STUDIA MATHEMATICA, T. XXII. (1962)

The limiting behaviour of indecomposable branching processes

by

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Let $\mathfrak{B}^N$ denote the set of all vectors $\boldsymbol{n} = \langle n_1, n_2, \ldots, n_N\rangle$ in an N-dimensional Euclidean space, whose components $n_1, n_2, \ldots, n_N$ are non-negative integers. By $\boldsymbol{e}_j$ $(j = 1, 2, \ldots, N)$ we shall denote the unit vector, whose j-th component is equal to 1 and the others are equal to 0.

Let us consider a physical cascade in a homogeneous medium consisting of N types of particles in which the decomposition of particles is a random event. For physical reasons we assume that

(i) future of a particle does not depend on its past and depends only on its actual state,

(ii) the destiny of a particle and its progeny does not depend on the future of the actually existing particles.

It is customary to treat mathematically such a cascade as a $\mathfrak{B}^N$-valued homogeneous Markov process $\boldsymbol{X}(t) = \langle X_1(t), X_2(t), \ldots, X_N(t)\rangle$, where the scalar component $X_j(t)$ represents the number of particles of type j in the cascade at the time t. Let $P(t, \boldsymbol{n}, \boldsymbol{m})$ be the transition probability from the state $\boldsymbol{n}$ to the state $\boldsymbol{m}$ in the time interval t. In particular, a particle of the type j has the probability $P(t, \boldsymbol{e}_j, \boldsymbol{m})$ of producing m_1 particles of the type 1, m_2 particles of the type $2, \ldots,$ and m_N particles of the type N in the time interval t. In the language of transition probabilities the conditions (i) and (ii) can be written as follows:

$$P(t, \boldsymbol{n}, \boldsymbol{m}) = \sum \prod_{j=1}^{N} \prod_{i=1}^{n_j} P\big(t, \boldsymbol{e}_j, \boldsymbol{k}(i, j)\big), \tag{1}$$

where the summation is extended over all systems $\boldsymbol{k}(i,j)$ $(i = 1, 2, \ldots, n_j;\ j = 1, 2, \ldots, N)$ of vectors from $\mathfrak{B}^N$, satisfying the condition $\sum_{j=1}^{N} \sum_{i=1}^{n_j} \boldsymbol{k}(i,j) = \boldsymbol{m}$. Since in every finite time interval only a finite number of decompositions of particles can happen, we assume that almost all sample functions of the process $\boldsymbol{X}(t)$ are step functions, i. e. they have only finitely many jumps in every finite interval and are identically constant in every open interval of continuity points. Every $\mathfrak{B}^N$-valued homogeneous Markov process satisfying (1) whose almost all sample functions are step functions will be called a *branching process*.

There exists now a rather complete theory of branching processes, which gives a simple mathematical model for the development of physical cascades and for the growth of populations involving several types of individuals (see [1], [2], [4], [5], [7], [10], [11] and [12]).

The aim of the present article is to study the limiting behaviour at infinity of sample functions of some branching processes.

A branching process is called *indecomposable* if each type of particles can produce any other type. More precisely, a branching process is indecomposable if and only if for every pair i, j of types there exists a vector $\boldsymbol{m} \in \mathfrak{B}^N$ such that $m_j \geqslant 1$ and $P(t, \boldsymbol{e}_i, \boldsymbol{m}) > 0$ for $t > 0$.

A branching process is called *trivial* if for every $t \geqslant 0$, for every type i and for every vector $\boldsymbol{m} \in \mathfrak{B}^N$ such that $\boldsymbol{m} \neq \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_N$ we have the equality $P(t, \boldsymbol{e}_i, \boldsymbol{m}) = 0$.

In the sequel by $\boldsymbol{0}$ and $\boldsymbol{\infty}$ we shall denote the vectors whose all components are equal to 0 and ∞ respectively.

THEOREM. *Let $\boldsymbol{X}(t)$ be an indecomposable branching process. If it is non-trivial, then for almost all sample functions the limit $\lim\limits_{t\to\infty} \boldsymbol{X}(t)$ exists and is equal either to $\boldsymbol{0}$ or to $\boldsymbol{\infty}$.*

If $\boldsymbol{X}(t)$ is a trivial process, then for almost all sample functions and all $t \geqslant 0$ the equality

$$\sum_{j=1}^{N} \boldsymbol{X}_j(t) = \sum_{j=1}^{N} \boldsymbol{X}_j(0)$$

holds. Moreover, if $N \geqslant 2$, then

$$\varliminf_{t\to\infty} X_i(t) = 0 \quad \text{and} \quad \varlimsup_{t\to\infty} X_i(t) = \sum_{j=1}^{N} X_j(0) \quad (i = 1, 2, \ldots, N)$$

with probability 1.

The case of one type of particles has been considered in [13] and [14]. It should be noted that the assumption of indecomposability of the process is essential. We quote an example due to Sevastyanov [11]. Let $\boldsymbol{Y}(t)$ denote a branching process of two types of particles. Suppose that $\boldsymbol{Y}(0) = \langle 0, 1\rangle$ and each particle of the first type at the time t has the probabilities $\alpha\Delta t + o(\Delta t), 1-(\alpha+\beta)\Delta t + 0(\Delta t)$ and $\beta \Delta t + o(\Delta t)$ $(\alpha > \beta > 0)$ of producing zero, one, or two particles of the first type respectively in the interval $(t, t+\Delta t)$. Further, suppose that each particle of the second type has the probabilities $1-\beta\Delta t + o(\Delta t)$ and $\beta\Delta t + o(\Delta t)$ of producing one particle of the second type, or one particle of the second type and one particle of the first type, respectively. It can be shown that

$$\lim_{t\to\infty} P(t, \boldsymbol{e}_2, k\boldsymbol{e}_1) = \left(1-\frac{\beta}{\alpha}\right)\left(\frac{\beta}{\alpha}\right)^k \quad (k = 0, 1, \ldots).$$

Thus for every positive integer k we have the inequality

$$\Pr\Big(\bigcap_{s=0}^{\infty} \bigcup_{s<t} \{Y_1(t) = k\}\Big) > 0.$$

Before proving the Theorem we shall prove some Lemmas. In the sequel $\boldsymbol{X}(t)$ will denote an indecomposable branching process. Since almost all sample functions of $\boldsymbol{X}(t)$ are step functions, the limits

$$q(\boldsymbol{n}, \boldsymbol{m}) = \lim_{t\to 0+} \frac{P(t, \boldsymbol{n}, \boldsymbol{m})}{t} \quad (\boldsymbol{n} \neq \boldsymbol{m}),$$

$$q(\boldsymbol{n}, \boldsymbol{n}) = \lim_{t\to 0+} \frac{P(t, \boldsymbol{n}, \boldsymbol{n}) - 1}{t}$$

exist and satisfy the following conditions:

$$q(\boldsymbol{n}, \boldsymbol{n}) \leqslant 0, \quad q(\boldsymbol{n}, \boldsymbol{m}) \geqslant 0 \quad \text{if} \quad \boldsymbol{n} \neq \boldsymbol{m}, \tag{2}$$

$$\sum_{\boldsymbol{m} \in \mathfrak{B}^N} q(\boldsymbol{n}, \boldsymbol{m}) = 0 \tag{3}$$

(see [3], p. 258-261, [6]). The limits $q(\boldsymbol{n}, \boldsymbol{m})$ are called *intensities*, or *infinitisimal transition probabilities*, of the process $\boldsymbol{X}(t)$. If $q(\boldsymbol{n}, \boldsymbol{n}) < 0$ and if $\boldsymbol{X}(t_0) = \boldsymbol{n}$, there is with probability 1 a sample function discontinuity for some $t > t_0$. The probability that the first jump is to $\boldsymbol{m}$ is $q(\boldsymbol{n}, \boldsymbol{m})/|q(\boldsymbol{n}, \boldsymbol{n})|$ $(n \neq m)$. We shall often write $q_j(\boldsymbol{m})$ instead of $q(\boldsymbol{e}_j, \boldsymbol{m})$.

