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A simple proof of the uniqueness of the extremal measure

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Let E be a plane set such that the boundary of the complement of E consists of a finite number of analytic Jordan curves. In the present paper we shall give a simple proof of the following

THEOREM. *The extremal measure, i.e. the measure realising*

$$\inf_{\nu\geqslant 0}\int_E\int_E \log\frac{1}{|z-\zeta|}\,d\nu(\zeta)\,d\nu(z) \qquad \big(\nu(E)=1,\ \nu(E\cdot e)=0 \text{ if } E\cdot e=0\big),$$

is unique.

In the general case this theorem was proved by Frostman [1] and his proof was based on the following

LEMMA. *If* $\nu(E)=0$, *then the integral* $\int_E\int_E \log\frac{1}{|z-\zeta|}\,d\nu(\zeta)\,d\nu(z)$ *is non-negative.*

M. Tsuji [2] proved this theorem without the use of the above lemma, but his proof is incomplete.

Proof of the theorem. Suppose that there exist two different extremal measures μ_1 and μ_2. Then there exists an open circle K such that $\mu_1(E\cdot K)\neq\mu_2(E\cdot K)$, for example $\mu_1(E\cdot K)>\mu_2(E\cdot K)$ and

$$(1)\qquad \mu_1[E\cdot(\bar{K}-K)]=\mu_2[E\cdot(\bar{K}-K)]=0\,,\qquad C[E\cdot(\bar{K}-K)]=0$$

where $C(e)$ denotes the capacity defined for a closed set e. Since [1], [2]

$$u_1(z)\overset{\mathrm{df}}{=}\int_E \log\frac{1}{|z-\zeta|}\,d\mu_1(\zeta) \qquad\text{and}\qquad u_2(z)\overset{\mathrm{df}}{=}\int_E \log\frac{1}{|z-\zeta|}\,d\mu_2(\zeta)$$

are continuous and equal to $\log(1/C(E))$ for $z\in E$ and since $u_1(z)-u_2(z)\to 0$ as $z\to\infty$, we have

$$(2)\qquad u_1(z)\equiv u_2(z)\,.$$

From (1) and (2) we have

$$\int_{\bar{K}\cdot E}\log\frac{1}{|z-\zeta|}\,d[\mu_1(\zeta)-\mu_2(\zeta)]=\int_{E-K}\log\frac{1}{|z-\zeta|}\,d[\mu_2(\zeta)-\mu_1(\zeta)]\,.$$

Hence the function $\int_{\overline{K}\cdot E} \log(1/|z-\zeta|)d[\mu_1(\zeta)-\mu_2(\zeta)]$ is not constant and bounded in the whole plane and harmonic except in a set with the capacity equal to zero, which is absurbed. Hence we obtain our theorem, q.e.d.

Remark. It is easy to see that by the same method we can prove a similar theorem for the potentials in three-dimensional space.

References

[1] O. Frostman, *Potentiel d'équilibre et capacité des ensembles*, Thèse, Lund 1935.

[2] M. Tsuji, *Fundamental theorems in the potential theory*, Jour. Math. Soc. Jap. 1952, p. 70.

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