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REMARKS ON INDEPENDENCE IN FINITE ALGEBRAS

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I. In this note we adopt the definitions and notations given by E. Marczewski in [1] and [2]. Our purpose is to prove two theorems about independent elements in finite algebras without algebraic constants. Let $(A; \boldsymbol{F})$ be an *algebra*, i. e. a set A of elements and a class $\boldsymbol{F}$ of fundamental operations consisting of A-valued functions of several variables running over A. If $A = \{a, b, \ldots\}$ and $\boldsymbol{F} = \{f, g, \ldots\}$, we shall sometimes write $(A; f, g, \ldots)$ or $(a, b, \ldots; f, g, \ldots)$ instead of $(A; \boldsymbol{F})$. We denote by $\boldsymbol{A}^{(n)}$ $(n = 1, 2, \ldots)$ the class of all *algebraic operations* of n variables, i. e. the smallest class of operations containing so called *trivial* operations

$$e_k^{(n)}(x_1, x_2, \ldots, x_n) = x_k \quad (k = 1, 2, \ldots, n),$$

and closed under the composition with the fundamental operations. The values of constant algebraic operations are called *algebraic constants*. If all algebraic operations are trivial, then the algebra is called *trivial*.

Following Marczewski, we say that elements $a_1, a_2, \ldots, a_n$ of A are *independent* if for any $f, g \in \boldsymbol{A}^{(n)}$ the equation

$$f(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n)$$

implies the identity of f and g in A. Henceforth, sets of independent elements will be called briefly *independent sets*.

THEOREM 1. *Let n and m be integers satisfying the inequalities $n > m$, $n > 3$ and let $(A; \boldsymbol{F})$ be a finite algebra without algebraic constants containing at least $n+m$ elements. Suppose that there exists an m-element subset M of A such that each n-element subset of $A \setminus M$ is independent. Then each n-element subset of A is independent.*

First of all we shall show by counter-examples that all assumptions of this theorem are essential.

1. Let N be the set of all non-negative integers and let f_0 be a one-to-one mapping of all ordered n-tuples $i_1, i_2, \ldots, i_n$ of different positive integers into the set of positive integers satisfying the condition

$$f_0(i_1, i_2, \ldots, i_n) > i_s \quad (s = 1, 2, \ldots, n).$$

For instance, as a mapping f_0 we can take the mapping

$$f_0(i_1, i_2, \ldots, i_n) = p_1^{i_1} p_2^{i_2} \ldots p_n^{i_n},$$

where $p_1, p_2, \ldots, p_n$ are primes. We extend the mapping f_0 over all n-tuples of non-negative integers by setting $f_0(i_1, i_2, \ldots, i_n) = i_1$ in all remaining cases. Consider the algebra $(N; f_0)$. Since $f_0 \neq e_1^{(n)}$ and $f_0(0, i_2, \ldots, i_n) = 0 = e_1^{(n)}(0, i_2, \ldots, i_n)$, we infer that all n-element subsets of N containing 0 are dependent. From the following Lemma it follows that the algebra in question contains no algebraic constant (formula (1)) and that every n-element set of positive integers is independent (formula (2)).

LEMMA. *If f is a non-trivial algebraic operation of n variables in $(N; f_0)$, then for any system $i_1, i_2, \ldots, i_n$, of different positive integers, the inequality*

$$f(i_1, i_2, \ldots, i_n) > i_s \quad (s = 1, 2, \ldots, n) \tag{1}$$

holds. Moreover, if g and h are different algebraic operations of n variables in $(N; f_0)$, then for any system $i_1, i_2, \ldots, i_n$ of different positive integers the inequality

$$g(i_1, i_2, \ldots, i_n) \neq h(i_1, i_2, \ldots, i_n) \tag{2}$$

holds.

Proof. The class $N^{(n)}$ of all algebraic operations of n variables in $(N; f_0)$ is the union $N^{(n)} = \bigcup_{k=0}^{\infty} N_k^{(n)}$, where the classes $N_k^{(n)}$ are defined recursively as follows

$$N_0^{(n)} = \{e_1^{(n)}, e_2^{(n)}, \ldots, e_n^{(n)}\},$$

$$N_{k+1}^{(n)} = N_k^{(n)} \cup \{f_0(f_1, f_2, \ldots, f_n): f_j \in N_k^{(n)}, j = 1, 2, \ldots, n\} \quad (k = 0, 1, \ldots)$$

