ZASTOSOWANIA MATEMATYKI
VI (1962)

H. STEINHAUS (Wrocław)

A BRIDGE WITH A HEXAGONAL FRAMEWORK

The objective aimed at by this Note is to show the advantages of a framework consisting of regular hexagons; we limit our considerations to the example of a bridge, or rather to a single arch of a bridge.

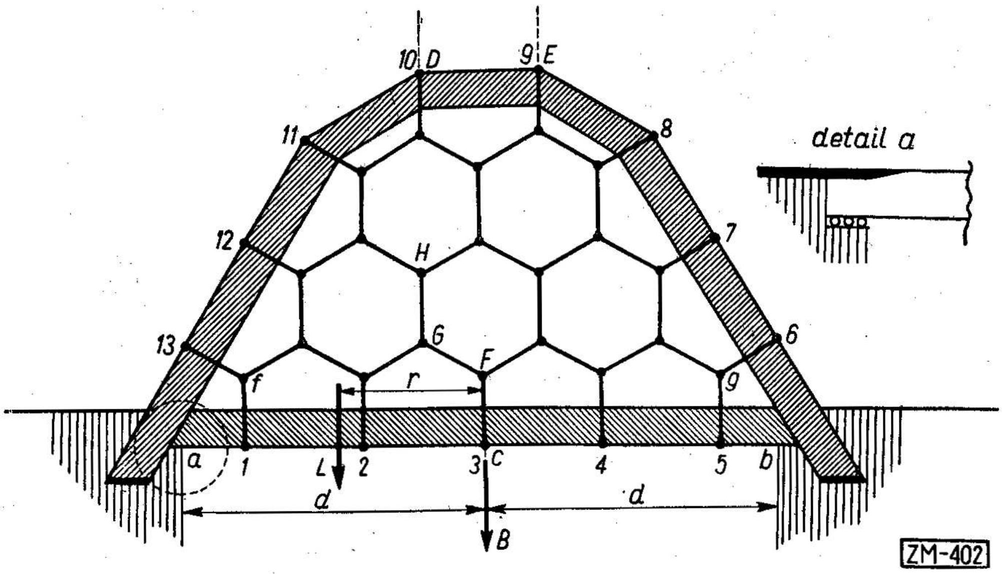

Fig. 1

The sketch (Fig. 1) simplifies the construction we have to describe; it exhibits only the essential parts: the upper belt or the proper *arch*, the lower belt, which we call the *beam*, and the hexagonal *framework* connecting these belts. As to the arch and the beam we are not concerned with details of their construction; the reader may suppose the arch to be made of concrete reinforced with iron and the beam to consist of a single I-bar; the only innovation we propose is the hexagonal framework connecting the upper belt with the lower one — most of the customary bridges employ long vertical rods for this purpose.

The arch is considered as rigid; it rests on a foundation of masonry; thus it is rigidly connected with the supports of the bridge. The rigid beam is supposed to hang on the framework; the framework being attached to the arch, the whole construction seems to be only a variant of familiar bridges, which, however, is not the case.

The framework consists of 38 bars, identical as to shape, length, and the material they are made of. They form a net of 16 meshes; the boundary of the net is the polygon *1-2-3-4-5-g-6-7-8-9-10-11-12-13-f-1* of 15 sides. Five of the meshes are regular hexagons; the eleven border meshes are truncated hexagons — the bars separating them are still the same. Thus the 15-gon P consists of 4 bars and of 11 other sides, which are diagonals of the basic hexagon; the points $1, 2, \ldots, 13$ are called the ends of the net.

The net has 34 knots, the ends included; they connect the bars with other bars or with the belts — in both cases they are pin-connected *joints* round which the bars are free to turn([1]). The knots $1,2,3,4,5$ are attached to the beam and cannot move relatively to it — the knots $6, 7, \ldots, 13$ are attached to the arch and thus absolutely fixed.