From (1), by simple computations, we get the formula

$$q(\boldsymbol{n}, \boldsymbol{m}) = \sum_{j=1}^{N} n_j q_j(\boldsymbol{m} - \boldsymbol{n} + \boldsymbol{e}_j). \tag{4}$$

Since the process $\boldsymbol{X}(t)$ is indecomposable, for every pair i, j of types there exists a chain of types $i_0 = i, i_1, \ldots, i_r = j$, and a system $\boldsymbol{m}(1), \boldsymbol{m}(2), \ldots, \boldsymbol{m}(r)$ of vectors from $\mathfrak{B}^N$ such that

$$m(k)_k \geqslant 1 \quad (k = 1, 2, \ldots, r) \tag{5}$$

and

$$q_k(\boldsymbol{m}(k+1)) > 0 \quad (k = 0, 1, \ldots, r-1). \tag{6}$$

Let us introduce the generating functions of the transition probabilities and the intensities:

(7) $$F_j(t,\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathfrak{B}^N} P(t,\boldsymbol{e}_j,\boldsymbol{n})x_1^{n_1}x_2^{n_2}\dots x_N^{n_N} \quad (j=1,2,\dots,N),$$

(8) $$f_j(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathfrak{B}^N} q_j(\boldsymbol{n})x_1^{n_1}x_2^{n_2}\dots x_N^{n_N} \quad (j=1,2,\dots,N),$$

where $\boldsymbol{x} = \langle x_1, x_2, \dots, x_N\rangle$, $|x_i| \leqslant 1$ $(i=1,2,\dots,N)$. The generating functions satisfy the fundamental equations

(9) $$\frac{\partial}{\partial t}F_i(t,\boldsymbol{x}) = f_i(\boldsymbol{F}(t,\boldsymbol{x})) \quad (i=1,2,\dots,N),$$

(10) $$\frac{\partial}{\partial t}F_i(t,\boldsymbol{x}) = \sum_{j=1}^N f_j(\boldsymbol{x})\frac{\partial}{\partial x_j}F_i(t,\boldsymbol{x}) \quad (i=1,2,\dots,N)$$

and the initial condition $\boldsymbol{F}(0,\boldsymbol{x}) = \boldsymbol{x}$, where $\boldsymbol{F}(t,\boldsymbol{x}) = \langle F_1(t,\boldsymbol{x}), F_2(t,\boldsymbol{x}), \dots, F_N(t,\boldsymbol{x})\rangle$ (see [2] and [10]).

It is well-known that the limits

(11) $$\lim_{t\to\infty} P(t,\boldsymbol{e}_j,\boldsymbol{0}) = Q_j \quad (j=1,2,\dots,N)$$

exist. The limiting probability Q_j is the probability that the cascade will become extinct when initially one particle of the type j is present. More precisely,

(12) $$Q_j = \Pr\big(\lim_{t\to\infty}\boldsymbol{X}(t) = 0 \,|\, \boldsymbol{X}(0) = \boldsymbol{e}_j\big) \quad (j=1,2,\dots,N).$$

The vector of the extinction probabilities $\boldsymbol{Q} = \langle Q_1, Q_2, \dots, Q_N\rangle$ satisfies the system of equations

(13) $$f_j(\boldsymbol{Q}) = 0 \quad (j=1,2,\dots,N)$$

(see [10], p. 87).

For every vector $\boldsymbol{n} = \langle n_1, n_2, \dots, n_N\rangle$ from $\mathfrak{B}^N$ we put $|\boldsymbol{n}| = \sum_{j=1}^N n_j$. Let J be the set of all types j for which there exists a vector $\boldsymbol{n}$ in $\mathfrak{B}^N$ such that $|\boldsymbol{n}| \geqslant 2$ and $q_j(\boldsymbol{n}) > 0$.

LEMMA 1. *If the process is non-trivial and the extinction probabilities satisfy the inequalities $Q_1 < 1, Q_2 < 1, \dots, Q_N < 1$, then the set J is non-empty.*

Proof. Contrary to this, let us suppose that the set J is empty, i. e. $q_j(\boldsymbol{n}) = 0$ whenever $|\boldsymbol{n}| \geqslant 2$ and $j = 1,2,\dots,N$. Since the process is non-trivial, there exists a type j_0 such that

(14) $$q_{j_0}(\boldsymbol{0}) > 0.$$

Equality (3) can be rewritten in the form

(15) $$q_j(\boldsymbol{0}) + \sum_{k=1}^N q_j(\boldsymbol{e}_k) = 0 \quad (j=1,2,\dots,N).$$

Further, according to (8) and (13), we have the equalities

(16) $$q_j(\boldsymbol{0}) + \sum_{k=1}^N q_j(\boldsymbol{e}_k)Q_k = 0 \quad (j=1,2,\dots,N).$$

Let M be the set of all types j for which $Q_j = \min_{1\leqslant k\leqslant N} Q_k$. First we consider the case when M contains all types $1, 2, \dots, N$, i. e. when $Q_1 = Q_2 = \dots = Q_N$. From (15) and (16) we obtain the equality

$$0 = q_{j_0}(\boldsymbol{0}) + \sum_{k=1}^N q_{j_0}(\boldsymbol{e}_k)Q_k - Q_1\Big(q_{j_0}(\boldsymbol{0}) + \sum_{k=1}^N q_{j_0}(\boldsymbol{e}_k)\Big) = q_{j_0}(\boldsymbol{0})(1-Q_1),$$

which contradicts the inequality $Q_1 < 1$ and formula (14).

Now let us assume that there exists a type which does not belong to M. Since the process in question is indecomposable, we can chooset in view of (5) and (6), a pair of types j_1, j_2 in such a way tha, t $\in M$, $j_2 \notin M$ and

(17) $$q_{j_1}(\boldsymbol{e}_{j_2}) > 0.$$

Obviously,

(18) $$Q_{j_1} < Q_{j_2}$$

and, according to (2), (15) and (16), the inequality

$$0 = q_{j_1}(\boldsymbol{0}) + \sum_{k=1}^N q_{j_1}(\boldsymbol{e}_k)Q_k - Q_{j_1}\Big(q_{j_1}(\boldsymbol{0}) + \sum_{k=1}^N q_{j_1}(\boldsymbol{e}_k)\Big)$$
$$= q_{j_1}(\boldsymbol{0})(1-Q_{j_1}) + \sum_{k\neq j_1} q_{j_1}(\boldsymbol{e}_k)(Q_k - Q_{j_1}) \geqslant q_{j_1}(\boldsymbol{e}_{j_2})(Q_{j_2} - Q_{j_1})\cdot$$

holds. But this contradicts (17) and (18). The Lemma is thus proved.

LEMMA 2. Let $\mathscr{A} = (a_{ij})$ $(i,j = 1,2,\dots,p+q;\ p\geqslant 1, q\geqslant 0)$ *be a matrix whose elements satisfy the conditions*

(19) $$a_{ij} \leqslant 0 \quad \text{if} \quad i \neq j \quad (i,j = 1,2,\dots,p+q),$$

(20) $$a_{ii} > -\sum_{j\neq i} a_{ij} \quad (i = 1,2,\dots,p)$$

and for every index i satisfying the inequality $p < i \leqslant p+q$ there exists *an index* k_i *such that*

$$1 \leqslant k_i < i, \tag{21}$$

$$a_{ik_i} < 0, \quad a_{ii} = -a_{ik_i}. \tag{22}$$

and

$$a_{ij} = 0 \quad \textit{otherwise}. \tag{23}$$

Then $\det \mathscr{A} > 0$.