(see [2], p. 47). Let f, g, and h belong to $N_k^{(n)}$. We shall prove the Lemma by induction with respect to k. If $k = 0$ or 1, then our assertion is a direct consequence of the definition of the mapping f_0. Suppose now that the Lemma is true for all operations from $N_k^{(n)}$, where $k > 0$. For any operation f from $N_{k+1}^{(n)} \setminus N_k^{(n)}$ there exist operations $f_1, f_2, \ldots, f_n$, belonging to $N_k^{(n)}$, such that

$$f = f_0(f_1, f_2, \ldots, f_n). \tag{3}$$

Since $f \notin N_k^{(n)}$, all operations $f_1, f_2, \ldots, f_n$ are different and at least one of them, say f_j, is non-trivial. Hence it follows that all numbers $f_1(i_1, i_2, \ldots, i_n)$, $f_2(i_1, i_2, \ldots, i_n), \ldots, f_n(i_1, i_2, \ldots, i_n)$ are different and $f_j(i_1, i_2, \ldots, i_n) > i_s$ $(s = 1, 2, \ldots, n)$, whenever $i_1, i_2, \ldots, i_n$ are different positive integers. Consequently, by (3) and by the definition of the mapping f_0 we have the inequality

$$f(i_1, i_2, \ldots, i_n) > f_j(i_1, i_2, \ldots, i_n) > i_s \quad (s = 1, 2, \ldots, n).$$

Thus formula (1) holds for all non-trivial operations in $N_{k+1}^{(n)}$. By inductive assumption and formula (1) it suffices to prove (2) for non-trivial operations g and h from $N_{k+1}^{(n)}$. Therefore we may assume that $g = f_0(g_1, g_2, \ldots, g_n)$ and $h = f_0(h_1, h_2, \ldots, h_n)$, where the operations $g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_n$ belong to $N_k^{(n)}$, all the operations $g_1, g_2, \ldots, g_n$ are different, and all the operations $h_1, h_2, \ldots, h_n$ are different. Moreover, since $g \neq h$, there exists an index r $(1 \leqslant r \leqslant n)$ such that

$$g_r \neq h_r. \tag{4}$$

Given a system $i_1, i_2, \ldots, i_n$ of different positive integers, we put

$$u_s = g_s(i_1, i_2, \ldots, i_n), \quad v_s = h_s(i_1, i_2, \ldots, i_n) \quad (s = 1, 2, \ldots, n).$$

By inductive assumption $u_1, u_2, \ldots, u_n$ are different positive integers and $v_1, v_2, \ldots, v_n$ are different positive integers. Moreover, by (4), $u_r \neq v_r$. Thus, by the definition of the mapping f_0, we have the inequality

$$g(i_1, i_2, \ldots, i_n) = f_0(u_1, u_2, \ldots, u_n) \neq f_0(v_1, v_2, \ldots, v_n) = h(i_1, i_2, \ldots, i_n),$$

which completes the proof.

The algebra $(N; f_0)$ shows that the assumption of finiteness of the algebra in Theorem 1 is essential.

S. Świerczkowski proved in [6] that *if $n > 3$ and all n-element subsets of a finite algebra are independent, then all algebraic operations of n variables are trivial.* We note that in the subalgebra $(1, 2, \ldots; f_0)$ all n-element subsets are independent. However, f_0 is a non-trivial operation of n variables. Thus the assumption of finiteness of the algebra in Świerczkowski's Theorem is also essential.

2. Let A be an arbitrary set containing at least four elements. For a fixed element a_0 in A we define two symmetric operations f_1 and f_2 of two and three variables respectively by means of the formulas $f_1(x, x) = x$, $f_1(x, y) = a_0$ if x and y are different; $f_2(x, y, x) = x$, $f_2(x, y, z) = a_0$ if x, y and z are different. Consider the algebras $(A; f_1)$ and $(A; f_2)$. It is very easy to verify that f_1 is the only non-trivial algebraic operation of two variables in $(A; f_1)$ and f_2 is the only non-trivial algebraic operation of three variables in $(A; f_2)$. Hence it follows that all two-element subsets of A which do not contain a_0 are independent in $(A; f_1)$ and all three-element subsets of A which do not contain a_0 are independent in $(A; f_2)$. On the other hand, for elements a_1 different from a_0 and elements a^2

different from a_0 and a_1 we have the equations $f_1(a_0, a_1) = a_0 = e_1^{(2)}(a_0, a_1)$, $f_2(a_0, a_1, a_2) = a_0 = e_1^{(3)}(a_0, a_0, a_2)$, which show that all pairs of elements of A containing a_0 are dependent in $(A; f_1)$ and all triplets of elements of A containing a_0 are dependent in (A, f_2). Thus the assumption $n > 3$ in Theorem 1 is essential.