Kinematics. The number of interior knots is $34-13=21$; if we fasten the ends of the beam to the supports, the interior joints represent 21 points with 42 degrees of freedom; the 38 rods give 38 constraints. In spite of the inequality $42 > 38$, the framework is completely fixed. To prove it let us remark that no movement of the framework is possible if the border meshes do not participate in it. Let us suppose that some border meshes participate in the movement; as the polygon $1, 2, \ldots, 13, 1$ is fixed, any displacement of a border mesh changes its shape and diminishes its area, because this area attains its unique maximum when the mesh in question is inscribed in a circle, which is the case for every mesh in its initial shape. The same applies to the interior meshes which are regular hexagons. Thus any such movement implies the decrease of the area of some meshes but in no case an increase. The polygon P being invariable, its area is constant — this contradiction shows the stability of the framework([2]).

Let us now study the movements of the system when we cut off the prolongations of the beam so that it does not reach beyond a and b, and is not hindered in its movement by the supports. Suppose a displacement of both extremities of the beam downwards. It starts by a small shift of 1 and of 5 downwards, so that the area of P increases — this implies the increase of the area of at least one of the meshes of the net, against the

([1]) Hexagonal frameworks are applied in modern architecture in roof systems. They are neither vertical nor pin-connected; cf. Z. S. Makowski, *Space Structures*: The Polytechnic, 309 Regent Street, London, W. 1.

([2]) The same property of stability still holds if the length of the outer bars (ex. *13-f* or *1-7*) is not equal to the length of the sides of the inner regular hexagons. However, this generalization will not be used in the sequel.

statement above that every mesh attains the maximum area in the initial position. For the same reason no clockwise turning of the beam round any of its points lying between a and c is possible — as to the opposite movement, it implies an increase of the potential energy of the heavy beam, which is incompatible with the initial zero-velocity (we assume the weight of the framework to be negligible). The same applies to the points between b and c. This considerations do not, however, exclude (see Suppl. I) the turning of the part *1-5* of the beam round c if the extensions of the beam are cut off and the supports cannot stop such movement. None the less, symmetry is a sufficient reason for the equilibrium — it would even be stable if the mathematical line *1-5* were considered as the upper face of the I-iron beam.

Of course the points $1, 2, \ldots, 5$ may describe identical circular arcs but — once again — it would increase the potential energy of the beam.

Statics. We have to compute the internal forces in the bars of the framework in the case described above, i.e. in the case of a heavy beam of weight B hanging freely on the framework attached to the arch.

THEOREM I. *Under the conditions stated above every bar of the framework is submitted to a tension equal to* $B/5$.

Proof. The system being in equilibrium, every knot is at rest. The knot k being at rest, the forces acting on k along ki, kj, km must be equal and either all directed away from k (tensions) or all directed towards k (pressures). Suppose that a tension T draws k in the direction km. Then m is subjected to the same tension T in the direction mk. This implies that knot *2* (Fig. 1) is also acted upon by a tension T, i.e. by a vertical force directed upwards. Consequently, all bars are submitted to tension T. In particular, the knots *1,2,3,4,5* are drawn upwards with a force T each. The beam being in equilibrium under the force of gravity and the five parallell forces T, we must have $5T = B$, which implies $T = B/5$. It is clear that a supposition of the internal forces being pressures would be incompatible with the equilibrium of the beam subjected to gravity.

Let us suppose now that the beam is extended in both directions — the weight is still B and the beam is in equilibrium, scarcely touching the supports. What happens if a load L (see Fig. 1) appears on the left side of the midpoint c? The equilibrium would be disturbed if there were no supports. In fact, the tensions in the five bars carrying B being equal, the equilibrium becomes impossible, because the momentum of the tensions with respect to c is 0 and the same is true for the momentum of the gravity acting on B; consequently, the total momentum is $Lr \neq 0$ and directed counterclockwise. The left support, however, immediately stops the rotation and furnishes a reaction R acting upwards.

