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NOTE ON APPROXIMATE DERIVATIVE

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Z. Zahorski proved in [4] that if the approximate derivative $f'_{\text{ap}}(x)$ of a function $f(x)$ $(a < x < b)$ exists at every point, except perhaps for a countable set, then it is of Baire class 3 with respect to the set of its existence. He also posed the question of whether it is always of Baire class 2. After his note had been printed, he solved this question affirmatively but he has not published the proof (1).

In this note we shall prove the following slightly stronger

THEOREM. *Let f be a finite function defined on the whole real line and let R be the set of all points x at which the approximate derivative $f'_{\text{ap}}(x)$ exists. If every point of R is a point of outer density of R, then*

(a) *there exists a countable set $Z \subset R$ such that f is of Baire class 1 with respect to $R-Z$* (2),

(b) *f'_{ap} is of Baire class 2 with respect to R.*

The proof will require the following lemmas.

LEMMA 1. *Let E be any set on the real line. Then the set $E_{(r)}$ $(E_{(l)})$ of all points at which outer right-hand (left-hand) upper density of E is positive, is a $G_{\delta\sigma}$-set.*

Proof. We may assume that E is measurable and we may confine ourselves to the case of right-hand upper density. Then, let $E_{n,k}$, for $n, k = 1, 2, \ldots$, denote the set of all points x such that for every interval $[x, \bar{x}]$, the inequality $\bar{x}-x < 1/k$ implies $\dfrac{|E \cap [x, \bar{x}]|}{\bar{x}-x} \leqslant 1/n$. Since each set $E_{n,k}$ is closed and since $E_{(r)} = (\bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{n,k})'$, the lemma is proved.

LEMMA 2. *If functions f and g are upper (lower) approximately semi-*

(1) The author has been informed by Prof. Z. Zahorski that the proof of his theorem had also been given by A. Matysiak in [2].

(2) (a) was given without proof in [4] in a slightly weaker form.

-continuous on the right at x_0 *then* $\max(f, g)$ *and* $\min(f, g)$ *are also upper (lower) approximately semi-continuous on the right at* x_0. *The same is true, if we replace "the right" by "the left".*

The proof is quite similar to that of the corresponding assertion for ordinary semi-continuity.

LEMMA 3. *Let* f *be a bounded measurable function and let* $F(x) = \int_a^x f(t)\,dt$. *If* f *is upper (lower) approximately semi-continuous on the right at* x_0, *then*

$$\overline{F}^+(x_0) \leqslant f(x_0) \qquad (\underline{F}^+(x_0) \geqslant f(x_0)).$$

The corresponding assertion is true if we replace "the right" by "the left".

The proof is similar to that of Theorem (10.7), p. 132 in [3].

LEMMA 4. *Let* f *be a finite function. Then the set* R *of all points* x *at which the approximate derivative* $f'_{\mathrm{ap}}(x)$ *exists is measurable, and* f *is of Baire class* 2 *with respect to* R.

Proof. Let us remark that for every real a the sets $\{x\colon \overline{\lim\mathrm{ap}}_{t\to x+0} f(t) > a\}$, $\{x\colon \overline{\lim\mathrm{ap}}_{t\to x-0} f(t) > a\}$, $\{x\colon \underline{\lim\mathrm{ap}}_{t\to x+0} f(t) < a\}$, and $\{x\colon \underline{\lim\mathrm{ap}}_{t\to x-0} f(t) < a\}$ are $G_{\delta\sigma}$. It is sufficient to show this for the first set. For this purpose let E_n, for each integer $n > 0$, denote the set of all points at which outer right-hand upper density of the set $\{t\colon f(t) > a+1/n\}$ is positive. From the definition of approximate limits it follows that

$$\Big\{x\colon \overline{\lim\mathrm{ap}}_{t\to x+0} f(t) > a\Big\} = \bigcup_{n=1}^{\infty} E_n.$$

Hence, in view of Lemma 1, we obtain the required result. Further, let us observe that the sets $\{x\colon f(x) > \overline{\lim\mathrm{ap}}_{t\to x+0} f(t)\}$, $\{x\colon f(x) > \overline{\lim\mathrm{ap}}_{t\to x-0} f(t)\}$, $\{x\colon f(x) < \underline{\lim\mathrm{ap}}_{t\to x+0} f(t)\}$, and $\{x\colon f(x) < \underline{\lim\mathrm{ap}}_{t\to x-0} f(t)\}$ are of measure zero. It is sufficient to show this for the first set. For this purpose let us note that

$$\Big\{x;\, f(x) > \overline{\lim\mathrm{ap}}_{t\to x+0} f(t)\Big\} = \bigcup_{n=1}^{\infty} \Big\{x;\, f(x) > w_n > \overline{\lim\mathrm{ap}}_{t\to x+0} f(t)\Big\},$$

where $\{w_n\}_{n=1,2,\ldots}$ is a sequence of all rational numbers. From the definition of approximate limits and from Lebesgue's density theorem, it follows that each set $\{x\colon f(x) > w_n > \overline{\lim\mathrm{ap}}_{t\to x+0} f(t)\}$ is of measure zero and therefore $\{x\colon f(x) > \overline{\lim\mathrm{ap}}_{t\to x+0} f(t)\}$ is also of measure zero. Let us now put

$$R_0 = \Big\{x\colon \overline{\lim\mathrm{ap}}_{t\to x+0} f(t) \leqslant \underline{\lim\mathrm{ap}}_{t\to x-0} f(t)\Big\} \cup \Big\{x\colon \overline{\lim\mathrm{ap}}_{t\to x-0} f(t) \leqslant \underline{\lim\mathrm{ap}}_{t\to x+0} f(t)\Big\}.$$