Proof. We prove our Lemma by induction with respect to q. For $q = 0$ the assertion is well-known (see e. g. [9], p. 108). Now let us suppose that $q \geqslant 1$ and that the assertion of the Lemma is true for indices less than q. Let $\mathscr{B} = (b_{ij})$ denote the matrix obtained from A by adding the last column to the k_{p+q}-th one. Evidently,

$$b_{ij} = a_{ij} \text{ if } j \neq k_{p+q}, \quad b_{ik_{p+q}} = a_{ik_{p+q}} + a_{i,p+q} \quad (i = 1, 2, \ldots, p+q) \tag{24}$$

and

$$\det \mathscr{A} = \det \mathscr{B}. \tag{25}$$

From (21), (22), (23) and (24) we get the equalities $b_{p+q,p+q} = a_{p+q,p+q}$, $b_{p+q,j} = 0$ $(j = 1, 2, \ldots, p+q-1)$. Consequently, by the development of $\mathscr{B}$ with respect to the last row we get the formula

$$\det \mathscr{B} = a_{p+q,p+q} \det \mathscr{B}_0, \tag{26}$$

where $\mathscr{B}_0 = (b_{ij})$ $(i, j = 1, 2, \ldots, p+q-1)$. Further, it is very easy to verify that the matrix $\mathscr{B}_0$ satisfies the conditions of the Lemma. Consequently, by induction assumption $\det \mathscr{B}_0 > 0$. Thus, by (22), (25), and (26), $\det \mathscr{A} > 0$, which completes the proof.

Lemma 3. *If the process is non-trivial and the extinction probabilities satisfy the inequalities* $Q_1 < 1, Q_2 < 1, \ldots, Q_N < 1$, *then the Jacobian* $\dfrac{\partial(f_1, f_2, \ldots, f_N)}{\partial(x_1, x_2, \ldots, x_N)}$ *is different from* 0 *at the point* $\boldsymbol{Q} = \langle Q_1, Q_2, \ldots, Q_N \rangle$.

Proof. Introducing the notation $\mathfrak{I}_i = \{\boldsymbol{n} : \boldsymbol{n} \neq \boldsymbol{0}, \boldsymbol{n} \neq \boldsymbol{e}_i, \boldsymbol{n} \in \mathfrak{B}^N\}$ $(i = 1, 2, \ldots, N)$, we have, according to (3), (8), and (13), the equations

$$q_i(\boldsymbol{0}) + q_i(\boldsymbol{e}_i) + \sum_{\boldsymbol{n} \in \mathfrak{I}_i} q_i(\boldsymbol{n}) = 0,$$

$$q_i(\boldsymbol{0}) + q_i(\boldsymbol{e}_i) Q_i + \sum_{\boldsymbol{n} \in \mathfrak{I}_i} q_i(\boldsymbol{n}) Q_1^{n_1} Q_2^{n_2} \ldots Q_N^{n_N}.$$

Hence we obtain the formula

$$q_i(\boldsymbol{e}_i) = (1-Q_i)^{-1} \sum_{\boldsymbol{n} \in \mathfrak{I}_i} q_i(\boldsymbol{n}) (Q_1^{n_1} Q_2^{n_2} \ldots Q_N^{n_N} - 1).$$

Setting this expression into the formula

$$\frac{\partial}{\partial x_i} f_i(\boldsymbol{x}) \Big|_{\boldsymbol{x}=\boldsymbol{Q}} = q_i(\boldsymbol{e}_i) + \sum_{\boldsymbol{n} \in \mathfrak{I}_i} n_i q_i(\boldsymbol{n}) Q_1^{n_1} Q_2^{n_2} \ldots Q_{i-1}^{n_{i-1}} Q_i^{n_i - 1} Q_{i+1}^{n_{i+1}} \ldots Q_N^{n_N}$$

we get the equality

$$\frac{\partial}{\partial x_i} f_i(\boldsymbol{x}) \Big|_{\boldsymbol{x}=\boldsymbol{Q}}$$

$$= (1-Q_i)^{-1} \sum_{\boldsymbol{n} \in \mathfrak{I}_i} q_i(\boldsymbol{n}) (Q_1^{n_1} Q_2^{n_2} \ldots Q_N^{n_N} - 1 - n_i Q_1^{n_1} Q_2^{n_2} \ldots Q_{i-1}^{n_{i-1}} Q_i^{n_i - 1} Q_{i+1}^{n_{i+1}} \ldots Q_N^{n_N})$$

Furthermore, for $i \neq j$ the equality

$$\frac{\partial}{\partial x_j} f_i(\boldsymbol{x}) \Big|_{\boldsymbol{x}=\boldsymbol{Q}} = \sum_{\boldsymbol{n} \in \mathfrak{I}_i} q_i(\boldsymbol{n}) n_j Q_1^{n_1} Q_2^{n_2} \ldots Q_{j-1}^{n_{j-1}} Q_j^{n_j - 1} Q_{j+1}^{n_{j+1}} \ldots Q_N^{n_N}$$

holds. Hence, by simple computations, we obtain the expansion

$$(-1)^N \frac{\partial(f_1, f_2, \ldots, f_N)}{\partial(x_1, x_2, \ldots, x_N)} \Big|_{\boldsymbol{x}=\boldsymbol{Q}} \tag{27}$$

$$= \prod_{k=1}^{N} (1-Q_k)^{-1} \sum_{\boldsymbol{n}(1) \in \mathfrak{I}_1} \sum_{\boldsymbol{n}(2) \in \mathfrak{I}_2} \ldots$$

$$\ldots \sum_{\boldsymbol{n}(N) \in \mathfrak{I}_N} q_1(\boldsymbol{n}(1)) q_2(\boldsymbol{n}(2)) \ldots q_N(\boldsymbol{n}(N)) \det \big(a_{ij}(\boldsymbol{n}(1), \boldsymbol{n}(2), \ldots, \boldsymbol{n}(N))\big),$$

where

$$a_{ii}(\boldsymbol{n}(1), \boldsymbol{n}(2), \ldots, \boldsymbol{n}(N)) = 1 - Q_1^{n(i)_1} Q_2^{n(i)_2} \ldots Q_N^{n(i)_N} - n(i)_i Q_1^{n(i)_1} Q_2^{n(i)_2} \ldots Q_{i-1}^{n(i)_{i-1}} Q_i^{n(i)_i - 1} Q_{i+1}^{n(i)_{i+1}} \ldots Q_N^{n(i)_N} (1-Q_i) \quad (i = 1, 2, \ldots, N), \tag{28}$$

$$a_{ij}(\boldsymbol{n}(1), \boldsymbol{n}(2), \ldots, \boldsymbol{n}(N)) = -n(i)_j Q_1^{n(i)_1} Q_2^{n(i)_2} \ldots Q_{j-0}^{n(i)_{j-1}} Q_j^{n(i)_j - 1} Q_{j+1}^{n(i)_{j+1}} \ldots Q_N^{n(i)_N} (1-Q_j) \quad (i \neq j;\ i, j = 1, 2, \ldots, N), \tag{29}$$

