$.

There exists a directly conformal map K of $\mathring{J}S_1$ onto $\mathring{J}M_e$. Since M_e is regular and analytic, K is conformally extendable over an open neighborhood of JS_1. The C^∞-diffeomorphism TK maps JS_1 onto JM preserving orientation, and is extendable as a C^∞-diffeomorphism over an open neighborhood of JS_1. Thus TK is a solution of the Schoenflies closed cell problem posed by the giving of M.

Proof that $2 \in \sigma_2$. There is given a C^∞-diffeomorphism f of S_1 onto S_1 which we seek to extend over JS_1. Without loss of generality we can suppose that f maps S_1 onto S_1 preserving sense on S_1. Were this not the case a reflection of S_1 in a diameter, composed with f, would give a mapping g of S_1 onto S_1 preserving sense, and f would be extendable over JS_1 if g were so extendable.

Let (r, θ) be polar coordinates in E_2. Under f a point $(r, \theta) = (1, \theta)$ on S_1 corresponds to a point with polar coordinates $(1, k(\theta))$ on S_1, where $\theta \to k(\theta)$ can be taken as a C^∞-diffeomorphism of the θ-axis onto itself such that $k(\theta+2\pi) = k(\theta)+2\pi$ and $k'(\theta) > 0$. Let $t \to \mu(t)$ be a C^∞-mapping of the t-axis onto the interval $[0, 1]$ such that $\mu(t) = 0$ for $t \leqslant 0$, and $\mu(t) = 1$ for $t \geqslant 1$. A C^∞-differentiable isotopy of f into the identity is defined by the mappings

$$\theta \to \big(1-\mu(t)\big)k(\theta)+\mu(t)\,\theta \qquad (r = 1),$$

applied to S_1 for each value of t. That 2 is in σ_2 now follows from Theorem 2.2.

The case $n = 3$. A proof that $n = 3$ is in σ_2 is relatively simple. Our proof that $n = 3$ is in τ is nore movel and involves the theory of critical points of non-degenerate functions.

The removal of the restriction to C^∞-diffeomorphisms, insofar as it occurs in this paper, is facilitated by the following theorem on "elevating manifold differentiability". See Ref. [16].

THEOREM 3.2. *Corresponding to a compact, regular, differentiable $(n-1)$-manifold of class C^m in E, $m > 0$, there exists an orientation-preserving C^m-diffeomorphism T of E onto E such that $T(M)$ is a regular differentiable $(n-1)$-manifold of class C^∞ in E.*

§ 4. Extensions of families of diffeomorphisms. Ref. [10] Theorem 2.3 concerns the extension over JS of the family of mappings f_t, $-\infty < t < \infty$, of a differentiable isotopy. In this section we consider extensions of families of mappings of much more general character. We begin with notation.

Product spaces. Let U and V be two spaces, and $U \times V$ their product. If X is a subset of $U \times V$ let $\mathrm{pr}_1 X$ and $\mathrm{pr}_2 X$ be defined as in Ref. [18]. For $v \in \mathrm{pr}_2 X$ set

$$X^v = \{u \mid (u, v) \in X\}$$

and term X^v the *v-section* of X. We have no need for u-sections of X.

Let $(u, v) \to F(u, v)$ be a mapping of X into E. For fixed $v \in \mathrm{pr}_2 X$ the partial mapping

$$u \to F(u, v) = F^v(u) \qquad (u \in X^v)$$

will be called the *v-section* F^v of F.

Γ-problems. The topological case. Let Γ be an arbitrary paracompact space with points p, q, etc. Let E be an euclidean n-space, with $n > 1$, with points x, y, etc. Let S be the unit sphere in E with center at the origin. We consider the product $E \times \Gamma$, and its subsets, such as $S \times \Gamma$.

The data required to define a Schoenflies Γ-problem is a neighborhood L of $S \times \Gamma$, open relative to $E \times \Gamma$, and a continuous mapping

$$\Phi\colon L \to E\colon (x, p) \to \Phi(x, p),$$

whose p-sections

$$\Phi^p\colon L^p \to E\colon x \to \Phi^p(x) \qquad (p \in \Gamma)$$

are homeomorphisms of L^p into E such that Φ^p maps points of L^p, interior (exterior) to S, into points which are interior (exterior) to the topological $(n-1)$-sphere $\Phi^p(S)$ in E. The fundamental extension theorem in the *topological* case is the following. Ref. [10].