3. Consider a $2n$-element set $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$. Put $f(a_j) = b_j$ and $f(b_j) = b_j$ $(j = 1, 2, \ldots, n)$. Of course, the elements $a_1, a_2, \ldots, a_n$ are independent and the elements $b_1, b_2, \ldots, b_n$ are dependent in the algebra $(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n; f)$. Thus the assumption $n > m$ in Theorem 1 is also essential.

4. Let A be an arbitrary set containing at least $n+1$ elements. For a fixed element c in A we define a constant operation $c(x) = c$ $(x \in A)$. Obviously, every n-element subset of A which does not contain the element c is independent, and every subset of A containing c is dependent in the algebra $(A; c)$. Thus the assumption that there is no algebraic constant in the algebra is essential.

From Theorem 1 we obtain some simple corollaries. In [5] a sufficient condition is given for a hereditary class of subsets to be a class of independent sets in an algebra. As a direct consequence of Theorem 1 we obtain examples of hereditary classes of subsets of a finite set which are not classes of independent sets in any algebra.

COROLLARY 1. *Let n and m be integers satisfying the inequalities $n > m$, $n > 3$ and let A be a finite set containing at least $n+m$ elements. Further, let M be an m-element subset of A. No hereditary class of subsets of A containing all one-point sets and all n-element subsets of $A \setminus M$ which does not contain all n-element subsets of A can be a class of independent sets in any algebra over A.*

COROLLARY 2. *If an algebra without algebraic constants and with an n-element basis has less than $2n$ elements, where $n > 3$, then it is trivial.*

Indeed, taking as the set M in Theorem 1 the complement of the basis, we infer that all n-element subsets of the algebra are independent. Thus, by Theorem 1 in [3], p. 749, each n-element subset is a basis of the whole algebra. Consequently, by Theorem 2 in [4], p. 94, the algebra in question is trivial.

We have seen that the assumption $n > 3$ in Theorem 1 is essential. For $n = 2$ or 3 we obtain the same result under an additional assumption.

THEOREM 2. *Let $n = 2$ or 3, $n > m$, and let $(A; \boldsymbol{F})$ be a finite algebra without algebraic constants containing at least $n+m$ elements. Suppose that there exists an m-element subset M of A such that every n-element subset of $A \setminus M$ is independent. Moreover, suppose that M is contained in an n-element independent set. Then every n-element subset of A is independent.*

II. Before proving the Theorems we shall prove some Lemmas. By $[a_1, a_2, \ldots, a_n]$ we shall denote henceforth the subalgebra generated by elements $a_1, a_2, \ldots, a_n$.

LEMMA 1. *Let n and m be integers satisfying the inequalities $n > m$, $n > 1$ and let $(A; \boldsymbol{F})$ be an algebra without algebraic constants containing at least $n+m$ elements. Suppose that there exists an m-element subset M of A such that every n-element subset of $A \setminus M$ is independent. Then all operations from $\boldsymbol{A}^{(n-1)}$ are trivial.*

Proof. Contrary to this let us suppose that there exists a non-trivial algebraic operation f of s variables depending on every variable, where $1 \leqslant s \leqslant n-1$. Let $a_1, a_2, \ldots, a_s$ be an arbitrary system of different elements of $A \setminus M$. By assumption, the elements $a_1, a_2, \ldots, a_s$ are independent. Since the operation f is non-trivial, we have the inequality $f(a_1, a_2, \ldots, a_s) \neq a_j$ $(j = 1, 2, \ldots, s)$. Thus the $s+1$-element set $\{f(a_1, a_2, \ldots, a_s), a_1, a_2, \ldots, a_s\}$ is dependent. But this is possible only when

$$f(a_1, a_2, \ldots, a_s) \in M. \tag{5}$$

Now we define auxiliary algebraic operations $f_1, f_2, \ldots, f_n$ of n variables:

$$f_j(x_1, x_2, \ldots, x_n) = \begin{cases} f(x_j, x_{j+1}, \ldots, x_{s+j-1}) & \text{if } 1 \leqslant j \leqslant n+1-s, \\ f(x_j, x_{j+1}, \ldots, x_n, x_1, x_2, \ldots, x_{s+j-n-1}) & \\ & \text{if } n+1-s < j \leqslant n. \end{cases} \tag{6}$$