We shall first prove that

$$\det \big(a_{ij}(\boldsymbol{n}(1), \boldsymbol{n}(2), \ldots, \boldsymbol{n}(N))\big) \geqslant 0 \tag{30}$$

for all systems $\boldsymbol{n}(1) \in \mathfrak{I}_1, \boldsymbol{n}(2) \in \mathfrak{I}_2, \ldots, \boldsymbol{n}(N) \in \mathfrak{I}_N$. To prove this it is sufficient to show that for all such systems the inequalities

$$a_{ii}\big(\boldsymbol{n}(1), \boldsymbol{n}(2), \dots, \boldsymbol{n}(N)\big) \geqslant \sum_{j\neq i}\Big|a_{ij}\big(\boldsymbol{n}(1), \boldsymbol{n}(2), \dots, \boldsymbol{n}(N)\big)\Big| \quad (i=1,2,\dots, N)$$

hold (see e. g. [9], p. 108). These inequalities, according to (28) and (29), are equivalent to the following ones

$$1-Q_1^{n(i)_1}Q_2^{n(i)_2}\dots Q_N^{n(i)_N} \geqslant \sum_{j=1}^{N} n(i)_j Q_1^{n(i)_1}Q_2^{n(i)_2}\dots Q_{j-1}^{n(i)_{j-1}}Q_j^{n(i)_j-1}Q_{j+1}^{n(i)_{j+1}}\dots Q_N^{n(i)_N}(1-Q_j)$$

$$(i = 1, 2, \dots, N).$$

Consequently, to prove (30) it suffices, for every $\boldsymbol{n}\,\epsilon\,\mathfrak{B}^N$, to prove the equality

$$1-Q_1^{n_1}Q_2^{n_2}\dots Q_N^{n_N} \geqslant \sum_{j=1}^{N} n_j Q_1^{n_1}Q_2^{n_2}\dots Q_{j-1}^{n_{j-1}}Q_j^{n_j-1}Q_{j+1}^{n_{j+1}}\dots Q_N^{n_N}(1-Q_j). \tag{31}$$

In order to establish the above inequality we multiply by $Q_1^{n_1}Q_2^{n_2}\dots Q_{j-1}^{n_{j-1}}$ the obvious inequality

$$1-Q_j^{n_j} = (1-Q_j)\sum_{k=0}^{n_j-1} Q_j^k \geqslant n_j Q_j^{n_j-1}(1-Q_j) \geqslant n_j Q_j^{n_j-1}Q_{j+1}^{n_{j+1}}\dots Q_N^{n_N}(1-Q_j)$$

and sum over j. This completes the proof of (30).

By Lemma 1 the set J of types is non-empty. Since the process is idecomposable, all types, say $s_1, s_2, \dots, s_q$, which do not belong to J can be ordered in such a way that $q_{s_i}(\boldsymbol{e}_{k_i}) > 0$ $(i = 1, 2, \dots, q)$, where $k_i\,\epsilon\, J\cup\{s_1, s_2, \dots, s_{i-1}\}$ (compare (5) and (6)). Without loss of generality we may suppose that $J = \{1, 2, \dots, p\}$ $(p\geqslant 1)$ and $q^i(\boldsymbol{e}_{k_i}) > 0$ $(i = p+1, p+2, \dots, N)$, where

$$k_i < i \quad (i = p+1, p+2, \dots, N). \tag{32}$$

By the definition of the set J for every type i $(1\leqslant i\leqslant p)$ we can choose a vector $\boldsymbol{m}(i)\,\epsilon\,\mathfrak{B}^N$, with $|\boldsymbol{m}(i)|\geqslant 2$, such that $q_i\big(\boldsymbol{m}(i)\big)>0$. Setting $\boldsymbol{m}(i) = \boldsymbol{e}_{k_i}$ for $i = p+1, p+2, \dots, N$ we have the inequality

$$q_i\big(\boldsymbol{m}(i)\big) > 0 \quad (i = 1, 2, \dots, N). \tag{33}$$

Denoting briefly by a_{ij} the matrix elements $a_{ij}\big(\boldsymbol{m}(1), \boldsymbol{m}(2), \dots, \boldsymbol{m}(N)\big)$ defined by (28) and (29), we note that, according to (2), (27), (30), and (33), in order to prove our Lemma it suffices to show that $\det(a_{ij}) > 0$.

Starting from (28) and (29) we deduce the relations

$$a_{ij} \leqslant 0 \text{ if } i\neq j \qquad (i, j = 1, 2, \dots, N),$$

$$a_{ik_i} = Q_{k_i}-1 < 0 \qquad (i = p+1, p+2, \dots, N),$$

$$a_{ii} = 1-Q_{k_i} = -a_{ik_i} \qquad (i = p+1, p+2, \dots, N)$$

and

$$a_{ij} = 0 \text{ if } j\neq i, k_i \qquad (i = p+1, p+2, \dots, N).$$

To prove inequality (20) it suffices, by virtue of (28) and (29), to show that for $\boldsymbol{n}\,\epsilon\,\mathfrak{B}^N$, with $|\boldsymbol{n}|\geqslant 2$, the inequality

$$1-Q_1^{n_1}Q_2^{n_2}\dots Q_N^{n_N} > \sum_{j=1}^{N} n_j Q_1^{n_1}Q_2^{n_2}\dots Q_{j-1}^{n_{j-1}}Q_j^{n_j-1}Q_{j+1}^{n_{j+1}}\dots Q_N^{n_N}(1-Q_j) \tag{34}$$

holds. Without loss of generality we may assume that either $n_1\geqslant 2$ or $n_1 = 1$ and $n_2 = 1$. In the first case we have the inequality

$$1-Q_1^{n_1} = (1-Q_1)\sum_{k=0}^{n_1-1} Q_1^k > n_1 Q_1^{n_1-1}(1-Q_1) \geqslant n_1 Q_1^{n_1-1}Q_2^{n_2}\dots Q_N^{n_N}(1-Q_1)$$

and in the second one

$$1-Q_1^{n_1} = 1-Q_1 > (1-Q_1)Q_2 \geqslant n_1 Q_1^{n_1-1}Q_2^{n_2}\dots Q_N^{n_N}(1-Q_1).$$

Further, applying the same arguments as in the proof of (31), we finally get (34) and, consequently, (20). Thus, we have proved that the matrix (a_{ij}) fulfils the conditions of Lemma 2, which completes the proof.

LEMMA 4. *If the process is non-trivial and the extinction probabilities satisfy the inequalities* $Q_1 < 1, Q_2 < 1, \dots, Q_N < 1$, *then for all vectors* $\boldsymbol{n}, \boldsymbol{m}\,\epsilon\,\mathfrak{B}^N$ $(\boldsymbol{m}\neq \boldsymbol{0})$ *the transition probabilities* $P(t, \boldsymbol{n}, \boldsymbol{m})$ *are integrable on the right half-line.*

Proof. By formula (1) it suffices to prove our Lemma in the case $\boldsymbol{n} = \boldsymbol{e}_1$ $(i = 1, 2, \dots, N)$. We prove this assertion by induction with respect to $|\boldsymbol{m}|$. First let us suppose that $|\boldsymbol{m}| = 1$, i. e. $\boldsymbol{m} = \boldsymbol{e}_k$ $(k = 1, 2, \dots, N)$. By differentiating (9) with respect to x_k and putting $\boldsymbol{x} = \boldsymbol{0}$ we obtain the equality

$$\frac{d}{dt}P(t, \boldsymbol{e}_i, \boldsymbol{e}_k) = \sum_{j=1}^{N} c_{ij}(t)P(t, \boldsymbol{e}_j, \boldsymbol{e}_k) \quad (i = 1, 2, \dots, N), \tag{35}$$

where $c_{ij}(t)$ is the derivative $\dfrac{\partial}{\partial x_j}f_i(\boldsymbol{x})$ at the point $\langle P(t, e_1, 0), P(t, e_2, 0), \dots, P(t, e_N, 0)\rangle$. Obviously, by (11),

$$\lim_{t\to\infty}\det\big(c_{ij}(t)\big) = \frac{\partial(f_1, f_2, \dots, f_N)}{\partial(x_1, x_2, \dots, x_N)}\bigg|_{\boldsymbol{x}=\boldsymbol{Q}}.$$

and, consequently, by Lemma 3, for sufficiently large t, $|\det(c_{ij}(t))|$ is greater than a positive constant. Thus, in view of (35), for sufficiently large t, the transition probabilities $P(t, \boldsymbol{e}_i, \boldsymbol{e}_k)$ are linear combinations with bounded coefficients of derivatives $\frac{d}{dt}P(t, \boldsymbol{e}_j, \boldsymbol{e}_k)$. Hence it follows that $P(t, \boldsymbol{e}_i, \boldsymbol{e}_k)$ are integrable on the right half-line.

