For any subset Y of X we shall assume the notation

$$+1\cdot Y = Y, \quad -1\cdot Y = -Y = X-Y.$$

(ix) *If for some elements $A_i \in \mathfrak{S}$ and numbers $\varepsilon_i = \pm 1$ the intersection*

$$\varepsilon_1\cdot h(A_1) \cap \ldots \cap \varepsilon_n \cdot h(A_n) \tag{5}$$

is a non-empty set, then there exists an element $C \in \mathfrak{S}$ such that $h(C)$ is a subset of (5).

Since the set (5) is not empty, there exists a function $f_0 \in X$ such that

$$f_0(A_i) = \varepsilon_i \quad \text{for} \quad i = 1, \ldots, n.$$

By (2),

$$\varepsilon_1\cdot A_1 \wedge \ldots \wedge \varepsilon_n \cdot A_n \neq 0,$$

i. e. there exists an element $C \in \mathfrak{S}$ such that $C \subset \varepsilon_i \cdot A_i$ for $i = 1, \ldots, n$. If $f \in h(C)$, i. e. $f(C) = 1$, then $f(A_i) = \varepsilon_i$ by (v), i.e. f belongs to (5). This proves that $h(C)$ is a subset of (5).

To prove theorem (B′), let as assume that $\mathfrak{A}$ is the field (of subsets of X) generated by all the sets $h(A)$, $A \in \mathfrak{S}$. Thus (c) follows directly from the definition of $\mathfrak{A}$. The field $\mathfrak{A}$ is the class of all finite unions of intersections of the form (5). This, by (ix), proves (a). Property (b) follows directly from (vi) and (viii).

Note that in the case where $\mathfrak{S}$ is a dense subset of a given Boolean algebra, the above proof yields the Stone representation theorem. Incidentally it shows also that the Stone space X is a subset of the Cantor space $H^{\mathfrak{S}}$.

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AXIOMS AND SOME PROPERTIES OF POST ALGEBRAS

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Introduction. The notion of n-valued logic was introduced first by E. L. Post, [5], in 1921. A special case of this notion, the 3-valued logic, was formulated earlier by J. Łukasiewicz, [3], in 1920.

It is well known that there is a Boolean algebra corresponding to the two-valued logic (see, e. g. S. Mazurkiewicz [4], p. 55). P. C. Rosenbloom [6] published in 1942 the first system of axioms of the algebra corresponding to the n-valued logic of E. L. Post. He has called this algebra *Post algebra*. However, Rosenbloom's system of axioms was a very difficult one.

G. Epstein [1] was the first who simplified this theory by making use of the existence of a Boolean algebra underlying a given Post algebra.

P. C. Rosenbloom has already noticed that the theory of the Post algebra may be applied in other branches of mathematics, not only in logic.

The purpose of the present paper is to give a few simple systems of axioms of the Post algebra and to formulate some of its properties, similar to those of a Boolean algebra.

In section 1 a distributive lattice called P_0-lattice is examined, some properties of which give us a good position to formulate in section 3 a few simple systems of axioms of the Post algebra. Section 2 contains Epstein's difinition of a Post algebra and some lemmas rewritten from Epstein paper [1]. Section 4 contains some simple lemmas on the extension of Boolean homomorphisms to Post homomorphisms and some properties of m-valued Post homomorphisms. In section 5 a normed measure on a given Post algebra is defined. Section 6 contains a set-theoretical representation of a Post algebra. In section 7 a congruence relation is defined which makes it possible to obtain a Post algebra from a P_0-lattice.

The most essential results of the present paper were published earlier, [8], without proofs.

I should like to remark, finally, that the paper is almost self-contained; I have only used one or two results of other authors.

Notation. If $\{x_t : t \in T\}$ is an indexed subset of a lattice L then, as usually, the symbols $\bigcup_{t\in T} x_t$ and $\bigcap_{t\in T} x_t$ will denote, respectively, the join and meet of all x_t in L. In particular cases the join of x and y will be denoted by $x\cup y$ and the meet of x and y by $x\cap y$. Sometimes, however, it is convenient in Post algebras to write xy instead of $x\cap y$. The complement of x, if it exists in L, will be denoted by $-x$. For a partially ordering relation in lattices the symbol $\leqslant$ is provided; for the set-inclusion the symbol $\subset$ is used, as usually. Instead of $x\cap -y$ we shall write $x-y$.

1. P_0-lattices. Definition 1.1. Let us consider a distributive lattice L with zero-element 0, and unit-element 1. Let a sublattice $B\subset L$ be a Boolean algebra of complemented elements of L. If there exists an ascending sequence

(1) $\quad 0 = e_0 \leqslant e_1 \leqslant \ldots \leqslant e_{n-1} = 1, \quad$ where n is an integer $\geqslant 2$,

of elements of L such that every $x\in L$ can be written in the form

$$x = b_1e_1\cup b_2e_2\cup\ldots\cup b_{n-1} = \bigcup_{i=1}^{n-1} b_ie_i,$$

where $b_1, \ldots, b_{n-1}\in B$, then L will be called a *P_0-lattice.*

The P_0-lattice L being determined by the sequence $e_0, e_1, \ldots, e_{n-1}$ and the Boolean algebra B, it is convenient to write

$$L = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle.$$

From definition 1.1 it follows immediately that every element x of the P_0-lattice L can be written also in the form

$$(*)\qquad x = de_1\cup d_2e_2\cup\ldots\cup d_{n-1}$$

where $d_i\in B$, $i = 1, \ldots, n-1$, and $d_1\geqslant d_2\geqslant\ldots\geqslant d_{n-1}$. (This can be proved by putting $d_i = \bigcup_{j=i}^{n-1} b_j$, $i = 1, \ldots, n-1$).

Every representation such as $(*)$ will be called a *monotonic representation* of x.