Of course, all the operations $f_1, f_2, \ldots, f_n$ are different. For any system $b_1, b_2, \ldots, b_n$ of different elements of $A \setminus M$ we have, by (5) and (6), the relation $f_j(b_1, b_2, \ldots, b_n) \in M$ $(j = 1, 2, \ldots, n)$. Since the set M has less than n elements, there exists a pair p, q of different indices such that $f_p(b_1, b_2, \ldots, b_n) = f_q(b_1, b_2, \ldots, b_n)$. But this contradicts the independence of $b_1, b_2, \ldots, b_n$. The Lemma is thus proved.

LEMMA 2. *Let $(A; \boldsymbol{F})$ be an algebra for which all operations from $\boldsymbol{A}^{(n-1)}$ are trivial and $n > 3$. Then for any operation f from $\boldsymbol{A}^{(n)}$ there exists an index k $(1 \leqslant k \leqslant n)$ such that*

$$f(x_1, x_2, \ldots, x_n) = e_k^{(n)}(x_1, x_2, \ldots, x_n)$$

whenever at least two elements among $x_1, x_2, \ldots, x_n$ are equal.

Proof. Replacing x_j by x_i in $f(x_1, x_2, \ldots, x_n)$, where $i \neq j$, we obtain an operation of $n-1$ variables which, by assumption, is trivial. Thus there exists an index $r(i, j)$ $(1 \leqslant r(i, j) \leqslant n)$ such that

$$f(x_1, x_2, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n) = e_{r(i,j)}^{(n)}(x_1, x_2, \ldots, x_n) \tag{7}$$

and, of course, $r(i,j) \neq j$. We note first that the equations $r(1,2) = 1$ and $r(3,4) = 3$ never hold simultaneously. Indeed, by (7) they would imply the equations

$$f(x_1, x_1, x_3, x_3, \ldots, x_n) = e^{(n)}_{r(1,2)}(x_1, x_1, x_3, x_3, \ldots, x_n) = x_1,$$

$$f(x_1, x_1, x_3, x_3, \ldots, x_n) = e^{(n)}_{r(3,4)}(x_1, x_1, x_3, x_3, \ldots, x_n) = x_3,$$

which gives a contradiction. Thus there exists a pair p, q $(p \neq q)$ of indices for which $r(p,q) \neq p$. Put $k = r(p,q)$. According to (7) to prove the Lemma it is sufficient to show that

$$(8) \qquad r(i,j) = k \quad \text{if} \quad j \neq k.$$

Suppose that $j \neq k$. Replacing the p-th and the q-th variable in (7) by x_j and taking into account the inequality $k \neq p, q, j$, we infer that the left-hand side of (7) is equal to $e^{(n)}_k(x_1, x_2, \ldots, x_n)$ and the right-hand side of (7) is equal to

$$e^{(n)}_{r(i,j)}(x_1, x_2, \ldots, x_{p-1}, x_j, x_{p+1}, \ldots, x_{q-1}, x_j, x_{q+1}, \ldots, x_n),$$

which, by simple reasoning, leads to formula (8). The Lemma is thus proved.

Proof of Theorem 1. We shall prove the theorem by induction with respect to m. For $m = 0$ the theorem is obvious. Suppose that $m > 0$. First consider the case when $A \setminus M$ contains n elements $a_1, a_2, \ldots, a_n$ such that $M \setminus [a_1, a_2, \ldots, a_n] \neq 0$. The set $M_0 = M \cap [a_1, a_2, \ldots, a_n]$ contains less than m elements. Moreover, the set $[a_1, a_2, \ldots, a_n] \setminus M_0$ contains at least n elements, all its n-element subsets are independent, in A and, consequently, in $[a_1, a_2, \ldots, a_n]$. Of course, the algebra $[a_1, a_2, \ldots, a_n]$ contains no algebraic constants. Thus, by inductive assumption, all n-element subsets of $[a_1, a_2, \ldots, a_n]$ are independent in the algebra $[a_1, a_2, \ldots, a_n]$ and, consequently, by Świerczkowski's theorem [6], all algebraic operations of n variables are trivial on $[a_1, a_2, \ldots, a_n]$. Hence, by independence of $a_1, a_2, \ldots, a_n$ in A, it follows that all operations from $A^{(n)}$ are trivial and, consequently, all n-element subsets of A are independent.