Now let us suppose that $|\boldsymbol{m}| \geqslant 2$ and that the assertion of the Lemma is true for all vectors $\boldsymbol{k}$, with $|\boldsymbol{k}| < |\boldsymbol{m}|$. By differentiating $\frac{\partial^{|\boldsymbol{m}|}}{\partial x_1^{m_1}\partial x_2^{m_2}\ldots\partial x_N^{m_N}}$ of (9) and putting $\boldsymbol{x} = \boldsymbol{0}$ we get the equality

$$\frac{d}{dt}P(t, \boldsymbol{e}_i, \boldsymbol{m}) = \sum_{j=1}^{N} c_{ij}(t)P(t, \boldsymbol{e}_j, \boldsymbol{m}) + u_i(t) \qquad (i = 1, 2, \ldots, N),$$

where the functions $u_i(t)$ are linear combinations with bounded coefficients of the transition probabilities $P(t, \boldsymbol{e}_j, \boldsymbol{k})$ $(|\boldsymbol{k}| < |\boldsymbol{m}|;\ j = 1, 2, \ldots, N)$. By induction assumption the functions $u_i(t)$ are integrable on the right half-line. Applying now the same arguments as above in the case $|\boldsymbol{m}| = 1$, we obtain the integrability of $P(t, \boldsymbol{e}_i, \boldsymbol{m})$ $(i = 1, 2, \ldots, N)$. The Lemma is thus proved.

By a theorem of Lévy ([8], p. 362) the limits $\lim\limits_{t\to\infty} P(t, \boldsymbol{n}, \boldsymbol{m})$ $(\boldsymbol{n}, \boldsymbol{m} \in \mathfrak{B}^N)$ exist. Thus, from Lemma 4 we get the following

Corollary. *If the process satisfies the conditions of Lemma 4, then* $\lim\limits_{t\to\infty} P(t, \boldsymbol{n}, \boldsymbol{m}) = 0$, *whenever* $\boldsymbol{m} \neq 0$.

Let us introduce the notation

$$M_i(t, k, s) = \sum_{n_1 = k} n_s P(t, \boldsymbol{e}_i, \boldsymbol{n}), \tag{36}$$

where the summation is over all vectors $\boldsymbol{n} \in \mathfrak{B}^N$, with $n_1 = k$.

Lemma 5. *If the process is non-trivial and the probabilities of extinction satisfy the inequalities* $Q_1 < 1, Q_2 < 1, \ldots, Q_N < 1$, *then the functions* $M_i(t, k, s)$ $(i = 1, 2, \ldots, N;\ s = 2, 3, \ldots, N;\ k = 0, 1, \ldots)$ *are integrable on the right half-line.*

Proof. From (7) and (36) we obtain the formula

$$M_i(t, k, s) = \frac{1}{k!}\frac{\partial^{k+1}}{\partial x_1^k \partial x_s} F_i(t, \boldsymbol{x})\Big|_{\boldsymbol{x}=\boldsymbol{e}_s}, \tag{37}$$

We shall first prove by induction with respect to k that all functions $M_i(t, k, s)$ $(i, s = 1, 2, \ldots, N)$ are bounded on the right half-line. Setting $\boldsymbol{x} = \boldsymbol{e}_s$ into (10) we get the equation

$$\frac{d}{dt}\sum_{m=0}^{\infty} P(t, \boldsymbol{e}_i, m\boldsymbol{e}_s) = f_s(\boldsymbol{e}_s)M_i(t, 0, s) + \tag{38}$$
$$+ \sum_{j\neq s} f_j(\boldsymbol{e}_s)\sum_{m=0}^{\infty} P(t, \boldsymbol{e}_i, \boldsymbol{e}_j + m\boldsymbol{e}_s).$$

From the Kolmogorov equations for transition probabilities in Markov processes

$$\frac{d}{dt}\sum_{m=0}^{\infty} P(t, \boldsymbol{e}_i, m\boldsymbol{e}_s) = \sum_{\boldsymbol{n}\in\mathfrak{B}^N} q_i(\boldsymbol{n})\sum_{m=0}^{\infty} P(t, \boldsymbol{n}, m\boldsymbol{e}_s)$$

and from (2) and (3) we get the inequality

$$\left|\frac{d}{dt}\sum_{m=0}^{\infty} P(t, \boldsymbol{e}_i, m\boldsymbol{e}_s)\right| \leqslant \sum_{\boldsymbol{n}\in\mathfrak{B}^N} |q_i(\boldsymbol{n})| = 2|q_i(\boldsymbol{e}_i)|.$$

Hence and from (38) it follows that all functions $M_i(t, 0, s)$ $(i, s = 1, 2, \ldots, N)$ are bounded.

Now let us suppose that $k \geqslant 1$, and for all integers r $(0 \leqslant r < k)$ the functions $M_i(t, r, s)$ $(i, s = 1, 2, \ldots, N)$ are bounded. Differentiating (10) k times with respect to x_1 and putting $\boldsymbol{x} = \boldsymbol{e}_s$ we have, according to (37),

$$\frac{d}{dt}\sum_{m=0}^{\infty} P(t, \boldsymbol{e}_i, k\boldsymbol{e}_1 + m\boldsymbol{e}_s) = f_s(\boldsymbol{e}_s)M_i(t, k, s) + v_i(t),$$

where $v_i(t)$ are linear combinations of functions $M_i(t, r, s)$ $(0 \leqslant r < k)$ and probabilities $\sum\limits_{m=0}^{\infty} P(t, \boldsymbol{e}_i, \boldsymbol{e}_j + l\boldsymbol{e}_1 + m\boldsymbol{e}_s)$ $(0 \leqslant l \leqslant k)$. Now the proof can be made similarly to that in the case $k = 0$. The boundedness of $M_i(t, k, s)$ is thus proved.

