such that $\lim\|x_n\|_p = 0$, and $\|f * x_n\|_r \geqslant C$, $n = 1, 2, \ldots$ Taking $|x_n(t)|$ instead of $x_n(t)$ we have also $\lim\| |x_n(t)| \|_p = 0$ and $\|f * |x_n|\|_r \geqslant C$, so we may assume that $x_n(t) \geqslant 0$. Taking a suitable sequence of positive scalars a_n we obtain $\lim\|a_n x_n\|_p = 0$, and $\lim\|f * a_n x_n\|_r = \infty$, so by passing, if necessary, to a subsequence we may assume that

$$\|x_n\|_p \leqslant 1/2^n \quad \text{and} \quad \|f * x_n\|_r \geqslant n$$

for $n = 1, 2, \ldots$ Now let $y = \sum_{n=1}^{\infty} x_n$; we have $y \in L_p$, so $\|f * y\|_r < \infty$. On the other hand, $y \geqslant x_n$, and so $f * y \geqslant f * x_n \geqslant 0$. Consequently $\|f * y\|_r \geqslant \|f * x_n\|_r \geqslant n$ which is the contradiction mentioned above, q. e. d.

COROLLARY. *If $L_p(G)$, $p \geqslant 1$, is an algebra under the convolution, then it is a Banach algebra (i. e. there exists a submultiplicative norm equivalent to the norm $\|x\|_p$).*

We may formulate now our main result

THEOREM 1. *Let G be a locally compact group and $p > 2$; then the space $L_p(G)$ is an algebra under the convolution if and only if the group G is compact.*

We may rewrite also the main result of [3] in the following form:

THEOREM 2. *Let G be a locally compact Abelian group and $p > 1$; then the space $L_p(G)$ is an algebra under the convolution if and only if the group G is compact.*

The following problem is open:

P 392. Is the conclusion of theorem 1 true for $1 < p \leqslant 2$?

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COLLOQUIUM MATHEMATICUM

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ON DECOMPOSITION OF A COMMUTATIVE p-NORMED ALGEBRA INTO A DIRECT SUM OF IDEALS

BY

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1. In the theory of commutative complex Banach algebras it is known that a Banach algebra A is decomposable into a direct sum of its two non-trivial ideals

$$A = I_1 \oplus I_2, \tag{1.1}$$

if and only if the compact space $\mathfrak{M}$ of all multiplicative linear functionals of A may be written in the form

$$\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2, \tag{1.2}$$

where $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are disjoint closed subsets of $\mathfrak{M}$.

The decompositions (1.1) and (1.2) are equivalent to the decomposition of the unit $e \in A$ into a sum of two non-zero idempotents

$$e = e_1 + e_2, \tag{1.3}$$

where

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 e_2 = 0. \tag{1.4}$$

When we have the decomposition (1.3) with (1.4) the decompositions (1.1) and (1.2) may be written by means of the formulas

$$I_1 = e_1 A, \quad I_2 = e_2 A, \tag{1.5}$$

and

$$\mathfrak{M}_1 = \{f \in \mathfrak{M}: f(e_1) = 1\}, \quad \mathfrak{M}_2 = \{f \in \mathfrak{M}: f(e_2) = 1\}. \tag{1.6}$$

This result was obtained by Šilov [4], who used analytic functions of several variables of elements of A. Here is presented a similar result for the class of p-normed algebras.

2. A *p-normed algebra* A is a metric algebra complete in the norm $\|x\|$ satisfying

$$\|xy\| \leqslant \|x\|\|y\|, \quad \|ax\| = |a|^p\|x\|,$$

where $x, y \in A$, a is a scalar, and p is a fixed real number satisfying $0 < p \leqslant 1$. Every locally bounded complete metric algebra is isomorphic and homeomorphic with a p-normed algebra ([5], theorem 1), and the class of p-normed algebras is more wide then that of Banach algebras. In this paper we assume A to be a commutative complex p-normed algebra with unit e.