Now suppose that for every n-tuple $a_1, a_2, \ldots, a_n$ of different elements of $A \setminus M$ the inclusion

$$(9) \qquad M \subset [a_1, a_2, \ldots, a_n]$$

holds. First we shall prove that every n-element subset of A containing exactly one element of M is independent. Contrary to this let us suppose that there exists a dependent n-element set $\{a_0, a_1, \ldots, a_{n-1}\}$, where $a_0 \in M$ and $a_1, a_2, \ldots, a_{n-1} \in A \setminus M$. Let a_n be an element of $A \setminus M$ different from $a_1, a_2, \ldots, a_{n-1}$. Obviously, the elements $a_1, a_2, \ldots, a_n$ are independent and, by (9), there exists a non-trivial operation f in $A^{(n)}$ such that $f(a_1, a_2, \ldots, a_n) = a_0$. Since $a_1, a_2, \ldots, a_n$ are independent and $f(a_1, a_2, \ldots, a_n)$, $a_1, a_2, \ldots, a_{n-1}$ dependent, for any permutation $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$ of elements $a_1, a_2, \ldots, a_n$, by Marczewski's Theorem (ii) in [2], p. 60, the elements $f(a_{i_1}, a_{i_2}, \ldots, a_{i_n}), a_{i_1}, a_{i_2}, \ldots, a_{i_{n-1}}$ are also dependent. But this is possible only when $f(a_{i_1}, a_{i_2}, \ldots, a_{i_n}) \in M$ for any permutation of elements $a_1, a_2, \ldots, a_n$. Since the set M contains less than n elements, there exist an element $c \in M$ and a family $\mathcal{I}$ containing at least $(n-1)!+1$ permutations $i_1, i_2, \ldots, i_n$ such that $f(a_{i_1}, a_{i_2}, \ldots, a_{i_n}) = c$ whenever $i_1, i_2, \ldots, i_n$ belongs to $\mathcal{I}$. Hence, by the independence of $a_1, a_2, \ldots, a_n$, for all $x_1, x_2, \ldots, x_n \in A$ we obtain the equations

$$(10) \qquad f(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) = f(x_{j_1}, x_{j_2}, \ldots, x_{j_n})$$

whenever $i_1, i_2, \ldots, i_n$ and $j_1, j_2, \ldots, j_n$ belong to $\mathcal{I}$. Of course, we may assume that $\mathcal{I}$ forms a group of permutations. Thus the identity permutation belongs to $\mathcal{I}$. By Lemmas 1 and 2 there exists an index k $(1 \leqslant k \leqslant n)$ such that

$$f(x_1, x_2, \ldots, x_n) = e^{(n)}_k(x_1, x_2, \ldots, x_n)$$

whenever at least two elements among $x_1, x_2, \ldots, x_n$ are equal. Thus, by (10),

$$e^{(n)}_k(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) = e^{(n)}_k(x_1, x_2, \ldots, x_n)$$

whenever $i_1, i_2, \ldots, i_n$ belongs to $\mathcal{I}$ and at least two elements among $x_1, x_2, \ldots, x_n$ are equal. In other words, all permutations from $\mathcal{I}$ preserve the k-th index. Thus $\mathcal{I}$ contains at most $(n-1)!$ permutations, which gives a contradiction. Thus every n-element subset of A containing exactly one element of M is independent. Therefore, setting $M_0 = M \setminus \{a_0\}$, where a_0 is an element of M, we have the independence of each n-element subset of $A \setminus M_0$. Hence, by inductive assumption, all n-element subsets of A are independent, which completes the proof of Theorem 1.

LEMMA 3. *Let $n > 1$ and let $(A; \boldsymbol{F})$ be an algebra containing a finite subalgebra $(B; \boldsymbol{F})$ in which all n-element subsets are independent. If there exists an n-element subset of B independent in $(A; \boldsymbol{F})$, then all n-element subsets of A are independent.*

Proof. Contrary to this let us suppose that there exists a dependent n-element set $\{a_1, a_2, \ldots, a_n\}$ in A. Consequently, there are different operations f and g in $A^{(n)}$ such that

$$(11) \qquad f(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n).$$

Of course, without loss of generality, we may assume that the following inequalities hold:

$$(12) \qquad f \neq e^{(n)}_k, \quad g \neq e^{(n)}_k \quad (k = 1, 2, \ldots, n-2).$$