THEOREM II. *Let us call* $2d$ *the span of the bridge,* r *the distance* Lc, L *the weight of the load* L, $T(r)$ *the tension in the bars, and* $R(r)$ *the reaction of the left support. Then we get*

$$T(r) = T_d + \frac{L}{5} \cdot \frac{d-r}{d}, \tag{1}$$

$$R(r) = Lr/d; \tag{2}$$

here $T_d = B/5$.

Proof. The necessary and sufficient conditions for the equilibrium are as follows:

$5T(r)+R(r) = B+L$ — the conditions that the total force is 0.

$R(r)d = Lr$ — the condition that the total momentum is 0.

Both conditions are satisfied by (1) and (2). The condition for the total momentum takes the form $Rd = Lr$ because the total momentum of the tensions computed for the point c is zero.

Formulae (1) and (2) can be read as follows: When the load appears at point a, i.e when $r = d$, the reaction of the left support is L and the tensions in all bars are $B/5$: the load is carried exclusively by the left support and the beam is carried exclusively by the framework. When the load travels from a to c, the distance r diminishes from d to 0 and the reaction of the left support decreases linearly from L to 0; as to the tensions, they are initially $B/5$ in every bar and they grow linearly, becoming $(L+B)/5$ when the moving load reaches the midpoint c of the beam. Thus we have to employ for our construction bars strong enough to be subjected to a tension $(L+B)/5$, L being the maximum load permitted to move along the beam (it is hardly necessary to remark that the right half of the beam behaves exactly like the left one).

It is possible that the slackening of the joints and the increase of the length of the bars exposed for a long time to stress diminishes their initial tension so that a part B' of the weight of the beam is carried by the supports and the rest, $B-B' = B''$, by the framework. Then we must write in (1) $T_d = B''/5$ instead of $T_d = B/5$, and to introduce on the right of (2) a supplementary term $+B'/2$.

The advantages of the hexagonal framework:

1. At any time the tensions in all bars are equal. They are maximum when the moving load is midway of the beam: $(L+B)/5$. The shape of the arch and the number of the meshes can be altered arbitrarily — the only change required is to replace 5 in the formulae by n, n being the number of joints in the beam.

2. The tensions are linear functions of $|r|$. This guarantees the absence of shocks when the moving load travels along: its passing above the joints *1,2,3,4,5* does not affect the bar carrying this joint.

3. An impact of a jumping load, causing a sudden displacement h of a bar downwards, displaces other bars too, but these movements are halved at every junction: for $h = 64$ mm the top joints are displaced no more than 1 mm. This property saves the arch from the shocks suffered by the beam: the framework as a whole behaves like a system of springs in spite of the rigidity of the bars.

4. The tension in the 8 bars ending in *6,7,8,9,10,11,12,13* being equal and directed orthogonally to the arch the weight of the beam and the load helps to hold the arch together: it prevents the arch from being flattened by its own weight and its "legs" from being stretched asunder. Thus, under the action of the weight of the beam and the load, the whole construction becomes similar to a homogeneous plate submitted to equal tensions in all outward directions.

5. The slackening of the bars can be counteracted by screws in D and E.

6. Formulae (1) and (2) are easy to handle.

7. The hinged connection of the bars presents no difficulty.

8. The tension T rises continuously when the load travels from a to c, and then falls continuously when it travels from c to b.

Now let us compare the hexagonal framework with the customary vertical bars:

Ad 1. The tensions in the bars of the ordinary bridge (OB) are not equal — thus we must compute the maximum charge. It could happen (though this is rather improbable) that the whole weight of the beam plus the weight of the load acts on a single bar — thus every bar must be capable of standing a tension equal to $B+L$ instead of $(B+L)/5$, as in our construction. The reason of this phenomenon is the indeterminacy of the customary framework: the slackening of all the bars, except one, is sufficient to cause this undesirable effect. This circumstance is usually taken into account by studying the elastic deformations of the beam and of the bars — in our framework these deformations have only an infinitesimal influence.

Ad 2. The remark ad 1 shows the difficulty of calculating the tensions in the bars for a load moving along OB — if we succeed in that computation, we get different formulae for two bars whose distance from the midpoint c is different. The passing of the load above a joint in OB affects essentially the bar ending at that joint.