The greatest part of the theory of Banach algebras is true, also for p-normed algebras. We list out some of these facts, which will be needed in the sequel.

2.1. If $\mathfrak{M}$ is set of all multiplicative linear functionals of a p-normed algebra A (topologized as in [1], so it forms a compact space), then

$$\lim \sqrt[n]{\|x^n\|} = \max_{f \in \mathfrak{M}} |f(x)|^p, \tag{2.1.1}$$

and the radical of A may be defined (see [6]) as

$$\operatorname{rad} A = \{x \in A: \lim \sqrt[n]{\|x^n\|} = 0\}. \tag{2.1.2}$$

2.2. If $x \in A$, then the spectrum $\sigma(x)$ of x is defined as

$$\sigma(x) = \{f(x): f \in \mathfrak{M}\}. \tag{2.2.1}$$

It is a compact subset of the complex plane, and for every analytic function $\Phi(z)$ defined in an open neighbourhood U of $\sigma(x)$ there exists in A an element y such that (see [7])

$$f(y) = \Phi(f(x)) \quad \text{for every} \quad f \in \mathfrak{M}. \tag{2.2.2}$$

3. Now we proceed to prove that if A is a commutative complex p-normed algebra with unit e, then (1.1) is equivalent to (1.2). But if (1.1) holds, then taking the subsequent decomposition of the unit e, we get two idempotents satisfying (1.4), so the decomposition (1.2) is given by (1.6). It is only to be shown, that (1.2) implies (1.1), or in view of the formula (1.5), that (1.2) implies the existence of an idempotent e_1 such that

$$\mathfrak{M}_1 = \{f \in \mathfrak{M}: f(e_1) = 1\}. \tag{3.1}$$

Note, that if a is an idempotent, then either $f(a) = 1$, or $f(a) = 0$ for every $f \in \mathfrak{M}$. Setting $e_2 = e - e_1$ we get the second idempotent, and by (1.5) we get (1.1). So we prove the existence of e_1 such that the implication (1.2) $\to$ (3.1) holds.

Let the space $\mathfrak{M}$ satisfy (1.2). Put

$$\|x\|_s = \max_{f \in \mathfrak{M}} |f(x)| = (\lim \sqrt[n]{\|x^n\|})^{1/p}.$$

By 2.1 it is a submultiplicative homogeneous norm in the algebra with unit $A' = A/\operatorname{rad} A$. The space of multiplicative linear functionals is the same for A' as that of A. The completion $\bar{A}$ of A' in the norm $\|x\|_s$ is a semisimple Banach algebra, and its space $\mathfrak{M}$ satisfies (1.2). So there exists in $\bar{A}$ an idempotent $\bar{e}_1$ such that

$$\mathfrak{M}_1 = \{f \in \mathfrak{M}: f(\bar{e}_1) = 1\},$$

and

$$\mathfrak{M}_2 = \{f \in \mathfrak{M}: f(\bar{e}_1) = 0\}$$

(we use here the same symbols for elements of $\mathfrak{M}$ treated as functionals on A, and as those for $\bar{A}$).

But A' is dense in $\bar{A}$, so we may choose such $X \in A'$, that

$$|f(X) - 1| < 1/3 \quad \text{for} \quad f \in \mathfrak{M}_1, \tag{3.2}$$

$$|f(X)| < 1/3 \quad \text{for} \quad f \in \mathfrak{M}_2. \tag{3.3}$$

This holds for any element of A' sufficiently close to $\bar{e}_1$. The same hold for any $x \in X$, i. e. $x \in A$. Hence

$$\sigma(x) \subset K(0, 1/3) \cup K(1, 1/3),$$

where $K(z_0, r)$ is a disc with centre z_0 and radius r. The function

$$\Phi(z) = \begin{cases} 1 & \text{for} \quad z \in K(1, 1/3), \\ 0 & \text{for} \quad z \in K(0, 1/3) \end{cases}$$

is analytic in $K(0, 1/3) \cup K(1, 1/3)$, so, by 2.2, there exists an element $e' \in A$ such that

$$f(e') = \begin{cases} 1 & \text{for} \quad f \in \mathfrak{M}_1, \\ 0 & \text{for} \quad f \in \mathfrak{M}_2. \end{cases}$$