Let $b_1, b_2, \ldots, b_n$ be an n-tuple of elements of B which is independent in the algebra $(A; \boldsymbol{F})$. By assumption, the subalgebra $[b_1, b_2, \ldots, b_n]$ is finite and each of its n-element subsets is independent. Thus, by (12), the n-element set $\{b_1, b_2, \ldots, b_{n-2}, f(b_1, b_2, \ldots, b_n), g(b_1, b_2, \ldots, b_n)\}$ is independent and, by Theorem 1 in [3], p. 749, is a basis of $[b_1, b_2, \ldots, b_n]$. Consequently, there exist operations h_1 and h_2 in $\boldsymbol{A}^{(n)}$ such that

$$h_1\big(b_1, b_2, \ldots, b_{n-2}, f(b_1, b_2, \ldots, b_n), g(b_1, b_2, \ldots, b_n)\big) = b_{n-1},$$
$$h_2\big(b_1, b_2, \ldots, b_{n-2}, f(b_1, b_2, \ldots, b_n), g(b_1, b_2, \ldots, b_n)\big) = b_n.$$

Since $b_1, b_2, \ldots, b_n$ are independent in the algebra $(A; \boldsymbol{F})$, the elements $b_1, b_2, \ldots, b_n$ in the last equations can be replaced by the elements $a_1, a_2, \ldots, a_n$. Hence and from (11) we get the equations

$$h_1\big(a_1, a_2, \ldots, a_{n-2}, f(a_1, a_2, \ldots, a_n), f(a_1, a_2, \ldots, a_n)\big) = a_{n-1}, \tag{13}$$

$$h_2\big(a_1, a_2, \ldots, a_{n-2}, f(a_1, a_2, \ldots, a_n), f(a_1, a_2, \ldots, a_n)\big) = a_n. \tag{14}$$

By Lemma 1 all operations of $n-1$ variables are trivial in the algebra $[b_1, b_2, \ldots, b_n]$ and, consequently, by independence of $b_1, b_2, \ldots, b_n$ in $(A; \boldsymbol{F})$, are trivial on A. Thus, by (13) and (14),

$$a_{n-1}, a_n \in \{a_1, a_2, \ldots, a_{n-2}, f(a_1, a_2, \ldots, a_n)\},$$

which is impossible, because $a_1, a_2, \ldots, a_n$ are different. The Lemma is thus proved.

An operation $f \in \boldsymbol{A}^{(n)}$ is said to be *alternating* if for every system $x_1, x_2, \ldots, x_n$ of elements of A the equation

$$f(x_1, x_2, \ldots, x_n) = f(x_n, x_1, \ldots, x_{n-1})$$

holds.

Lemma 4. *Let f be an alternating operation of three variables in an algebra $(A; \boldsymbol{F})$. Then for every triplet a_1, a_2, a_3 of independent elements of A the relation*

$$f\big(a_1, a_2, f(a_1, a_2, a_3)\big) \notin \{a_1, a_2\}$$

holds.

Proof. Let us assume that $f\big(a_1, a_2, f(a_1, a_2, a_3)\big) = a_j$, where $j = 1$ or 2. Hence, by independence of a_1, a_2 and a_3, for all elements x_1, x_2 and x_3 in A we get the equation

$$f\big(x_1, x_2, f(x_1, x_2, x_3)\big) = x_j. \tag{15}$$

Thus $f(x_1, x_2, x_n) = f\Big(x_1, x_2, f\big(x_1, x_2, f(x_1, x_2, x_3)\big)\Big) = x_j$. Since the operation f is alternating, the last equation implies that $f(x, y, x) = f(y, x, x) = f(x, x, y) = x$. Hence and from (15) it follows that

$$x = f(y, x, x) = f\big(y, x, f(y, x, x)\big) = y \quad \text{if} \quad j = 1$$

and

$$x = f(x, y, x) = f\big(x, y, f(x, y, x)\big) = y \quad \text{if} \quad j = 2,$$

which gives a contradiction.

Proof of Theorem 2. Let us suppose that there exists an n-element subset $\{a_1, a_2, \ldots, a_n\}$ of $A \setminus M$ for which $M \cap [a_1, a_2, \ldots, a_n] = 0$. Thus, by assumption, all n-element subsets of $[a_1, a_2, \ldots, a_n]$ are independent in the algebra $(A; \boldsymbol{F})$. Hence and from Lemma 3 it follows that every n-element subset of A is independent.

