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COMMENTS ON SOME WALLACE'S PROBLEMS ON TOPOLOGICAL SEMIGROUPS

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In [20] Wallace lists nine problems on topological semigroups (P 326-334). This note is intended to review the present status of this latest set, and to indicate directions in which the author feels some of the more interesting might take. I shall state each problem and follow it with my comments. In the following, *semigroup* will always mean *topological semigroup* (i. e., a Hausdorff space with a continuous associative multiplication). We shall use S to denote the semigroup, E its set of idempotents, and K the kernel (minimal ideal) when it exists.

P 326. Is it possible to construct a semigroup on the closed n-cell, $n \geqslant 2$, such that E is the boundary?

Comments on P 326. The answer is still far from known, although in the case $n = 2$, a number of results have been obtained, mostly by participants in a seminar of R. J. Koch's during the past year. For example, one can easily show that every element of S has a square root, and from this one obtains (using the methods of A. Lester Hudson [6]) that every element lies on an I-semigroup with end points on the boundary. Further properties can be established using these subsemigroups. Again, in [6] it is shown that the boundary of S cannot be a subsemigroup, for this implies the existence of idempotents in the interior.

P 327. Is it possible to construct a continuous associative multiplication on an n-sphere in such a way that (i) every element is the product of two elements, (ii) there is a zero element.

Comments on P 327. It is generally conjectured that there is no non-trivial semigroup on a sphere X with $X^2 = X$ except the groups on S^1 and S^3. In dimension 1, this was proved by Koch and Wallace [11]. If there exists a structure with non-trivial multiplication (i. e., not such that $xy = x$ or $xy = y$ for $x, y \in X$ such that $X^2 = X$, one can show that there exists one with zero, so that the problem is more general than it appears. It has been shown by Mostert and Shields [16] that if $X = S^2$ has a non-trivial connected subgroup, then $X^2 \neq X$. Wallace's generalization [21] of a portion of that paper, together with the further techniques of [16] prove the following: If X is an n-sphere with a subgroup H_e which has a subset cutting X, then either $H_e = X$, or H_e cuts X, and eXe is an L-semigroup [14] with zero whose boundary is H_e. (Hence H_e is an $n-1$ sphere and thus either $n = 2$ or $n = 4$). Further $X^2 \neq X$. The full known structure can be read from [16] by simply reading "S^4" for "S^2" and "S^3" for "circle". It is still unknown whether there is a relatively simple description for $S - eSe$ beyond that given.

P 328. If G is a compact totally disconnected metrizable group, does there exist a tree (compact, connected, locally connected, acyclic, one-dimensional, metrizable space) which is a semigroup S with identity such that the maximal subgroup is precisely G which is also the set of endpoints of S?

Comments on P 328. This has been solved in the affirmative by Hunter and Rothman [9]. One takes the unit interval $[0,1]$ under the usual multiplication (or any other with 0 acting as a zero and 1 the identity). Let $x_i \to 1$, $i = 0, 1, \ldots$, be a convergent sequence with $x_0 = 0$ and $H_0, H_1, H_2, \ldots$ be normal closed subgroups of G such that $H_0 = G \neq \neq H_l, H_i \supset H_{i+1}$, and G/H_i is finite, $i = 0, 1, 2, \ldots$ One then takes $[0,1] \times G$ modulo the following equivalence relation: $(x, g) \sim (x', g')$ if $x = x' \neq 0$, and $g'g^{-1} \epsilon H_i$ whenever $x_{i-1} < x \leqslant x_i$, $(0, g) \sim (0, g')$ for all g, g'.
The problem here is slightly more restrictive than Wallace's statement and the example is hence also a solution of that statement. In the case of the Cantor group (and another closely related question solved by Koch and McAuley — unpublished) the answer was already known.

P 329. Suppose S is a semigroup with identity which is topologically Euclidean n-space. Can compact connected subgroups containing the identity be self-linked?