Since for $s \geqslant 2$

$$\sum_{m=0}^{\infty} P(t, \boldsymbol{e}_i, m\boldsymbol{e}_s) \leqslant P(t, \boldsymbol{e}_i, \boldsymbol{0}) + \sum_{m=1}^{\infty} mP(t, \boldsymbol{e}_i, m\boldsymbol{e}_s)$$
$$\leqslant P(t, \boldsymbol{e}_i, \boldsymbol{0}) + M_i(t, 0, s),$$

we can change the order of summation and, passing to the limit, we have

$$\lim_{t\to\infty}\sum_{m=0}^{\infty} P(t, \boldsymbol{e}_i, m\boldsymbol{e}_s) = \sum_{m=0}^{\infty}\lim_{t\to\infty} P(t, \boldsymbol{e}_i, m\boldsymbol{e}_s).$$

Hence, by (11) and by the Corollary to Lemma 4, we obtain the equality

$$\lim_{t\to\infty}\sum_{m=0}^{\infty}P(t,\boldsymbol{e}_i,m\boldsymbol{e}_s)=Q_i\qquad(i=1,2,\ldots,N;\ s=2,3,\ldots,N),$$

which, by virtue of (7) can be rewritten in the following form:

$$\lim_{t\to\infty}F_i(t,\boldsymbol{e}_s)=Q_i\qquad(i=1,2,\ldots,N;\ s=2,3,\ldots,N).\tag{39}$$

We proceed now to the proof of the Lemma by induction with respect to k. Differentiating (9) with respect to x_s $(s\geqslant 2)$ and putting $\boldsymbol{x}=\boldsymbol{e}_s$ we have

$$\frac{d}{dt}M_i(t,0,s)=\sum_{j=1}^{N}d_{ij}(t)M_j(t,0,s)\qquad(i=1,2,\ldots,N),\tag{40}$$

where $d_{ij}(t)$ is the derivative $\dfrac{\partial}{\partial x_j}f_i(x)$ at the point $\langle F_1(t,\boldsymbol{e}_s),F_2(t,\boldsymbol{e}_s),\ldots,F_N(t,\boldsymbol{e}_s)\rangle$. Obviously, by (39),

$$\lim_{t\to\infty}\det\big(d_{ij}(t)\big)=\frac{\partial(f_1,f_2,\ldots,f_N)}{\partial(x_1,x_2,\ldots,x_N)}\bigg|_{x=Q}$$

and, consequently, by Lemma 3, for sufficiently large t, $|\det\big(d_{ij}(t)\big)|$ is greater than a positive constant. Thus, in view of (40), for sufficiently large t, the function $M_i(t,0,s)$ is a linear combination with bounded coefficients of derivatives of bounded functions $M_j(t,0,s)$ $(j=1,2,\ldots,N)$ which implies the integrability of $M_i(t,0,s)$ $(i=1,2,\ldots,N;\ s=2,3,\ldots,N)$ on the right half-line.

Now let us suppose that $k\geqslant 1$ and that the assertion of the Lemma is true for all indices less than k. By differentiating (9) with respect to x_s, and k times with respect to x_1, and putting $\boldsymbol{x}=\boldsymbol{e}_s$ $(s\geqslant 2)$ we have the equality

$$\frac{d}{dt}M_i(t,k,s)=\sum_{j=1}^{N}d_{ij}(t)M_j(t,k,s)+w_i(t)\qquad(i=1,2,\ldots,N),$$

where $w_i(t)$ are linear combinations with bounded coefficients of $M_j(t,r,s)$ $(0\leqslant r<k)$. Applying the same arguments as in the case $k=0$, we get the integrability of $M_i(t,k,s)$ $(i=1,2,\ldots,N;\ s=2,3,\ldots,N)$, which completes the proof of the Lemma.

Lemma 6. *For every integer k and every vector $\boldsymbol{n}\in\mathfrak{V}^N$ we have the equality*

$$\Pr\Big(\bigcup_{t\leqslant u\leqslant T}\{X_1(u)=k\}\,|\,\boldsymbol{X}(0)=\boldsymbol{n}\Big)=\Pr\big(X_1(T)=k\,|\,\boldsymbol{X}(0)=\boldsymbol{n}\big)+$$
$$+\sum_{\boldsymbol{m}\in\mathfrak{A}_k}\sum_{\boldsymbol{r}\notin\mathfrak{A}_k}\int_t^T\Pr\Big(\bigcap_{0\leqslant v\leqslant T-u}\{X_1(T-v)\neq k\}\,|\,\boldsymbol{X}(0)=\boldsymbol{r}\Big)\,q(\boldsymbol{m},\boldsymbol{r})P(u,\boldsymbol{n},\boldsymbol{m})\,du,$$

where

$$\mathfrak{A}_k=\{\boldsymbol{m}\colon m_1=k,\ \boldsymbol{m}\in\mathfrak{V}^N\}.\tag{41}$$

Proof. Suppose that $t=u_1<u_2<\ldots<u_q=T$. From the equality

$$\bigcup_{i=1}^{q}\{X_1(u_i)=k\}=\{X_1(T)=k\}\cup\bigcup_{i=1}^{q-1}\bigcap_{i\leqslant j\leqslant q}\{X_1(u_j)\neq k\}\cap\{X_1(u_i)=k\}$$
$$=\{X_1(T)=k\}\cup\bigcup_{\boldsymbol{m}\in\mathfrak{A}_k}\bigcup_{\boldsymbol{r}\notin\mathfrak{A}_k}\bigcup_{i=1}^{q-1}\bigcap_{i+1<j\leqslant q}\{X_1(u_j)\neq k\}\cap\{\boldsymbol{X}(u_{i+1})=\boldsymbol{r}\}\cap$$
$$\cap\{\boldsymbol{X}(u_i)=\boldsymbol{m}\}$$

we get the formula

$$\Pr\Big(\bigcup_{i=1}^{q}\{X_1(u_i)=k\}\,|\,\boldsymbol{X}(0)=\boldsymbol{n}\Big)=\Pr\big(X_1(T)=k\,|\,\boldsymbol{X}(0)=\boldsymbol{n}\big)+$$
$$+\sum_{\boldsymbol{m}\in\mathfrak{A}_k}\sum_{\boldsymbol{r}\notin\mathfrak{A}_k}\sum_{i=1}^{q-1}\Pr\Big(\bigcap_{i+1<j\leqslant q}\{X_1(u_j-u_i)\neq k\}\,|\,\boldsymbol{X}(0)=\boldsymbol{r}\Big)$$
$$P(u_{i+1}-u_i,\boldsymbol{m},\boldsymbol{r})P(u_i,\boldsymbol{n},\boldsymbol{m}).$$

It can be shown that, when $\max\limits_{1\leqslant j\leqslant q}(u_j-u_{j-1})\to 0$, the last sum approaches the series of integrals

$$\sum_{\boldsymbol{m}\in\mathfrak{A}_k}\sum_{\boldsymbol{r}\notin\mathfrak{A}_k}\int_t^T\Pr\Big(\bigcap_{0\leqslant v\leqslant T-u}\{X_1(T-v)\neq k\}\,|\,\boldsymbol{X}(0)=\boldsymbol{r}\Big)q(\boldsymbol{m},\boldsymbol{r})P(u,\boldsymbol{n},\boldsymbol{m})\,du.$$

Moreover, the left-hand side of the last equality approaches the probability $\Pr\Big(\bigcup\limits_{t\leqslant u\leqslant T}\{X_1(u)\neq k\}\,|\,\boldsymbol{X}(0)=\boldsymbol{n}\Big)$. We note that this reasoning is justified by the fact that almost all sample functions of the process are step functions.