Similarly: every representation

$$x = c_1e_1\cup c_2e_2\cup\ldots\cup c_{n-1},$$

where $c_i\in B$, $i = 1, \ldots, n-1$, and $c_i\cap c_j = 0$ for $i\neq j$, will be called a *disjoint representation* of $x\in P$.

From now on in this section $L = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle$ denotes a fixed P_0-lattice.

LEMMA 1.2. *If the elements x and y of L have monotonic representations*

$(\mathrm{r}_1)\quad x = a_1e_1\cup a_2e_2\cup\ldots\cup a_{n-1},$

$(\mathrm{r}_2)\quad y = b_1e_1\cup b_2e_2\cup\ldots\cup b_{n-1},$

then

$$(a_1\cup b_1)e_1\cup(a_2\cup b_2)e_2\cup\ldots\cup(a_{n-1}\cup b_{n-1})$$

is a monotonic representation of $x\cup y$, and

$$a_1b_1e_1\cup a_2b_2e_2\cup\ldots\cup a_{n-1}b_{n-1}$$

is a monotonic representation of $x\cap y$.

Proof. It follows from (1) and from the distributivity of L.

LEMMA 1.3. *If I is a prime ideal of L, then the set $I_0 = I\cap B$ is a prime ideal of the Boolean algebra B.*

Proof. If $a, b\in I_0$, then obviously $a\cup b\in I_0$. If $a\leqslant b\in I_0$ and $a\in B$, then $a\in I$. Thus $a\in I_0$. Therefore I_0 is an ideal of B. If $a\cap b\in I_0$ and $a, b\in B$, then $a\in I$ or $b\in I$, I being a prime ideal of L. This implies $a\in I_0$ or $b\in I_0$, q. e. d.

THEOREM 1.4. *If I_0 is an ideal of the Boolean algebra B, then the set $I_i\subset L$, defined by the equivalence*

$$x\in I_i \Longleftrightarrow \begin{cases}\text{there exists a monotonic representation}\\ x = d_1e_1\cup d_2e_2\cup\ldots\cup d_{n-1} \text{ such that } d_i\in I_0,\end{cases}$$

is an ideal of L for $i = 1, 2, \ldots, n-1$.

Proof. If $x, y\in I_i$ and $(\mathrm{r}_1), (\mathrm{r}_2)$ are their respective monotonic representations, such that $a_i\in I_0$ and $b_i\in I_0$, then $a_i\cup b_i\in I_0$. Thus $x\cup y\in I_i$, by 1.2.

Now, if x and y have monotonic representations (r_1) and (r_2) respectively, $x\leqslant y$, and $b_i\in I_0$, then

$$x = x\cap y = a_1b_1e_1\cup a_2b_2e_2\cup\ldots\cup a_{n-1}b_{n-1}$$

is a monotonic representation of x such that $a_ib_i\in I_0$. This proves that $x\in I_i$, q. e. d.

THEOREM 1.5. *If the ideal I_0 is a prime ideal of the Boolean algebra B, then the ideal I_i defined in 1.4 is a prime ideal of L.*

Proof. We have only to prove that the relation $x\cap y\in I_i$ implies $x\in I_i$ or $y\in I_i$.

For this purpose let us assume that there is a monotonic representation

$$x\cap y = c_1e_1\cup c_2e_2\cup\ldots\cup c_{n-1}$$

such that $c_i\in I_0$. Further, let (r_1) and (r_2) be arbitrary monotonic representations of x and y, respectively.

Now let us put

$$(\mathrm{r}_3)\qquad \tilde{x} = \Big(\bigcup_{j=1}^{n-1}(a_j-b_j)\cup c_1\Big)e_1\cup\Big(\bigcup_{j=2}^{n-1}(a_j-b_j)\cup c_2\Big)e_2\cup\ldots\cup(a_{n-1}-b_{n-1}\cup c_{n-1}),$$

$$(\mathrm{r}_4)\qquad \tilde{y} = \Big(\bigcup_{j=1}^{n-1}(b_j-a_j)\cup c_1\Big)e_1\cup\Big(\bigcup_{j=2}^{n-1}(b_j-a_j)\cup c_2\Big)e_2\cup\ldots\cup(b_{n-1}-a_{n-1}\cup c_{n-1}).$$

It is evident that (r_3) and (r_4) are monotonic representations.

Further

$$a_je_j = (a_j-b_j)e_j\cup a_jb_je_j \leqslant \tilde{x}\cup x\cap y = \tilde{x},\qquad j=1,\ldots,n-1,$$

whence $x\leqslant\tilde{x}$. Since the contrary inequality is obvious, it follows that $x=\tilde{x}$. In the same way $y=\tilde{y}$.

We also see at once that $(a_j-b_j)\cap(b_i-a_i)=0$ for $i,j=1,\ldots,n-1$, the representations (r_1) and (r_2) being monotonic. Hence

$$\Big(\bigcup_{j=i}^{n-1}(a_j-b_j)\cup c_i\Big)\cap\Big(\bigcup_{j=i}^{n-1}(b_j-a_j)\cup c_i\Big) = c_i\,\epsilon\, I_0.$$

Consequently

$$\Big(\bigcup_{j=i}^{n-1}(a_j-b_j)\cup c_i\Big)\,\epsilon\, I_0\quad\text{ or }\quad\Big(\bigcup_{j=i}^{n-1}(b_j-a_j)\cup c_i\Big)\,\epsilon\, I_0,$$

I_0 being prime.

This implies that $x\,\epsilon\, I_i$ or $y\,\epsilon\, I_i$, which completes the proof of the theorem.