From the preceding remarks, it follows that R_0 is measurable, and f is measurable on R_0. Since at each point of R, at least one of the following conditions

$$\text{(1)} \qquad \overline{\lim\mathrm{ap}}_{t\to x+0} f(t) \leqslant f(x) \leqslant \underline{\lim\mathrm{ap}}_{t\to x-0} f(t),$$

$$\text{(2)} \qquad \overline{\lim\mathrm{ap}}_{t\to x-0} f(t) \leqslant f(x) \leqslant \underline{\lim\mathrm{ap}}_{t\to x+0} f(t)$$

is satisfied, we get $R \subset R_0$. Put

$$h(x) = \begin{cases} f(x) & \text{for} \quad x \in R_0, \\ 0 & \text{for} \quad x \notin R_0. \end{cases}$$

Since h is measurable, it follows from Theorem (11.2), p. 299 in [3], that the set Q of all points x at which $h'_{\mathrm{ap}}(x)$ exists, is measurable. Hence, since $Q \cap R_0$ and R differ at most by a set of measure zero, R is also measurable. We now proceed to the proof of the second part of the lemma. For this purpose, put

$$P = \Big\{x\colon \overline{\lim\mathrm{ap}}_{t\to x+0} f(t) < \underline{\lim\mathrm{ap}}_{t\to x-0} f(t)\Big\} \cup \Big\{x\colon \overline{\lim\mathrm{ap}}_{t\to x-0} f(t) < \underline{\lim\mathrm{ap}}_{t\to x+0} f(t)\Big\}.$$

Since P is countable (see [1]), it is sufficient to show that f is of Baire class 2 with respect to the set $A = R-P$. For this purpose observe that

$$\text{(3)} \qquad \{x;\, f(x) \leqslant a,\, x\in A\}$$
$$= \{x\colon \overline{\lim\mathrm{ap}}_{t\to x+0} f(t) \leqslant a,\, x\in A\} + \{x\colon \overline{\lim\mathrm{ap}}_{t\to x-0} f(t) \leqslant a,\, x \in A\},$$

where a is any finite number. In view of the remark made at the beginning of the proof, (3) implies that $\{x;\, f(x) \leqslant a,\, x\in A\}$ is an $F_{\sigma\delta}$-set with respect to A. By symmetry, we obtain the same for the set $\{x\colon f(t) \geqslant a,\, x\in A\}$. This completes the proof.

For any real a, any finite function f defined on a set X, and any point $x \in X$, let

$$M(f, X, a, x) = \Big\{t\colon \frac{f(t)-f(x)}{t-x} > a,\, t\in X,\, t \neq x\Big\}.$$

LEMMA 5. *Let* f *be a finite function of Baire class* 1 *with respect to a measurable set* X *of finite measure. Then, for arbitrary* a *and* β, *the set* $\{x\colon |M(f, X, a, x)| > \beta,\, x \in X\}$ *is an* F_σ*-set with respect to* X.

Proof. There exists a sequence $\{f_n\}_{n=1,2,\ldots}$ of finite functions continuous on X and such that $\lim_{n\to\infty} f_n(x) = f(x)$, for $x \in X$. Let us put for the integers $n, k > 0$

$$N_k(f_n, X, a, x) = \Big\{t\colon \frac{f_n(t)-f_n(x)}{t-x} \geqslant a + \frac{1}{k},\, t\in X,\, t \neq x\Big\}.$$

It is easy to see that

$$\left\{x\colon |M(f, X, a, x)| > \beta, x \in X\right\} = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \left\{x\colon |N_k(f_n, X, a, x)| \geqslant \beta + \frac{1}{k}, x \in X\right\}.$$

Hence, it is sufficient to show that each set $\left\{x\colon |N_k(f_n, X, a, x)| \geqslant \beta + \frac{1}{k}, x \in X\right\}$ is closed with respect to X. For this purpose, let x_s, $s = 1, 2, \ldots$, belong to the fixed set $\left\{x\colon |N_k(f_n, X, a, x)| \geqslant \beta + \frac{1}{k}, x \in X\right\}$, and let $\lim_{s\to\infty} x_s = x_0$, $x_0 \in X$. Then we have

$$\overline{\lim_{s\to\infty}} N_k(f_n, X, a, x_s) - \{x_0\} \subset N_k(f_n, X, a, x_0),$$

and therefore $\overline{\lim_{s\to\infty}} |N_k(f_n, X, a, x_s)| \leqslant |N_k(f_n, X, a, x_0)|$; hence we infer that $|N_k(f_n, X, a, x_0)| \geqslant \beta + \frac{1}{k}$. This completes the proof.