Now let us assume that for every n-element subset $\{a_1, a_2, \ldots, a_n\}$ of $A \setminus M$ the relation

$$M \cap [a_1, a_2, \ldots, a_n] \neq 0 \tag{16}$$

holds.

First we consider the case when M is a one-point set $\{c\}$. We have to prove that every n-element subset of A containing c is independent. Contrary to this let us suppose that there exists a dependent n-element set $\{c, b_1, b_2, \ldots, b_{n-1}\}$, where, of course, $b_1, b_2, \ldots, b_{n-1}$ belong to $A \setminus M$. Given an arbitrary element b_n, different from $b_1, b_2, \ldots, b_{n-1}$, there exists, by (16), an operation $f \in \boldsymbol{A}^{(n)}$ such that $f(b_1, b_2, \ldots, b_n) = c$. Of course, the operation f is non-trivial. Moreover, from the independence of $b_1, b_2, \ldots, b_n$ and the dependence of $f(b_1, b_2, \ldots, b_n), b_1, b_2, \ldots, b_{n-1}$ we obtain, by Theorem (ii) in [2], p. 60, the dependence of elements $f(a_1, a_2, \ldots, a_n)$, $a_1, a_2, \ldots, a_{n-1}$ for any system $a_1, a_2, \ldots, a_n$ of elements of $A \setminus M$. On account of the independence of each n-element subset of $A \setminus M$ this is possible only when $f(a_1, a_2, \ldots, a_n) \in M$. Thus $f(a_1, a_2, \ldots, a_n) = c$ and, consequently, for any system $a_1, a_2, \ldots, a_{n-1}$ of elements of $A \setminus M$ the set $\{c, a_1, a_2, \ldots, a_{n-1}\}$ is dependent. In other words, every n-element subset of A containing M is dependent, which contradicts the assumption of the Theorem. Consequently, for one-point sets M the Theorem is proved.

Now suppose that M is a two-element set $\{c_1, c_2\}$ and, of course, that relation (16) holds. In this case we have $n = 3$. By the assumption there exists an element $a_1 \in A \setminus M$ such that c_1, c_2, a_1 are independent. Let a_2, a_3 be a pair of elements of $A \setminus M$ different from a_1. Without loss of generality, in virtue of (16), we may assume that

$$c_1 \in [a_1, a_2, a_3]. \tag{17}$$

First we shall prove that every three-element subset of $[a_1, a_2, a_3]$ which does not contain the element c_2 is independent. Contrary to this let us suppose that there exists a pair b_1, b_2 of elements of $[a_1, a_2, a_3]\setminus M$ such that c_1, b_1, b_2 are dependent. Of course, we can choose an element b_3 in $[a_1, a_2, a_3]\setminus M$ different from b_1 and b_2. Since b_1, b_2, b_3 are independent and the subalgebra $[a_1, a_2, a_3]$ is generated by three elements, we have, by Theorem 1 in [3], p. 749, the equation $[b_1, b_2, b_3] = [a_1, a_2, a_3]$. Hence and from (17) it follows that there exists an operation $f \in \boldsymbol{A}^{(3)}$ such that

(18) $$f(b_1, b_2, b_3) = c_1.$$

It is clear that the operation f is non-trivial. Moreover, we shall prove that it is alternating. We know that the elements $f(b_1, b_2, b_3)$, b_1, b_2 are dependent and the elements b_1, b_2, b_3 are independent. Thus, by Theorem (ii) in [2], p. 60, for any triplet u_1, u_2, u_3 of different elements of $A\setminus M$ the elements $f(u_1, u_2, u_3), u_1, u_2$ are also dependent. By the independence of all three-element subsets of $A\setminus M$ this is possible only when

(19) $$f(u_1, u_2, u_3) \in M.$$

Hence, in particular, we obtain the relation $f(b_{i_1}, b_{i_2}, b_{i_3}) \in M$ for every permutation i_1, i_2, i_3 of indices $1, 2, 3$. Since M consists of two elements, there exist three permutations p_1, p_2, p_3, q_1, q_2, q_3 and r_1, r_2, r_3 for which the equation

$$f(b_{p_1}, b_{p_2}, b_{p_3}) = f(b_{q_1}, b_{q_2}, b_{q_3}) = f(b_{r_1}, b_{r_2}, b_{r_3})$$

holds. Hence, by the independence of b_1, b_2, b_3, for all elements x_1, x_2 and x_3 in A we get the equation

$$f(x_{p_1}, x_{p_2}, x_{p_3}) = f(x_{q_1}, x_{q_2}, x_{q_3}) = f(x_{r_1}, x_{r_2}, x_{r_3}).$$

Let $\mathscr{I}$ be the group of permutations of variables x_1, x_2, x_3 under which the operation f is invariant. We have proved that $\mathscr{I}$ contains at least three permutations. Thus, $\mathscr{I}$ is either the alternating group $\mathscr{A}_3$ or the symmetric group $\mathscr{S}_3$. In both cases we obtain the equation

$$f(x_1, x_2, x_3) = f(x_3, x_1, x_2) = f(x_2, x_3, x_1),$$

which shows that the operation f is alternating.