THEOREM 1.6. *If there exists a properly ascending chain $I_1\subset I_2\subset\ldots$ $\ldots\subset I_{n-1}$ of prime ideals of P_0-lattice L, then $e_{i-1}\,\epsilon\, I_i$ and $e_i\,\bar{\epsilon}\, I_i$ for $i=1,\ldots,n-1$.*

Proof. Suppose the theorem is not true. Then there exists an integer i_0 such that one of two next conditions holds:

1° $e_{i_0-1}\,\bar{\epsilon}\, I_{i_0}$; $\quad 2^\circ$ $e_{i_0}\,\epsilon\, I_{i_0}$.

In the case 1° let us suppose that i_0 is the least integer such that 1° holds. Evidently $i_0\geqslant 2$ and $e_{i_0-2}\,\epsilon\, I_{i_0-1}$. Let x_0 be an element of L which belongs to I_{i_0} and does not belong to I_{i_0-1} (such an element exists, as the ideals $I_1,\ldots,I_{n-1}$ are different and form an ascending sequence). We consider an arbitrary monotonic representation of x_0:

$$x_0 = d_1e_1\cup d_2e_2\cup\ldots\cup d_{n-1}.$$

The ideal I_{i_0} being prime, it follows that $d_{i_0-1}\,\epsilon\, I_{i_0}$, in view of 1°.

On the other hand, $x_0\,\bar{\epsilon}\, I_{i_0-1}$ and

$$d_1e_1\cup\ldots\cup d_{i_0-2}e_{i_0-2}\leqslant e_{i_0-2}\,\epsilon\, I_{i_0-1}.$$

Hence d_{i_0-1} does not belong to I_{i_0-1}, on account of the inequality

$$d_ie_i\leqslant d_{i_0-1}\quad\text{for}\quad i\geqslant i_0.$$

Consequently

$$d_{i_0-1}\,\bar{\epsilon}\, I_{i_0-1}\quad\text{and}\quad d_{i_0-1}\,\epsilon\, I_{i_0}.$$

Therefore

$$d_{i_0-1}\,\bar{\epsilon}\, I_{i_0-1}\cap B\quad\text{and}\quad d_{i_0-1}\,\epsilon\, I_{i_0}\cap B.$$

This leads, however, to a contradiction, because, by 1.3, the sets $I_{i_0-1}\cap B$, $I_{i_0}\cap B$ are prime ideals of the Boolean algebra B; besides $I_{i_0-1}\cap$ $\cap B\subset I_{i_0}\cup B$, and every prime ideal in a Boolean algebra is maximal.

The case 2° leads to a contradiction in a similar way if we suppose that i_0 is the greatest integer for which 2° holds.

Definition 1.7. An ideal I of P_0-lattice L, which contains e_{i-1} and does not contain e_i, will be called *of order i*.

The theorem 1.6 can be now formulated as follows:

If $I_1\subset I_2\subset\ldots\subset I_{n-1}$ is a properly ascending chain of prime ideals of L, then I_i is an ideal of order i, $i=1,\ldots,n-1$.

THEOREM 1.8. *If every prime ideal in L is a member of a chain of $n-1$ properly ascending prime ideals, then for every $a\,\epsilon\, B$ and every $i=1,\ldots,$ $n-1$ the inequality $ae_i\leqslant e_{i-1}$ implies $a=0$.*

Proof. If, on the contrary, for some $a\neq 0$, $a\,\epsilon\, B$, and some i the inequality $ae_i\leqslant e_{i-1}$ holds, then the meet ae_i belongs to every prime ideal of order i, and thus a belongs to every prime ideal of order i.

There exists a prime ideal of the Boolean algebra B, say I_0, such that $a\,\bar{\epsilon}\, I_0$. Let I_i be the prime ideal of L defined in theorem 1.4.

By assumption there exists a chain

$$I_1^0\subset I_2^0\subset\ldots\subset I_{n-1}^0$$

of properly ascending prime ideals such that I_i is a member of this chain, and I_j^0 is of order j, $j=1,\ldots,n-1$, by 1.6.

Since the set $I_j^0\cap B$ is a prime ideal of B (see 1.3) and every prime ideal of a Boolean algebra is maximal, it follows that

$$I_1^0\cap B = I_2^0\cap B=\ldots=I_{n-1}^0\cap B = I_i\cap B = I_0.$$

Consequently both: a belongs to I_0 and it does not, a contradiction.

2. Epstein's definition. The following definition is due to G. Epstein [1]:

Definition 2.1. Let n be a fixed integer $\geqslant 2$. A *Post algebra* is a distributive lattice P with zero and unit, in which the following two axioms are satisfied:

I. There exist in P n fixed elements $e_0, e_1, \ldots, e_{n-1}$ such that

(a) $$0 = e_0 \leqslant e_1 \leqslant \ldots \leqslant e_{n-1} = 1,$$

(b) $$\text{if } x \in P \text{ and } xe_1 = 0, \text{ then } x = 0,$$

(c) $$\text{if } x \in P \text{ and } x \cup e_{i-1} = e_i \text{ for some } i, \text{ then } x = e_i.$$

II. For every $x \in P$ there exists a sequence $\big(C_0(x), C_1(x), \ldots, C_{n-1}(x)\big) \subset P$ such that

(d) $$C_i(x) \cap C_j(x) = 0 \quad \text{for} \quad i \neq j,$$

(e) $$\bigcup_{i=0}^{n-1} C_i(x) = 1,$$

(f) $$x = C_1(x)e_1 \cup C_2(x)e_2 \cup \ldots \cup C_{n-1}(x)e_{n-1}.$$

Now we re-write from Epstein's paper the following properties of the Post algebra P:

2.2. *The elements $e_0, e_1, \ldots, e_{n-1}$ are distinct and unique.*

2.3. *For every $x \in P$ there exists only one sequence $C_0(x), C_1(x), \ldots, C_{n-1}(x)$ satisfying* II.