Ad 3. The impact of a jumping load in OB travels along the bars upwards and exposes the arch to the same shock at the top of the affected bar as at its extremity attached to the beam.

Ad 4. All the tensions in OB being vertical, the advantage 4 is lost.

Ad 5. No tuning of the framework is possible in OB — in our construction such possibility is described in 3: the tuning can be achieved by a screw turned by man-power.

Ad 6. Cf. remarks ad 1 and 2.

Ad 7. In OB we have no standard bars as in the hexagonal framework.

Ad 8. The behaviour of T depends on the elasticity of B and on the bar considered.

Supplements to kinematics. Let us study again the kinematics of the 15-gon *12345g67...13f1.*

SUPPLEMENT I. We have to prove the possibility of the turning of the base *1-5* about c as centre.

It is not easy to establish this possibility by pure kinematics: we shall resort to dynamics by supposing *1-2* to be a thin, rigid and very heavy beam, and L a small load. The sketch shows that there is no equilibrium because the momentum of the weight L is Lr relatively to c and the momentum of other forces (tensions an gravity) nil. Thus a movement of the beam must result from these forces. It cannot be a translation downwards, as we have already shown in the section on Kinematics, or a translation upwards because of the increase of the gravitational potential, which would result from such displacement of $B+L$. Consequently, B must turn about a certain point p of the line *1-5*. Let us suppose first that p lies to the left of c. Then a counterclockwise movement would contradict the potential principle for L sufficiently small. The clockwise turning would increase the area of the P-gon, as shown by the displacements of *5g6* and *13f1*. If p lies to the right of c, the same argument works when we write "clockwise" for "counterclockwise" and *vice versa*. Thus $p = c$ and the only movement possible is a rotation of *1-5* about c. As shown by dynamics, this movement must appear as a result of the forces acting on the beam; this is a proof of its kinematical possibility.

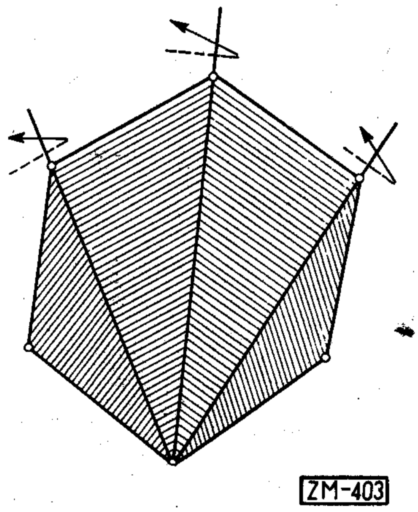

Fig. 2

The proof above is an example of establishing a mathematical theorem in which the terms "force, mass, gravity, external force" etc. do not appear by a reasoning based on these concepts.

To finish this supplement let us state that the rotation is infinitesimal: it is a displacement in the vicinity of the maximal area; for a finite load

L the centre c would rise, as it must do because of the decrease of the area of the polygon P.

SUPPLEMENT II. Suppose that each joint allows two degrees of freedom to every bar which is attached to it. Is it possible — the beam being attached to the supports — for a mesh to become skew? If that happened, we could connect a joint of such a mesh with all the other joints of the same mesh by diagonals, consider them as hinges (Fig. 2) and stretch out the mesh in the initial plane. The area of this new plane mesh is at most equal to that of the initial plane mesh, and the projection of the skew mesh on the plane of the sketch has a smaller area than the skew mesh itself. Thus the projection of the skew mesh is smaller than the initial mesh. The projections of the other meshes have at most the areas of initial meshes. Thus the projection of the whole distorted framework is smaller than the area of P, which is absurd. Thus no joint can leave the initial plane.

Dr W. Mromliński and Eng. K. Czapliński (of the Civil Eng. Dept., Wrocław Inst. of Technology) contributet by their criticisms to the paper above — the author states it here with due gratitude.