THEOREM 4.1. *Corresponding to (Φ, L, Γ) conditioned as above, and to any sufficiently small neighborhood N of $S \times \Gamma$, open relative to $E \times \Gamma$, there exists a continuous mapping into E*

$$(4.1) \qquad \Lambda_\Phi\colon N \cup (JS \times \Gamma); \qquad (x, p) \to \Lambda_\Phi(x, p)$$

whose p-section for $p \in \Gamma$, is a homeomorphism of $N^p \cup JS$ into E which extends $\Phi^p|N^p$.

Γ-problems. The differentiable case. In the differentiable case there is given an integer $m > 0$ and an arbitrary C^m-manifold Γ with a countable base. The data is an ensemble (Φ, L, Γ), with Φ and L conditioned as in the topological case, and in addition such that Φ is of class C^m on L and for each $p \in \Gamma$, Φ^p a C^m-diffeomorphism of L^p into E.

Conventions. Let X be a neighborhood of $JS \times \Gamma$, open relative to $E \times \Gamma$. By a C^m_Γ-mapping F, $m > 0$, of X into E we mean a continuous mapping of X into E whose restriction to $X - (0 \times \Gamma)$ is a C^m-mapping. By a C^m_Γ-mapping of $JS \times \Gamma$ into E we mean a mapping extendable as a C^m_Γ-mapping F into E of some neighborhood X of $JS \times \Gamma$ open relative to $E \times \Gamma$.

The first theorem in the differentiable case is as follows. See Ref. [10].

THEOREM 4.2. *Corresponding to (Φ, L, Γ) conditioned as above, and to any sufficiently small neighborhood $N \subset L$ of $S \times \Gamma$, open relative to $E \times \Gamma$, there exists a C^m_Γ-mapping Λ_Φ into E, of the form* (4.1) *whose p-section, for each $p \in \Gamma$, is a C^m_0-diffeomorphism of $N^p \cup JS$ into E which extends $\Phi^p|N^p$.*

In the differentiable case, $m > 0$, there is an equivalent theorem. Cf. Lemmas 6.1 and 6.2, Ref. [11].

THEOREM 4.3. *Given a C^m-mapping Φ of $S \times \Gamma$ into E, each of whose p-sections, for $p \in \Gamma$, is a C^m-diffeomorphism of S into E, there exists a C^m_Γ-mapping Λ_Φ of $JS \times \Gamma$ into E, whose p-section, for each $p \in \Gamma$, is a C^m_0-diffeomorphism of JS into E which extends Φ^p.*

Note that Theorem 4.2 is the analogue of Theorem 4.1 stated for the topological case. Theorem 4.3 has an analogue, Theorem 5.2, in the analytic case. Theorem 4.2 has no analogue in the analytic case, nor Theorem 4.3 in the topological case. For a more precise statement of these differences see § 6.

§ 5. The analytic case. We begin with the simple Schoenflies problem. One should recall the definition of an A_0-diffeomorphism as given in § 2. The first theorem is as follows.

THEOREM 5.1. *An analytic diffeomorphism f of S into E can be extended by an A_0-diffeomorphism F of JS onto $JF(S)$.*

This theorem was first stated and proved by Royden in Ref. [8], making use of Theorem 2.1 of Morse. Because of the many difficulties in extending this theorem to the case of analytic families Θ of diffeomorphisms f in Ref. [11], Huebsch and Morse found it necessary to give a different proof of Theorem 5.1, one that could serve as a model when Θ was given. This model is presented in Ref. [9]. Recently Theorem 5.1 has been greatly extended. Cor 7.1 [12].

Γ-problems. The analytic case. Theorem 4.3 has its analogue in the analytic case. Here Γ is a real analytic, regular, proper r-manifold in some euclidean space.

Convention. An A_Γ-mapping of $JS \times \Gamma$ into E is defined as was a C^m_Γ-mapping of $JS \times \Gamma$ into E, and a C^m-mapping by an analytic mapping.