2.4. *If $i \neq j$, then $C_j(e_i) = 0$, $C_i(e_i) = 1$.*

2.5. *For any $x \in P$, $C_i(x)$ $(i = 1, \ldots, n-1)$ belongs to the Boolean algebra of complemented elements of P.*

3. Some equivalent definitions. THEOREM 3.1. *A distributive lattice P with zero and unit is a Post algebra if and only if there exists a sublattice $B \subset P$ which is a Boolean algebra, and a sequence $(e_0, e_1, \ldots, e_{n-1}) \subset P$ such that*

(1) $$0 = e_0 \leqslant e_1 \leqslant \ldots \leqslant e_{n-1} = 1;$$

(2) *for any $x \in P$ there exists a sequence $(b_1, \ldots, b_{n-1}) \subset B$ such that $x = b_1e_1 \cup b_2e_2 \cup \ldots \cup b_{n-1}$;*

(3) *if $a \in B$ and $ae_i \leqslant e_{i-1}$ for some i, then $a = 0$.*

In other words: *A lattice P is a Post algebra if and only if it is a P_0-lattice satisfying* (3).

Proof of necessity. Conditions (1) and (2) are satisfied in P on account of 2.1 and 2.5.

To prove (3) let us suppose that

(*) $$ae_i \leqslant e_{i-1} \quad \text{for some } i,$$

where a belongs to the Boolean algebra of complemented elements of P. Since $-ae_i \leqslant e_i$ and $-ae_i \cup ae_i = e_i$ it follows, by (*) and (1), that $-ae_i \cup e_{i-1} = e_i$. Hence $-ae_i = e_i$ in view of I (c).

This, however, implies, by 2.3 and 2.4, that $-a = 1$. Consequently $a = 0$.

Proof of sufficiency. Now let us consider an arbitrary element x of a lattice P satisfying (1), (2), (3). Then we can write

(**) $$x = d_1e_1 \cup d_2e_2 \cup \ldots \cup d_{n-1},$$

where $d_i \in B$, $i = 1, \ldots, n-1$, and $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_{n-1}$, the lattice P being a P_0-lattice.

From the monotonic representation (**) we easily obtain, by (1), the following one:

$$x = (d_1 - d_2)e_1 \cup (d_2 - d_3)e_2 \cup \ldots \cup d_{n-1},$$

in which the Boolean coefficients are disjoint.

Let us put now

$$C_0(x) = -d_1, \quad C_i(x) = d_i - d_{i+1} \quad \text{for} \quad i = 1, 2, \ldots, n-2,$$
$$C_{n-1}(x) = d_{n-1}.$$

It is easily seen that conditions (d), (e), (f), are satisfied.

To prove axiom I (b) let us suppose that

$$x = \bigcup_{i=0}^{n-1} C_i(x)e_i \neq 0.$$

Then for some i_0 we have the inequality $C_{i_0}(x) \neq 0$. Therefore the inequality $C_{i_0}(x)e_i \leqslant e_{i-1}$ is false for every $i > 0$. In particular

$$C_{i_0}(x)e_1 \neq 0.$$

This, however, implies $C_{i_0}(x)e_1e_{i_0} \neq 0$ (in view of (1)) and consequently $xe_1 \neq 0$.

Now we are going to prove axiom I(c). For this purpose let us suppose that

$$x = b_1e_1 \cup b_2e_2 \cup \ldots \cup b_{n-1},$$

where $b_i \in B$, $i = 1, \ldots, n-1$, and

$$x \cup e_{i-1} = e_i$$

for some $i > 0$. Of course

(z_1) $$x \leqslant e_i.$$

In view of (1), (z_1) and of the distributivity of P, the representation of x can be written as follows:

(z_2) $$x = b_1e_1 \cup b_2e_2 \cup \ldots \cup b_{i-1}e_{i-1} \cup b_i'e_i,$$

where $b'_i = \bigcup_{j=i}^{n-1} b_j$. Therefore

$$e_i = x \cup e_{i-1} = e_{i-1} \cup b'_i e_i.$$

Hence

$$e_i = e_{i-1} \cup b'_i e_i.$$

On the other hand, $e_i = -b'_i e_i \cup b'_i e_i$. Then $-b'_i e_i \leqslant e_{i-1}$. Now it follows from (3) that $b'_i = 1$, and from (z_2) that $e_i \leqslant x$.

Combining the last inequality with (z_1) we have the equation $x = e_i$, q. e. d.

LEMMA 3.2. *If $ae_i = be_i$ for some i in a Post algebra $P = \langle e_0, e_1, \ldots, \ldots, e_{n-1}; B\rangle$ and $a, b \in B$, then $a = b$.*

Proof. By assumption, $-bae_i = -bbe_i = 0$. Hence $-ba = 0$, on account of (3). In the same way $-ab = 0$. Consequently $a = b$.

Now we are in a good position to give a simple proof of the following theorem of Epstein (cf. [1], p. 303, th. 7):

THEOREM 3.3. *For every element x of a Post algebra $P = \langle e_0, e_1, \ldots, \ldots, e_{n-1}; B\rangle$ there exists exactly one monotonic representation.*

Proof. Let

$$(t_1) \qquad x = a_1 e_1 \cup a_2 e_2 \cup \ldots \cup a_{n-1} = b_1 e_1 \cup b_2 e_2 \cup \ldots \cup b_{n-1}$$

be two monotonic representations of x. Then, by (1), $xe_1 = a_1 e_1 = b_1 e_1$. Hence $a_1 = b_1$ in view of 3.2.

Now we assume that $a_i = b_i$ for $0 < i < k \leqslant n-1$ and we take meets of e_k and of each of the two sides of the identity (t_1). We obtain

$$a_1 e_1 \cup \ldots \cup a_k e_k = b_1 e_1 \cup \ldots \cup b_k e_k.$$

Hence

$$-b_k a_k e_k \leqslant -b_k b_1 e_1 \cup \ldots \cup -b_k b_{k-1} e_{k-1} \cup -b_k b_k e_k \leqslant e_{k-1}.$$

It follows from (3) that $-b_k a_k = 0$.