Comments on P 329. For the special case $n = 3$, and any compact subgroup G (not necessarily containing the identity), Curtis [1] proved the statement in the negative. The author, with a simpler proof, has shown that there can be no self-linked subgroups of any contractible semigroups, and the existence of an identity is not assumed [13].

P 330. If S is a compact connected locally connected metrizable one-dimensional semigroup with identity, then it is known that S is either a dendrite or contains exactly one simple closed curve which coincides with the minimal ideal of S. Is there an analogous proposition for higher dimensions?

Comments on P 330. That anything analogous occurs in higher dimensions would seem remote — both in probability and closeness of the analogy. Certainly it would seem impossible to prove that there are a small number of cases where S is, say, acyclic, and a small number of cases where it is not, except in dimension one. Some of the interesting examples of Hunter [10] and Hunter and Rothman [11] show the diversity even in dimension two.

P 331. If S is a compact connected commutative semigroup with identity all of whose elements are idempotent, does S have the fixed point property?

Comments on P 331. In a conversation with Wallace, J. L. Kelley pointed out an unpublished result of his which solves this in the affirmative in case S is finite dimensional and locally connected. Kelley's theorem is as follows:

Let X be a finite dimensional locally connected continuum. If there is a retract f of $X \times X$ onto the diagonal such that $f(x, y) = f(y, x)$, then X is an absolute retract.

The general situation is still open despite the author's statement to the contrary in his review of Wallace's problems for the American Mathematical Reviews.

P 332. Let S be a compact connected semigroup, and E the set of idempotents. If $S = ESE$, does the minimal ideal K and S have the same cohomology?

Comments on P 332. If S has an identity, this is known to be true [18], and a number of even weaker conditions are sufficient. However, recently A. Lester Hudson has constructed an example of a semigroup with zero and $ESE = E$ on a 2-sphere with four "whiskers" at the zero [8].

P 333. It is a corollary to the result of P 332 that a compact connected semigroup with zero and identity is unicoherent. Is there a proof of this using only set-theoretic topology?

Comments on P 333. As of the moment, no such proof is known.

P 334. Let S be a compact semigroup and let B denote the "boundary" of S in some suitable sense.

(a) If every element of S has a square-root in S does every element of B have a square root in B (Problem of H. H. Corson).

(b) Under some interpretations of "boundary" it is known that if S has an identity, then it lies in B. Are there other useful interpretations of "boundary" for which this is so?

(c) If one assumes multiplication on B is commutative, are there agreeable conditions under which it may be shown to be commutative on S?

Comments on P 334. (a) has still not been investigated formally and shows promise of interesting and important consequences. There are a number of incidental results that give examples in support of the conjective that this is so — at least if the boundary is regular [14], and [6].

It would seem that (b) has already been solved in its best possible form by Mostert and Shields in [15], for there it is solved for relative manifolds with boundary (i. e., S compact, B a closed set such that S/B is locally Euclidean), and since the proof only used cohomology properties of such spaces, it is true for spaces (S, B) where S is compact, B closed, and S/B a cohomology manifold. It would be difficult to conceive of a more general form of "boundary".

In special cases, most notably in [15], (c) is known to be true. However, the general problem seems not yet to have been studied.

One of the most useful tools so far in virtually all the questions concerning the boundary (and also in boundary-like objects [2], [3]) has been the construction of one-parameter subsemigroups [18]. However, in the questions posed above, there seem definite limitations to their use, and it would appear other strong tools will be needed and developed in their solutions. Actually, it appears that it is not "boundary" that is important in these questions, but the existence of certain distinguished subsets which are sufficiently large (boundary-like for certain open subsets) [2, 3, 4, 5, 6, 7, 12, 14, 15, 16, 17, 21] and this is usually the boundary in case S is compact [6, 14]. The compactness restriction is not always necessary even for this distinguished set [4, 5, 12], but certainly makes life easier.

In this connection, one has the problem of Mostert and Shields [15], if S is a semigroup with identity on a manifold, and L is the boundary of the maximal connected subgroup G (which is open [15]), does L contain an idempotent? This is solved if S is the plane, but is unknown otherwise.

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