Lemma 7. *For all integers k, j $(k=0,1,\ldots;\ j=2,3,\ldots,N)$ and for all vectors $\boldsymbol{n}\in\mathfrak{V}^N$ we have the evaluation*

$$\sum_{\boldsymbol{m}\in\mathfrak{A}_k}m_jP(t,\boldsymbol{n},\boldsymbol{m})\leqslant\sum_{i=1}^{N}\sum_{s=0}^{k}n_iM_i(t,s,j),$$

where the set $\mathfrak{A}_k$ and the functions $M_i(t,s,j)$ are defined by (41) *and* (36) *respectively.*

Proof. Let $\boldsymbol{m}(i,r)$ $(r=1,2,\ldots,n_i;\ i=1,2,\ldots,N)$ be a system of vectors from $\mathfrak{V}^N$ satisfying the condition

$$\sum_{i=1}^{N}\sum_{r=1}^{n_i}\boldsymbol{m}(i,r)=\boldsymbol{m}.$$

From (1), (36), and from the evaluation

$$m_j \prod_{i=1}^{N}\prod_{r=1}^{n_i} P\big(t, \mathbf{e}_i, \mathbf{m}(i,r)\big) = \Big(\sum_{i=1}^{N}\sum_{r=1}^{n_i} m(i,r)_j\Big)\prod_{i=1}^{N}\prod_{r=1}^{n_i} P\big(t, \mathbf{e}_i, \mathbf{m}(i,r)\big)$$
$$\leqslant \sum_{i=1}^{N}\sum_{r=1}^{n_i} m(i,r)_j P\big(t, \mathbf{e}_i, \mathbf{m}(i,r)\big)$$

we obtain the inequality

$$\sum_{\mathbf{m}\in\mathfrak{A}_k} m_j P(t, \mathbf{n}, \mathbf{m}) = \sum_{\mathbf{m}\in\mathfrak{A}_k}\sum_{\mathbf{m}(i,r)} m_j \prod_{i=1}^{N}\prod_{r=1}^{n_i} P\big((t, \mathbf{e}_i, \mathbf{m}(i,r)\big)$$
$$\leqslant \sum_{i=1}^{N}\sum_{r=1}^{n_i}\sum_{\mathbf{l}\in\mathfrak{A}_0\cup\mathfrak{A}_1\ldots\cup\mathfrak{A}_k} l_j\, P(t, \mathbf{e}_i, \mathbf{l}) = \sum_{i=1}^{N} n_i \sum_{s=0}^{k} M_i(t, s, j),$$

which completes the proof.

Proof of the theorem. First let us assume that the process $\mathbf{X}(t)$ is non-trivial. Put $I = \{i: Q_i = 1\}$. We shall now show that either $I = \{1, 2, \ldots, N\}$ or I is an empty set. Contrary to this let us suppose that $0 \neq I \neq \{1, 2, \ldots, N\}$. Since the process is indecomposable, we can choose a pair i, j of types such that $i \in I$, $j \notin I$, and according to (5) and (6), $q_i(\mathbf{m}) > 0$ for a vector $\mathbf{m} \in \mathfrak{B}^N$, with $m_j \geqslant 1$. Further, from (3), (8) and (13), we get the inequality

$$0 = \sum_{\mathbf{n}\in\mathfrak{B}^N} q_i(\mathbf{n}) Q_1^{n_1} Q_2^{n_2} \ldots Q_N^{n_N} < \sum_{\mathbf{n}\in\mathfrak{B}^N} q_i(\mathbf{n}) = 0,$$

which gives the contradiction. Thus, we have either $Q_1 = Q_2 = \ldots = Q_N = 1$ or $Q_1 < 1, Q_2 < 1, \ldots, Q_N < 1$. In the first case from (12) we deduce that for almost all sample functions of the process $\mathbf{X}(t)$ the equality $\lim_{t\to\infty} \mathbf{X}(t) = \mathbf{0}$ holds.

Now let us consider the second case: $Q_1 < 1$, $Q_2 < 1, \ldots, Q_N < 1$. Put

$$H_k(t, \mathbf{n}) = \sum_{\mathbf{m}\in\mathfrak{A}_k}\sum_{\mathbf{r}\notin\mathfrak{A}_k} q(\mathbf{m}, \mathbf{r}) P(t, \mathbf{n}, \mathbf{m}) \qquad (k = 0, 1, \ldots; \mathbf{n}\in\mathfrak{B}^N),$$

where the set $\mathfrak{A}_k$ is defined by (41). From (2), (3), (4), and Lemma 7, we obtain the inequality

$$H_k(t, \mathbf{n}) = k \sum_{\mathbf{m}\in\mathfrak{A}_k}\sum_{\mathbf{r}\notin\mathfrak{A}_k} q_1(\mathbf{r}-\mathbf{m}+\mathbf{e}_1) P(t, \mathbf{n}, \mathbf{m}) +$$
$$+\sum_{j=2}^{N}\sum_{\mathbf{m}\in\mathfrak{A}_k}\sum_{\mathbf{r}\notin\mathfrak{A}_k} m_j q_j(\mathbf{r}-\mathbf{m}+\mathbf{e}_j) P(t, \mathbf{n}, \mathbf{m})$$
$$\leqslant k|q_1(\mathbf{e}_1)| \sum_{\mathbf{m}\in\mathfrak{A}_k} P(t, \mathbf{n}, \mathbf{m}) + \sum_{j=2}^{N} |q_j(\mathbf{e}_j)| \sum_{\mathbf{m}\in\mathfrak{A}_k} m_j P(t, \mathbf{n}, \mathbf{m})$$
$$\leqslant k|q_1(\mathbf{e}_1)| P(t, \mathbf{n}, k\mathbf{e}_1) + k|q_1(\mathbf{e}_1)| \sum_{j=2}^{N}\sum_{\mathbf{m}\in\mathfrak{A}_k} m_j P(t, \mathbf{n}, \mathbf{m}) +$$
$$+\sum_{j=2}^{N} |q_j(\mathbf{e}_j)| \sum_{\mathbf{m}\in\mathfrak{A}_k} m_j P(t, \mathbf{n}, \mathbf{m}) \leqslant k|q_1(\mathbf{e}_1)| P(t, \mathbf{n}, k\mathbf{e}_1) +$$
$$+\sum_{j=2}^{N} |q_j(\mathbf{e}_j) + kq_1(\mathbf{e}_1)| \sum_{i=1}^{N}\sum_{s=0}^{k} n_i M_i(t, s, j).$$

Hence, by Lemmas 4 and 5, the functions $H_k(t, \mathbf{n})$ ($k = 0, 1, \ldots$; $\mathbf{n}\in\mathfrak{B}^N$) are integrable on the right half-line.

From Lemma 6 we obtain the inequality

$$\Pr\Big(\bigcup_{t\leqslant u\leqslant T} \{X_1(u) = k\} \,|\, \mathbf{X}(0) = \mathbf{n}\Big) \tag{42}$$
$$\leqslant \Pr\big(X_1(T) = k \,|\, \mathbf{X}(0) = \mathbf{n}\big) + \int_t^T H_k(u, \mathbf{n})\,du \qquad (k = 0, 1, \ldots; \mathbf{n}\in\mathfrak{B}^N).$$

Further, from Lemma 7, we get the evaluation

$$\Pr\big(X_1(T) = k \,|\, \mathbf{X}(0) = \mathbf{n}\big) = \sum_{\mathbf{m}\in\mathfrak{A}_k} P(T, \mathbf{n}, \mathbf{m})$$
$$P(T, \mathbf{n}, k\mathbf{e}_1) + \sum_{j=2}^{N}\sum_{\mathbf{m}\in\mathfrak{A}_k} m_j P(T, \mathbf{n}, \mathbf{m})$$
$$\leqslant P(T, \mathbf{n}, k\mathbf{e}_1) + \sum_{j=2}^{N}\sum_{i=1}^{N}\sum_{s=0}^{k} n_i M_i(t, s, j).$$

Hence, according to Lemma 5, Corollary to Lemma 4, and formula (12),

$$\lim_{T\to\infty} \Pr\big(X_1(T) = k \,|\, \mathbf{X}(0) = \mathbf{n}\big) = 0 \quad \text{if} \quad k \geqslant 1,$$
$$\lim_{T\to\infty} \Pr\big(X_1(T) = 0 \,|\, \mathbf{X}(0) = \mathbf{n}\big) \leqslant Q_1^{n_1} Q_2^{n_2} \ldots Q_N^{n_N}$$
$$= \Pr\big(\lim_{t\to\infty} \mathbf{X}(t) = \mathbf{0} \,|\, \mathbf{X}(0) = \mathbf{n}\big).$$