In the same way we obtain $-a_k b_k = 0$. Consequently $a_k = b_k$ for $k = 1, \ldots, n-1$, by induction.

Definition 3.4. The uniquely determined (by 3.3) Boolean coefficients of the monotonic representation of x will be denoted by $D_1(x), \ldots, D_{n-1}(x)$, respectively.

It is easy to see that the operations D_i $(i = 1, \ldots, n-1)$ are homomorphisms of the Post algebra P into the Boolean algebra B such that $D_i(x) = x$ for $x \in B$.

THEOREM 3.5. *Let $L = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle$ be a P_0-lattice. Then the following conditions are equivalent:*

(α) *L is a Post algebra;*

(β) *if $ae_i \leqslant e_{i-1}$ for some $a \in B$ and some $i > 0$, then $a = 0$;*

(γ) *every prime ideal of L is a member of a chain of $n-1$ properly ascending prime ideals of L;*

(δ) *for every $x \in L$ there exists exactly one monotonic representation;*

(ε) *for every $x \in L$ there exists exactly one disjoint representation, i. e. a representation $x = c_1 e_1 \cup c_2 e_2 \cup \ldots \cup c_{n-1}$ such that $c_i \in B$, $i = 1, \ldots, n-1$, and $c_i \cap c_j = 0$ for $i \neq j$.*

Proof. The equivalence (α) $\Longleftrightarrow$ (β) has been already proved (see 3.1). The equivalence (δ) $\Longleftrightarrow$ (ε) is obvious. We have proved the implication (β) $\Rightarrow$ (δ) in 3.4 and (γ) $\Rightarrow$ (β) in 1.8.

It remains to prove (δ) $\Rightarrow$ (γ). Let I be any prime ideal of L and let us assume the condition (δ) to be satisfied in L. Let i_0 denote the order of I, i. e. $i_0 = \min\{i: e_i \notin I\}$. Put

$$I_0 = I \cap B.$$

Evidently $I_0 \neq 0$ and it is a prime ideal of the Boolean algebra B (see 1.3). By assumption, for an arbitrary element $x \in L$ there exists exactly one representation

$$x = d_1 e_1 \cup d_2 e_2 \cup \ldots \cup d_{n-1}$$

such that $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_{n-1}$.

The monotonic representation of x being unique, it is convenient to denote its Boolean coefficients by $D_1(x), \ldots, D_{n-1}(x)$, respectively, as in Post algebras. It is easy to see that $D_i(e_i) = 1$ and $D_i(e_{i-1}) = 0$.

Now let us consider the sets

$$I_i = \{x \in L\colon D_i(x) \in I_0\}, \qquad i = 1, \ldots, n-1.$$

By 1.5, the set I_i $(i = 1, \ldots, n-1)$ is a prime ideal of the lattice L. Since $D_i(e_i) = 1$ and $D_i(e_{i-1}) = 0$, it follows immediately that the ideal I_i is of order i.

Now we shall have to prove that $I = I_{i_0}$. Since the ideal I is prime and of order i_0, the relation $x \in I$ implies $D_{i_0}(x) \in I \cap B = I_0$. Hence $x \in I_{i_0}$.

On the other hand, if $x \in I_{i_0}$, then $D_{i_0}(x) \in I_0$, thus $D_{i_0}(x) \in I$ and finally $x \in I$, I being of order i_0. This completes the proof of the theorem.

Let us remark that the implications (α) $\Rightarrow$ (γ) and (α) $\Rightarrow$ (δ) were first proved by G. Epstein [1].

COROLLARY 3.6. *A subset I of a Post algebra $P = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle$ is a prime ideal of order i of P if and only if there exists a prime ideal I_0 of the Boolean algebra B such that the conditions*

$$x \in I \quad \text{and} \quad D_i(x) \in I_0$$

are equivalent.

COROLLARY 3.7. *If I and J are two prime ideals of the same order of a Post algebra, and $I \subset J$, then $I = J$.*

4. Post homomorphisms. Let $P = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle$ and $P' = \langle e'_0, e'_1, \ldots, e'_{m-1}; B'\rangle$ be two Post algebras.

Definition 4.1. A lattice homomorphism h of P into P' is called *Post homomorphism* provided

(h_1) $h|B$ is a Boolean homomorphism of B into B',

(h_2) $h(e_i) \in (e'_0, e'_1, \ldots, e'_{m-1})$ for every $i = 1, 2, \ldots, n-1$.

A one-to-one Post homomorphism will be called *Post isomorphism.*

THEOREM 4.2. *Every Boolean homomorphism of B into B' can be extended to a Post homomorphism of P into P'.*

Proof. Let us choose of integers $0, 1, \ldots, m-1$ a non-descending sequence $k_1, k_2, \ldots, k_{n-2}$, and let h_0 be a Boolean homomorphism of B into B'. Put

(s_1) $$h(e_i) = e'_{k_i}, \quad i = 1, \ldots, n-2,$$

(s_2) $$h(x) = h_0(D_1(x))e'_{k_1} \cup \ldots \cup h_0(D_{n-2}(x))e'_{k_{n-2}} \cup h_0(D_{n-1}(x)).$$

Since P is a distributive lattice, and since D_i is a homomorphism of P into B (see 3.4), it follows that

$$h(x \cup y) = h(x) \cup h(y) \quad \text{and} \quad h(x \cap y) = h(x) \cap h(y),$$

which proves that h is a lattice homomorphism of P into P'. Evidently $h|B = h_0$. Consequently the mapping h is a Post homomorphism.

COROLLARY 4.3. *If $m \geqslant n$ and h_0 is an isomorphism of B into B', then it may be extended to a Post isomorphism of P into P'.*

Proof. Let h_0 be a Boolean isomorphism of B into B'. We choose a properly ascending sequence $k_1, \ldots, k_{n-1}$ of integers $0, 1, \ldots, m-1$ such that $k_{n-1} = m-1$, and we make use of formulas (s_1) and (s_2).