Praca wpłynęła 18. 5. 1961

H. STEINHAUS (Wrocław)

MOST Z SZEŚCIOBOCZNĄ KRATOWNICĄ

STRESZCZENIE

Celem Noty jest krótki opis kratownicy złożonej z umiarowych sześcioboków przegubowych; jej zalety pokażemy na przykładzie przęsła mostowego, jako jednym z możliwych zastosowań (rys. 1).

Jezdnia wisi na górnym pasie nośnym za pośrednictwem cięgien — są to pręty lub linki stalowe napięte tak, że tworzą sieć sześcioboków umiarowych. Górny pas jest sztywny i nieruchomo złączony z ziemią. Rysunek ilustruje układ 38 cięgien. Wykażemy, że każdy węzeł sieci, to jest każdy punkt, w którym łączą się przegubowo trzy pręty (na rysunku punkty grubo znaczone wyobrażają przeguby) jest nieruchomy przy dowolnym rozkładzie obciążeń jezdni: jak wiadomo, wszelka deformacja sześcioboku umiarowego w płaszczyźnie zmniejsza jego pole, jeżeli boki zachowują długość. Wobec tego wszelki ruch węzła zmniejsza łączne pole sieci, a więc podnosi jezdnię, co jest sprzeczne z grawitacją.

Z nieruchomości węzłów wynika, że napięcia wszystkich cięgien są równe. Ta właściwość kratownicy sześciobocznej ułatwia obliczenia i zmniejsza koszt konstrukcji w porównaniu z tymi kratownicami, w których, zależnie od rozkładu obciążeń na pasie

dolnym, raz jedno cięgno, innym razem drugie jest maksymalnie obciążone (twierdzenie II). Także pas górny korzysta z ekonomii, jaką daje równość napięć w cięgnach.

Należy podkreślić, że postać pasa górnego jest dowolna, byle można było umieścić na nim skrajne węzły sieci.

Umieszczanie ciężarów bezpośrednio na węzłach wewnętrznych (tj. takich jak F, G, H itd.) nie jest objęte powyższą teorią; może ono doprowadzić do zerwania konstrukcji już przy niedużych ciężarach.

Publikacje opisujące różne współczesne struktury heksagonalne pomijają sieci przegubowe.

Х. ШТЕЙНХАУС (Вроцлав)

МОСТ С ШЕСТИУГОЛЬНОЙ ФЕРМОЙ

РЕЗЮМЕ

Статья содержит описание фермы, составленной из правильных шарнирных шестиугольников; её преимущества будут показаны на примере мостовой фермы (черт. 1).

Мостовая прикреплена к верхнему несущему поясу при помощи тросов, или стержней, образующих сеть правильных шестиугольников. Верхний пояс жёсткий и неподвижно скреплён с землёй. На чертеже изображена система из 38 звеньев. Покажем, что любой узел, т. е. каждая точка, в которой соединяются три стержня (точки, жирно выделенные на чертеже, обозначают шарниры) неподвижна при любом распределении нагрузок на мостовой. Известно, что любая плоская деформация правильного шестиугольника, сохраняющая длину его сторон, уменьшает его площадь. Поэтому любое перемещение узла уменьшает общую площадь сети, вследствие чего мостовая поднимается, но это противоречит гравитации.

С неподвижности узлов следует равенство напряжений во всех стержнях фермы. Это свойство шестиугольной фермы облегчает расчёты и уменьшает стоимость конструкции по сравнению с фермами, у которых в зависимости от распределения нагрузок на нижнем поясе, либо один стержень, либо другой максимально нагружены (II теорема). Равенство напряжений в стержнях благоприятно отражается также на конструкции верхнего пояса.

Следует подчеркнуть, что ферма верхнего пояса произвольна, при условии, что можна на нём разместить крайние шарниры сети.

Теория не предусматривает нагрузок, приложенных непосредственно к внутренним узлам (например к узлам F, G, H и т. д.). Даже небольшие нагрузки такого рода могут привести к разрушению всей конструкции.

Публикации, описывающие современные шестиугольные конструкции, не рассматривают шарнирных сетей.