Hence, by 2.1,

$$e'^2 = e' \operatorname{mod}(\operatorname{rad} A).$$

It is sufficient to apply now the following

LEMMA. *Let e' be an element of a p-normed algebra A, which is idempotent modulo radical; then there exists in A an idempotent e_1 which is equal, modulo radical, to e'.*

This lemma is formulated in the Rickart's book ([3], theorem (2.3.9)) for Banach algebras, but its proof is also valid for p-normed algebras. So we have proved the following

THEOREM. *Let A be a commutative complex p-normed algebra with unit e. Let $\mathfrak{M}$ be the compact space of its multiplicative linear functionals. Then A may be written in the form* (1.1) *of direct sum of its ideals if and* *only if the space* $\mathfrak{M}$ *may be written in the form* (1.2) *of union of two disjoint closed subsets. The relations between these two decompositions are given by* (1.3), (1.5) *and* (1.6).

Remark. The proof given in [4] cannot be indirectly used here, because we do not know whether the theory of analytic functions of several variables known for the Banach algebras is true for the p-normed algebras. So we pose the following problem:

P 393. Let A be a p-normed algebra, $\mathfrak{M}$ its multiplicative linear functional space. Let $x_1, \ldots, x_n \in A$. The joint spectrum of the n-tuple $(x_1, \ldots, x_n)$ is defined as

$$\sigma(x_1, \ldots, x_n) = \{(f(x_1), \ldots, f(x_n)) \in C^n : f \in \mathfrak{M}\}.$$

Let $\Phi(z_1, \ldots, z_n)$ be an analytic function of n complex variables defined on an open subset $U \subset C^n$ containing the spectrum $\sigma(x_1, \ldots, x_n)$. Does there exist in A an element y such that

$$f(y) = \Phi(f(x_1), \ldots, f(x_n)) \quad \text{for every} \quad f \in \mathfrak{M}?$$

A similar problem may be posed also for the locally analytic operations in a p-normed algebra (for the definition cf. [2], § 13).

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*ON A NEW APPROACH TO CONTINUOUS METHODS OF SUMMATION**

BY

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Introduction. In my preceding paper [4] I gave a definition of the continuous methods of limitation as follows:

Definition 1. A functional method of limitation A described by the sequence $\{a_\nu(t)\}$ of functions $a_\nu(t)$ defined in the interval $t_0 \leqslant t < T$ $(T \leqslant +\infty)$ is called *continuous method* if

(i) all functions $a_\nu(t)$ are continuous in this interval $t_0 \leqslant t < T$,

(ii) there exists an increasing sequence $t_0, t_1, t_2, \ldots, t_m, \ldots$ tending to T such that for every sequence $x = \{\xi_\nu\}$ the convergence of the series

$$A(t, x) = \sum_{\nu=0}^{\infty} a_\nu(t)\, \xi_\nu$$

for $t = t_m$ and $t = t_{m+1}$ implies uniform convergence of the series $A(t, x)$ in the interval $t_m \leqslant t \leqslant t_{m+1}$.

Definition 2. The sequence $x = \{\xi_\nu\}$ is called *limitable by the continuous method A to the number ξ*, if

1° the series $A(t, x)$ is convergent for $t_0 \leqslant t < T$,

2° the limit $\lim\limits_{t \to T-} A(t, x) = \xi$ exists.

Definition 3. The set A^* of sequences $x = \{\xi_\nu\}$ limitable by the method A is called the *field of the method A*.

Now we shall give a new definition of a continuous method of limitation:

Definition 4. A functional method of limitation $A = \{a_\nu(t)\}$ $(t_0 \leqslant t < T)$ will be called *continuous method* (*in a new sense*) if this method satisfies the condition (i).

* written during my stay at Tulane University.