We obtain, by 4.2, a homomorphism h of P into P'. It remains to prove that $x \neq y$ implies $h(x) \neq h(y)$.

If $x \neq y$, then there exists an index i_0 such that $D_{i_0}(x) \neq D_{i_0}(y)$. Hence $h(D_{i_0}(x)) \neq h(D_{i_0}(y))$, $h|B$ being a Boolean isomorphism. But

$$h(D_i(x)) = D_i(h(x)), \quad i = 1, 2, \ldots, n-1,$$

in view of (s_2). Consequently

$$D_{k_{i_0}}(h(x)) \neq D_{k_{i_0}}(h(y)),$$

which implies inequality $h(x) \neq h(y)$, q. e. d.

If $h|B$ is a two-valued homomorphism, then $e'_0, e'_1, \ldots, e'_{m-1}$ may be the only values of the Post homomorphism h.

Definition 4.4. If a Post homomorphism h has exactly m values $e_0, e_1, \ldots, e_{m-1}$, then it will be called an *m-valued Post homomorphism.*

THEOREM 4.5. *If h is an m-valued Post homomorphism then the set $I = \{x \in P\colon h(x) = 0\}$ is a prime ideal of P.*

Proof. Obviously it is an ideal. We are going to prove that it is prime.

From $h(x \cap y) = 0$ it follows that

(p_1) $$h(D_i(x \cap y)) \cap h(e_i) = 0 \quad \text{for every } i.$$

Now let i_0 be the least index such that $h(e_{i_0}) \neq 0$. From (3) and (p_1) we obtain

(p_2) $$h(D_{i_0}(x \cap y)) = 0.$$

But

$$D_{i_0}(x \cap y) = D_{i_0}(x) \cap D_{i_0}(y) \in B.$$

Since the set

$$\{x \in B\colon h(x) = 0\} \subset I$$

is a prime ideal of the Boolean algebra B, the homomorphism $h|B$ being two-valued, it follows from (p_2) that

$$h(D_{i_0}(x)) = 0 \quad \text{or} \quad h(D_{i_0}(y)) = 0.$$

This means, however, by the definition of i_0, that $h(x) = 0$ or $h(y) = 0$, and this completes the proof.

THEOREM 4.6. *The set of all prime ideals of order 1 of a Post algebra $P = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle$ and the set of all n-valued Post homomorphisms of P into $P' = \langle e'_0, e'_1, \ldots, e'_{n-1}; B'\rangle$ have the same cardinal.*

Proof. If h is an n-valued Post homomorphism of P into P', then the set

(s) $$I = \{x \in P\colon h(x) = 0\}$$

is a prime ideal of P of order 1. On the other hand, if I is a prime ideal of order 1, then the set $I_0 = I \cap B$ is a prime ideal of the Boolean algebra B.

Let us put

$$h_0(x) = \begin{cases} e'_0 & \text{if} \quad x \in I_0, \\ e_{n-1} & \text{if} \quad x \in B \text{ but } x \notin I_0. \end{cases}$$

It is well known that h_0 is a two-valued homomorphism of B into B'.

Now, using 4.2, we extend h_0 to a Post homomorphism h, putting $k_1 = 1,\ k_2 = 2,\ \ldots,\ k_{n-2} = n-2$.

In view of 3.6, $h(x) = 0$ for every $x \in I$. It is easy to see that h is the only Post homomorphism having this property. Thus we have just established the one-to-one mapping of the set of all prime ideals of P of order 1 onto the set of all n-valued homomorphism of P into P'.

5. Measures on Post algebras. Let $P = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle$ be a fixed Post algebra with a normed measure m_0 defined on the Boolean algebra B, and let $a_0, a_1, \ldots, a_{n-1}$ be fixed real numbers such that

$$0 = a_0 \leqslant a_1 \leqslant \ldots \leqslant a_{n-1} = 1.$$

Definition 5.1. Let m be a real function defined on P as follows:

$$m(x) = m_0\big(C_1(x)\big)a_1 + m_0\big(C_2(x)\big)a_2 + \ldots + m_0\big(C_{n-1}(x)\big).$$

We shall prove that m is a measure.

LEMMA 5.2. *If $x, y \in P$ and $x \cap y = 0$, then $D_i(x) \cup D_j(y) = 0$ for every $i, j = 1, \ldots, n-1$.*

Proof. Suppose on the contrary that

$$b = D_1(x) \cap D_2(y) \neq 0.$$

Then $be_1 \neq 0$, by (3). Hence

$$0 \neq be_1 \leqslant D_1(x)e_1 \leqslant x$$

and

$$0 \neq be_1 \leqslant D_1(y)e_1 \leqslant y.$$

Contradiction, as $x \cap y = 0$.

Since $D_i(x) \leqslant D_1(x)$ and $D_i(y) \leqslant D_1(y)$ for $i > 1$, it follows that $D_i(x) \cap D_j(y) = 0$ for $i, j = 1, \ldots, n-1$.

LEMMA 5.3. *For every sequence $\{x_k\colon k = 1, 2, \ldots\}$ of disjoint elements of P, whose join is also in P, we have*

$$C_i\Big(\bigcup_{k=1}^{\infty} x_k\Big) = \bigcup_{k=1}^{\infty} C_i(x_k), \qquad i = 1, \ldots, n-1,$$

and $C_i(x_k)$, $k = 1, 2, \ldots$, are disjoint for every i.