Comparing this result with (42), we obtain for any $\boldsymbol{n}\in\mathfrak{B}^N$ the formulae

$$\Pr\Big(\bigcap_{s=0}^{\infty}\bigcup_{s\leqslant t}\{X_1(t)=k\}\,|\,\boldsymbol{X}(0)=\boldsymbol{n}\Big)=0 \quad \text{if} \quad k\geqslant 1,$$

$$\Pr\big(\varliminf_{t\to\infty}X_1(t)=0\,|\,\boldsymbol{X}(0)=\boldsymbol{n}\big)\leqslant\Pr\big(\lim_{t\to\infty}\boldsymbol{X}(t)=\boldsymbol{0}\,|\,\boldsymbol{X}(0)=\boldsymbol{n}\big).$$

Taking into account the inclusions

$$\{0<\lim_{t\to\infty}X_1(t)<\infty\}\subset\bigcup_{k=1}^{\infty}\bigcap_{s=0}^{\infty}\bigcup_{s\leqslant t}\{X_1(t)=k\},$$

$$\{\lim_{t\to\infty}\boldsymbol{X}(t)=\boldsymbol{0}\}\subset\{\varliminf_{t\to\infty}X_1(t)=0\},$$

we infer, in view of the last formulae, that

$$\Pr\big(0<\varliminf_{t\to\infty}X_1(t)<\infty\big)=0,\qquad \Pr\big(\{\varliminf_{t\to\infty}X_1(t)=0\}\setminus\{\lim_{t\to\infty}\boldsymbol{X}(t)=\boldsymbol{0}\}\big)=0.$$

By symmetry of our assumptions we obtain the same assertion for any other type j, i. e.

$$\Pr\big(0<\varliminf_{t\to\infty}X_j(t)<\infty\big)=0,\qquad \Pr\big(\{\varliminf_{t\to\infty}X_j(t)=0\}\setminus\{\lim_{t\to\infty}\boldsymbol{X}(t)=\boldsymbol{0}\}\big)=0$$
$$(j=1,2,\ldots,N).$$

Hence $\Pr\big(\lim_{t\to\infty}\boldsymbol{X}(t)=\boldsymbol{0}$ or $\infty\big)=1$, which completes the proof o the Theorem for non-trivial processes.

Now let us suppose that the process $\boldsymbol{X}(t)$ is trivial. Then from (4) the equality $q(\boldsymbol{n},\boldsymbol{m})=0$ follows, whenever $|\boldsymbol{n}|\neq|\boldsymbol{m}|$. Solving the Kolmogorov equations

$$\frac{d}{dt}\Pr\big(|\boldsymbol{X}(t)|=|\boldsymbol{n}|\,\big|\,\boldsymbol{X}(0)=\boldsymbol{n}\big)=\sum_{|\boldsymbol{m}|=|\boldsymbol{n}|}q(\boldsymbol{n},\boldsymbol{m})\Pr\big(|\boldsymbol{X}(t)|=|\boldsymbol{n}|\,\big|\,\boldsymbol{X}(0)=\boldsymbol{m}\big)$$

under the initial conditions $\Pr\big(|\boldsymbol{X}(0)|=|\boldsymbol{n}|\,\big|\,\boldsymbol{X}(0)=\boldsymbol{n}\big)=1$, we obtain $\Pr\big(|\boldsymbol{X}(t)|=|\boldsymbol{n}|\,\big|\,\boldsymbol{X}(0)=\boldsymbol{n}\big)=1$ $(t\geqslant 0;\ \boldsymbol{n}\in\mathfrak{B}^N)$. Hence, we have $|\boldsymbol{X}(t)|=|\boldsymbol{X}(0)|$ $(t\geqslant 0)$ with probability 1.

Finally, let us assume that $N\geqslant 2$. We note that $P(t,\boldsymbol{e}_i,\boldsymbol{e}_j)>0$ $(i,j=1,2,\ldots,N,\ t>0)$ because of the indecomposability of the process $\boldsymbol{X}(t)$. Hence and from (1), by simple reasoning, we get the inequality

$$P(t,\boldsymbol{n},\boldsymbol{m})>0\qquad(t>0;\ |\boldsymbol{n}|=|\boldsymbol{m}|). \tag{43}$$

Consider the sample functions of the process in question satisfying the initial condition $\boldsymbol{X}(0)=\boldsymbol{n}$. They form a homogeneous Markov process

with a finite number of states $\boldsymbol{m}$ $(|\boldsymbol{m}|=|\boldsymbol{n}|)$. It is well-known that the condition (43) implies the relation

$$\Pr\Big(\bigcap_{s=1}^{\infty}\{\boldsymbol{X}(t)\colon t\geqslant s\}=\{\boldsymbol{m}\colon|\boldsymbol{m}|=|\boldsymbol{n}|\}\,\Big|\,\boldsymbol{X}(0)=\boldsymbol{n}\Big)=1$$

(see [8], [13]). Hence in particular it follows that if $X(0)=n$, then $\varliminf_{t\to\infty}X_j(t)=0$ and $\varlimsup_{t\to\infty}X_j(t)=|\boldsymbol{n}|$ $(j=1,2,\ldots,N)$ with probability 1. In other words, for almost all sample functions we have the relations $\varliminf_{t\to\infty}X_j(t)=0$, $\varlimsup_{t\to\infty}X_j(t)=|\boldsymbol{X}(0)|$ $(j=1,2,\ldots,N)$. The Theorem is thus proved.

References

[1] N. Arley, *On the theory of stochastic processes and their applications to the theory of cosmic radiation*, New York 1948.

[2] N. A. Dmitriev and A. N. Kolmogorov, *Branching stochastic processes*, Compt. Rend. Acad. Sci. URSS, Doklady, 56 (1947), p. 5-8.

[3] J. L. Doob, *Stochastic processes*, New York-London 1953.

[4] C. I. Everett and S. Ulam, *Multiplicative systems in several variables*, I, II, III, Los Alamos Scientific Laboratory Declassified Documents, LADC- 534 (AECD-2164), LADC-533 (AECD-2165), LA-707, 1948.

[5] T. E. Harris, *Some mathematical models for branching processes*, Proc. Second Berkeley Symposium on Math. Statistics and Probability, 1951, p. 305-328.

[6] А. М. Колмогоров, *К вопросу о дифференцируемости переходных вероятностей в однородных по времени процессах Маркова со счётным числом состаяний*, Учён. зап. МГУ, Т. IV, вып. 148, (1951), p. 53-59.

[7] А. Н. Колмогоров и Б. А. Севастьянов, *Вычисление финальных. вероятностей для ветвящихся случайных процессов*, ДАН, 56 No 8 (1947), p. 783-786

[8] P. Lévy, *Systèmes markoviens et stationnaires, Cas dénombrable*, Ann. Ec. Norm. Sup. 68 (1951), p. 327-381.

[9] A. Mostowski and M. Stark, *Algebra wyższa I*, Warszawa 1953.

[10] Б. А. Севастьянов, *Теория ветвящихся случайных процессов*, Успехи матем. наук, VI, (1951), p. 47-99.

[11] — *Предельные теоремы для ветвящихся случайных процессов специального вида*, Теория вероятностей и её применения, II (1957), p. (339-348.

[12] — *Переходные явления в ветвящихся случайных процессах*, Теория вероятностей и её применения, IV, (1959), p. 121-135.

[13] K. Urbanik, *Własności graniczne procesów Markowa*. Rozprawy Matematyczne 13 (1957), p. 1-48.

[14] — *Limit properties of homogeneous Markoff processes with a denumerable set of states*, Bull. Acad. Polon. Sci. Cl. III, 2 (1954), p. 371-373.

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Reçu par la Rédaction le 12. 1. 1962