The principal theorem is as follows.

THEOREM 5.2. *Corresponding to a real analytic mapping Φ of $S \times \Gamma$ into E each of whose p-sections is an analytic diffeomorphism of S into E, there exists an A_Γ-mapping Λ_Φ of $JS \times \Gamma$ into E whose p-section for each $p \in \Gamma$ is an A_0-diffeomorphism of JS into E which extends Φ^p.*

§ 6. The data in simple extension problems. We shall bring out fundamental differences between the topological, differential and analytic cases of the simple extension problem.

In posing a simple extension problem the domain of definition of the given mapping f into E has been one of two types.

I. *The $(n-1)$-sphere S.* In this case an extension F of f over JS is tentatively admitted if $f = F|S$.

II. *An open neighborhood N of S.* In this case an extension F of f is tentatively admitted if $F|W = f$, where $W \subset N$ is some sufficiently small open neighborhood of S.

The question arises, is it possible in all three cases, the topological, the differentiable, and the analytic, to take the domain of f either of type I or II at pleasure, and be assured of an extension F of f over JS which is *admissible*, that is, one that is tentatively admissible in the sense of I or II and in addition such that F is a homeomorphism, a C_0^m-diffeomorphism, or an A_0-diffeomorphism of JS into E, respectively, according to the case at hand. The answer is no. Corresponding to the case the following table gives the type of domain of f for which f always has an extension F which is *admissible* in the above sense. We term such a domain admissible.

Case	Type of domain
Topological	II
Differentiable	I or II
Analytic	I

That S is not admissible as a domain in the topological case follows from Ref. [1]. We have seen in Theorem 1.2 that S is admissible as a domain in the differentiable case. A domain of type II is likewise admissible in the differentiable case, as is shown in Ref. [6].

In the analytic case a domain of type II is not admissible. That is, if the given mapping f is an analytic diffeomorphism into E of an open neighborhood of S, there will exist, in general, no A_0-extension of f which agrees with f in a neighborhood $W \subset N$ of S. The following example shows this.

EXAMPLE. Let (r, θ) be polar coordinates in a 2-plane E_2 of coordinates x, y. Set $z = x+iy$. Let a point $z = e^{i\theta}$ on the circle $|z| = 1$ be represented by θ and let this circle C be mapped diffeomorphically onto an ellipse G by setting

$$x = a\cos\theta\,, \quad y = b\sin\theta \quad (0 < a < b)\,,$$

the point $z = e^{i\theta} \in C$ corresponding to $(x, y) \in G$. Let this map of C onto G be denoted by g. Let g be extended over a small open neighborhood of C in such a manner that short normals to C and G at corresponding points, themselves correspond. Cf. Ref. [5], § 5. In particular let points in $\mathring{J}C$ and $\mathring{J}G$ on corresponding normals correspond if at the same distance from C and G. Let this correspondence be similarly defined for points exterior to C and G and near C and G. We understand that no normal to C extends to the origin and no normal to G extends to a focal point of G. Let the resulting analytic diffeomorphism of an open neighborhood of C onto an open neighborhood of G be denoted by f.

This mapping f does not admit an extension as an A_0-diffeomorphism of JC onto JG. In fact the analytic extension of f will fail to be a diffeomorphism at those points of JC which are antecedents of focal points of G.

Nevertheless the original analytic mapping g of C onto G admits an extension as an analytic diffeomorphism into E_2 of an open neighborhood of JC. This statement is a consequence of the following facts. The integer $n = 2$ is in σ_1 so that g can be extended as a C^∞-diffeomorphism over JC. The mapping g thus admits an extension as an analytic diffeomorphism into E_2 of an open neighborhood of JC, in accord with the following theorem. Cf. Huebsch and Morse, Ref. [9].

THEOREM 5.3. *If an analytic diffeomorphism f of S into E admits an extension as a C^m-diffeomorphism of JS into E, $m > 1$, then f admits an extension as an analytic diffeomorphism of JS into E.*

The hypothesis of this theorem may fail to be satisfied when $n > 3$.

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