Proof. G. Epstein has proved (see [1], p. 313) that $D_i\Big(\bigcup_{k=1}^{\infty} x_k\Big) = \bigcup_{k=1}^{\infty} D_i(x_k)$, $i = 1, \ldots, n-1$. Therefore, and by the definitions of operations C_i and D_i, we have the following equations:

$$C_i\Big(\bigcup_{k=1}^{\infty} x_k\Big) = D_i\Big(\bigcup_{k=1}^{\infty} x_k\Big) - D_{i+1}\Big(\bigcup_{k=1}^{\infty} x_k\Big) = \bigcup_{k=1}^{\infty} D_i(x_k) - \bigcup_{k=1}^{\infty} D_{i+1}(x_k).$$

Hence, by the infinite distributivity in Post algebras (see [1], p. 313),

$$C_i\Big(\bigcup_{k=1}^{\infty} x_k\Big) = \bigcup_{k=1}^{\infty}\Big(D_i(x_k) - \bigcup_{j=1}^{\infty} D_{i+1}(x_j)\Big).$$

But, on account of 5.2,

$$D_i(x_k) - \bigcup_{j=1}^{\infty} D_{i+1}(x_j) = D_i(x_k) - D_{i+1}(x_k).$$

In consequence

$$C_i\Big(\bigcup_{k=1}^{\infty} x_k\Big) = \bigcup_{k=1}^{\infty}\big(D_i(x_k) - D_{i+1}(x_k)\big) = \bigcup_{k=1}^{\infty} C_i(x_k).$$

Finally it is easy to see, in view of 5.2, that the assumption $x_k \cap x_k' = 0$ implies $C_i(x_k) \cap C_i(x_{k'}) = 0$ for $i = 1, \ldots, n-1$ and $k \neq k'$.

THEOREM 5.4. *The function m defined in 5.1 is a normed measure.*

Proof. Since $C_i(x) \cap C_j(x) = 0$ for $i \neq j$, it follows that

$$\sum_{i=1}^{n-1} m_0\big(C_i(x)\big) \leqslant 1 \quad \text{for every} \quad x \in P,$$

the measure m_0 being normed. Thus $\sum_{i=1}^{n-1} m_0\big(C_i(x)\big)a_i \leqslant 1$, too. So the inequality

$$0 \leqslant m(x) \leqslant 1$$

has been proved for every $x \in P$.

Now let $\{x_k\colon k = 1, 2, \ldots\}$ be a sequence of disjoint elements of P whose join is in P. In view of 5.1 and 5.3 we can write

$$m\Big(\bigcup_{k=1}^{\infty} x_k\Big) = m_0\Big(\bigcup_{k=1}^{\infty} C_1(x_k)\Big)a_1 + \ldots + m_0\Big(\bigcup_{k=1}^{\infty} C_{n-1}(x_k)\Big)a_{n-1}$$

$$= \sum_{k=1}^{\infty}\big\{m_0\big(C_1(x_k)\big)a_1 + \ldots + m_0\big(C_{n-1}(x_k)\big)a_{n-1}\big\}$$

$$= \sum_{k=1}^{\infty} m(x_k).$$

This completes the proof of the theorem.

THEOREM 5.5. *The set of all $x \in P$ such that $m(x) = 0$ is an ideal of P.*

Proof. Let $I = \{x \in P\colon m(x) = 0\}$. The set $I_0 = I \cap B$ is an ideal of the Boolean algebra B. Put $i_0 = \min\{i\colon a_i \neq 0\}$ and let x be an arbitrary element of I.

Then

$$m_0(C_i(x)) = 0 \quad \text{for every} \quad i \geqslant i_0,$$

which means that $C_i(x) \in I_0$ for each $i \geqslant i_0$. Therefore $D_{i_0}(x) \in I_0$.

On the other hand, if $D_{i_0}(x) \in I_0$, then $C_i(x) \in I_0$ for each $i \geqslant i_0$. Hence $m(x) = 0$ and $x \in I$.

In consequence, by 1.4, I is an ideal of P.

Definition 5.6. If m_0 is a two-valued measure on B, then the numbers $0 = a_0, a_1, \ldots, a_{n-1} = 1$ are the only values of m. In this case the measure m will be called *n-valued measure.*

THEOREM 5.7. *If m is an n-valued measure on P, then the ideal of all elements of measure zero is prime.*

Proof. It follows from 3.6, 5.4 and from the fact that the ideal of all elements of a Boolean algebra of two-valued measure zero is prime.

6. A representation theorem. The representation problem for Post algebras was solved first by G. Epstein in his paper [1]. In this section another solution of that problem will be presented.

Let $P = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle$ be an arbitrary but fixed Post algebra.

Let $\mathfrak{X}$ be a compact topological space such that

(s_1) the Boolean algebra B is isomorphic with the field $\boldsymbol{F}$ of all clopen subsets of $\mathfrak{X}$ (see [6], p. 22),

(s_2) each element $X \neq 0$ of the field $\boldsymbol{F}$ contains at least $n-1$ points (an isolated point of X may be split in $n-1$ parts if necessary).

Therefore there exists a sequence $E_1, E_1, \ldots, E_{n-1}$ of dense subsets of $\mathfrak{X}$ such that

(d_1) $$E_1 \subset E_2 \subset \ldots \subset E_{n-1} = \mathfrak{X},$$

(d_2) $$E_{i-1} \text{ is a boundary set in } E_i,$$

for $i = 2, 3, \ldots, n-1$.

THEOREM 6.1. *The class $\boldsymbol{R}$ of subsets of $\mathfrak{X}$ of the form*

$$A_1E_1 \cup A_2E_2 \cup \ldots \cup A_{n-1}E_{n-1},$$

where $A_i \in \boldsymbol{F}$, $i = 1, \ldots, n-1$, is a Post algebra with set-theoretical union and intersection as lattice operations.

Proof. Conditions (1), (2), (3) of 3.1 must be verified. Condition (1) is obviously satisfied with $e_0 = 0$ (the empty set), $e_1 = E_1, \ldots, e_{n-1} = E_{n-1} = \mathfrak{X}$; condition (2) is fulfilled by definition if we put $\boldsymbol{F} = B$; condition (3) follows from (d_2) and from the assumption that E_1 is dense.