We now proceed to the proof of the theorem. It follows from Lemma 4 that the set R is measurable and f is measurable on R. Since each point of R is a point of density for R, we may assume that f is measurable on the whole real line (3). Let us now put $f_n = \max\big(-n, \min(n, f)\big)$ for $n = 1, 2, \ldots$ By Lemma 2 and Lemma 3, in view of (1) and (2), we infer that for every integer $n > 0$ and for each point $x \in R$, one at least of the following conditions is satisfied:

$$\overline{F}_n^+(x) \leqslant f_n(x) \leqslant \underline{F}_n^-(x) \quad \text{or} \quad \overline{F}_n^-(x) \leqslant f_n(x) \leqslant \underline{F}_n^+(x), \tag{4}$$

where $F_n(x) = \int_0^x f_n(t)\,dt$. Now let

$$Z = \bigcup_{n=1}^{\infty} \{x\colon \overline{F}_n^+(x) < \underline{F}_n^-(x)\} \cup \{x\colon \overline{F}_n^-(x) < \underline{F}_n^+(x)\}.$$

By Theorem (1.1), p. 261 in [3], the set Z is countable. We shall prove that f is of Baire class 1 with respect to the set $Q = R - Z$. In view of (4) it follows that, for any a and $\beta > a$,

$$\begin{aligned} &\{x\colon f_n(x) < \beta, x \in Q\} = \{x\colon \overline{F}_n^+(x) < \beta, x \in Q\} \cup \{x\colon \overline{F}_n^-(x) < \beta, x \in Q\}, \\ &\{x\colon f_n(x) > a, x \in Q\} = \{x\colon \underline{F}_n^+(x) > a, x \in Q\} \cup \{x\colon \underline{F}_n^-(x) > a, x \in Q\}. \end{aligned} \tag{5}$$

(3) This assumption is made only in the proof of (a).

The sets on the right-hand side of (5) are F_σ-sets with respect to Q (see the proof of Theorem (4.1), p. 170, in [3]), and hence the sets $\{x\colon f_n(x) > a, x \in Q\}$ and $\{x\colon f_n(x) < \beta, x \in Q\}$ are also F_σ with respect to Q. It is easy to see that

$$\{x\colon a < f(x) < \beta, x \in Q\} = \bigcup_{n=1}^{\infty} \{x\colon a < f_n(x) < \beta, x \in Q\} \cap \{x\colon -n < f_n(x) < n, x \in Q\},$$

and therefore the set $\{x; a < f(x) < \beta, x \in Q\}$ is an F_σ-set with respect to Q. This completes the proof of the first part of the theorem (4). We now proceed to the second part. It is sufficient to show that $f_{\rm ap}$ is of Baire class 2 with respect to Q. For this purpose, let $\{I_s\}_{s=1,2,\ldots}$ denote a sequence of all closed different rational intervals, and $N(q)$ $(q = 1, 2, \ldots)$ the set of all integers $s > 0$ such that $|I_s| < 1/q$. It is easy to see that for every real a

$$\{x\colon f'_{\rm ap}(x) \geqslant a, x \in Q\} = \bigcap_{n=1}^{\infty} \bigcap_{q=1}^{\infty} \bigcup_{s \in N(q)} A_{n,q,s}, \tag{6}$$

where $A_{n,q,s}$ $(n, q = 1, 2, \ldots, s \in N(b))$ is the set of all points x belonging to $Q \cap I_s$ such that $|M(f, Q \cap I_s, a - 1/n, x)| > \frac{1}{2}|I_s|$. From Lemma 5 it follows that each set $A_{n,q,s}$ is F_σ with respect to $Q \cap I_s$, and therefore also an F_σ-set with respect to Q. By (6) the set $\{x\colon f'_{\rm ap}(x) \geqslant a, x \in Q\}$ is $F_{\sigma\delta}$ with respect to Q. By symmetry, the set $\{x\colon f'_{\rm ap}(x) \leqslant a, x \in Q\}$ is also $F_{\sigma\delta}$ with respect to Q, and therefore $f'_{\rm ap}$ is of Baire class 2 with respect to Q. This completes the proof of the theorem.

(4) From the proof it follows that (a) holds for a measurable function f without any assumption concerning the density of the set R.

REFERENCES

[1] S. Kempisty, *Sur les fonctions approximativement discontinues*, Fundamenta Mathematicae 6 (1924), p. 6-8.

[2] A. Matysiak, *O granicach i pochodnych aproksymatywnych*, thesis (in Polish), Łódź 1960.

[3] S. Saks, *Theory of the integral*, Warszawa-Lwów 1937.

[4] Z. Zahorski, *Sur la classe de Baire des dérivées approximatives d'une fonction quelconque*, Annales de la Société Polonaise des Mathématiques 21 (1948), p. 306-323.

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