THEOREM 6.2. *The isomorphism h_0 of B onto $\boldsymbol{F}$ may be extended to a Post isomorphism of P onto $\boldsymbol{R}$.*

Proof. It follows immediately from 4.3.

7. Factor algebras. Let $P = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle$ be a P_0-lattice, i. e. L is a distributive lattice with $0 = e_0$ and $1 = e_{n-1}$ such that for every $x \in L$ there exists a representation

$$x = b_1e_1 \cup b_2e_2 \cup \ldots \cup b_{n-1},$$

where $b_i \in B$, $i = 1, 2, \ldots, n-1$.

Let I be an ideal of the Boolean algebra B with the property:

(w) if $b \in B$ and $be_i \leqslant e_{i-1}$ for some index $i > 0$, then $b \in I$.

LEMMA 7.1. *If I is an ideal of the Boolean algebra B with the property* (w), *and an element $x \in L$ has two monotonic representations*

(p) $$x = b_1e_1 \cup b_2e_2 \cup \ldots \cup b_{n-1} = d_1e_1 \cup d_2e_2 \cup \ldots \cup d_{n-1},$$

then

$$b_i - d_i \cup d_i - b_i \in I \quad \textit{for} \quad i = 1, \ldots, n-1.$$

Proof. Since $b_i \geqslant b_j$ and $d_i \geqslant d_j$ for $1 \leqslant i \leqslant j \leqslant n-1$, it follows from (p) that $xe_1 = b_1e_1 = d_1e_1$. Hence

$$(b_1 - d_1)e_1 = (d_1 - d_1)e_1 = 0$$

and

$$(b_1 - b_1)e_1 = (d_1 - b_1)e_1 = 0.$$

Therefore, by (w), $b_1 - d_1 \in I$ and $d_1 - b_1 \in I$.

Now let us suppose that

$$b_i - d_i \cup d_i - b_i \in I$$

for $0 < i < k \leqslant n-1$, where k is a fixed but arbitrary integer > 1.

Taking meets of e_k and of each of the two sides of the identity (p) we obtain

$$b_1e_1 \cup \ldots \cup b_ke_k = d_1e_1 \cup \ldots \cup d_ke_k,$$

whence

$$(b_1 - b_k)e_1 \cup \ldots \cup (b_{k-1} - b_k)e_{k-1} = (d_1 - b_k)e_1 \cup \ldots \cup (d_k - b_k)e_k.$$

Then

$$(d_k - b_k)e_k \leqslant e_{k-1}.$$

But the last inequality implies $d_k - b_k \in I$, by property (w).

In the same way the relation $b_k - d_k \in I$ can be proved.

Consequently, by induction

$$b_k - d_k \cup d_k - b_k \in I$$

for $k = 1, \ldots, n-1$, q. e. d.

Definition 7.2. If I is an ideal of B with property (w) and if

$$x = a_1 e_1 \cup a_2 e_2 \cup \ldots \cup a_{n-1}, \quad y = b_1 e_1 \cup b_2 e_2 \cup \ldots \cup b_{n-1}$$

are monotonic representations of two elements of L, then we shall write

$$x \equiv y \quad \text{if and only if} \quad a_i - b_i \cup b_i - a_i \epsilon I$$

for $i = 1, \ldots, n-1$.

THEOREM 7.3. *The relation $\equiv$ is an equivalence relation, i. e. it is reflexive, transitive and symmetrical.*

Proof. It follows from 7.1 and from the known properties of the relation $a - b \cup b - a \epsilon I$ in a Boolean algebra (see e. g. [6], p. 27).

THEOREM 7.4. *If $x \equiv y$ and $u \equiv v$, then $x \cup u \equiv y \cup v$ and $x \cap u \equiv y \cap v$.*

The easy proof is ommitted.

The abstract class of the relation $\equiv$, containing an element x of L, will be denoted by $[x]$. The set of all classes $[x]$, where x runs over L, will be denoted by L/I.

The set L/I becomes a distributive lattice with zero $0 = [e_0]$ and unit $1 = [e_{n-1}]$ under the following definition of lattice operations:

$$[x] \cup [y] = [x \cup y], \quad [x] \cap [y] = [x \cap y].$$

B/I is, of course, a Boolean algebra of complemented elements of the lattice L/I.

THEOREM 7.5. *If L is a P_0-lattice, $L = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle$, and I is an ideal of B with the property* (w), *then the factor lattice L/I is a Post algebra.*

Proof. By the above remarks L/I is a P_0-lattice. By property (w) of I we get property (3) of 3.1, q. e. d.

COROLLARY 7.6. *If $\langle e_0, e_1, \ldots, e_{n-1}; B\rangle$ is a Post algebra and I is any ideal of B, then the factor lattice P/I is a Post algebra.*

EXAMPLE. Let $\boldsymbol{F}$ be the field of all Lebesgue measurable sets of the real line $\mathfrak{X}$. Let M denote a fixed non-measurable subset of $\mathfrak{X}$ such that for every element $E \epsilon \boldsymbol{F}$ of positive measure the intersection $E \cap M$ is not measurable. For the existence of such a set M see Halmos [2], p. 70. The family $\boldsymbol{L}$ of all subsets $X \subset \mathfrak{X}$ of the form $X = A_1 \cap M \cup A_2$, where $A_1, A_2 \epsilon \boldsymbol{F}$, is a P_0-lattice with $n = 3$, with set-theoretical union and intersection as lattice join and meet, respectively, with the empty set as e_0, M as e_1, and the whole space as $e_2 = 1$.

Let $\boldsymbol{I}$ denote the ideal of all sets of measure zero. Clearly, it has property (w). Therefore there exists the factor lattice $\boldsymbol{L}/\boldsymbol{I}$ and it is a Post algebra.

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