From (19) it follows that there exists an index i ($i = 1$ or 2) such that

(20) $$f(a_1, a_2, a_3) = c_i.$$

Now we shall prove that

(21) $$f(a_1, a_2, c_i) = c_i.$$

Since the operation f is alternating, we have, by (20) and Lemma 4, $f(a_1, a_2, c_i) \notin \{a_1, a_2\}$. Thus the relation $f(a_1, a_2, c_i) \in A\setminus M$ would imply the independence of $f(a_1, a_2, c_i), a_1, a_2$ and, consequently, by Theorem (ii) in [2], p. 60, the independence of $f(b_1, b_2, b_3), b_1, b_2$ which, by (18), contradicts the assumption of dependence of c_1, b_1, b_2. Consequently, $f(a_1, a_2, c_i) \in M$. Suppose that $f(a_1, a_2, c_i) = c_j$, where $j \neq i$ and, consequently, $\{c_i, c_j\} = \{c_1, c_2\}$. Since the operation f is alternating, we have then the equation $f(c_i, a_1, a_2) = c_j$. Thus the set $\{f(c_i, a_1, a_2), c_i, a_1\}$ being equal to $\{c_1, c_2, a_i\}$ is independent, which, by Theorem (ii) in [2], p. 60, implies the independence of $f(b_1, b_2, b_3), b_1, b_2$. But, according to (18), this contradicts the assumption of the dependence of c_1, b_1, b_2, which completes the proof of (21).

From (20) and (21), in view of the independence of a_1, a_2 and a_3, we get the equation

(22) $$f(x_1, x_2, f(x_1, x_2, x_3)) = f(x_1, x_2, x_3)$$

for all x_1, x_2 and x_3 in A. Since the elements c_1, c_2, a_1 are independent and the operation f is non-trivial, we have the inequality $f(a_1, c_1, c_2) \neq a_1, c_1, c_2$. Set

(23) $$d_1 = f(a_1, c_1, c_2).$$

Further, since the operation f is alternating, we have, by (22) and (23), the equation

(24) $$\begin{aligned} f(c_2, a_1, d_1) &= f(c_2, a_1, f(a_1, c_1, c_2)) = f(c_2, a_1, f(c_2, a_1, c_1)) \\ &= f(c_2, a_1, c_1) = f(a_1, c_1, c_2) = d_1. \end{aligned}$$

Let us take an element d_2 in $A\setminus M$, different from a_1 and d_1. Since all three elements a_1, d_1 and d_2 belong to $A\setminus M$, we have, by (19),

(25) $$f(a_1, d_1, d_2) \in M \quad \text{and} \quad f(d_1, a_1, d_2) \in M.$$

If

(26) $$f(a_1, d_1, d_2) = c_1,$$

then, by (22) and (23), we get the equation

$$\begin{aligned} f(c_1, a_1, f(a_1, c_1, c_2)) &= f(c_1, a_1, d_1) = f(a_1, d_1, c_1) \\ &= f(a_1, d_1, f(a_1, d_1, d_2)) = f(a_1, d_1, d_2) = c_1. \end{aligned}$$

Hence, by the independence of c_1, c_2, a_1, we get the equation

(27) $$f(x_1, x_2, f(x_2, x_1, x_3)) = x_1$$

for all $x_1, x_2, x_3 \in A$. In particular, we have the equation

$$f(a_1, d_1, f(d_1, a_1, d_2)) = a_1.$$

On the other hand, by Lemma 4,

$$f(a_1, d_1, f(a_1, d_1, d_2)) \neq a_1.$$

Thus, $f(a_1, d_1, d_2) \neq f(d_1, a_1, d_2)$, which, according to (25) and (26), implies the equation $f(d_1, a_1, d_2) = c_2$. Since the operation f is alternating, the last equation and (27) imply

$$f(c_1, a_1, d_1) = f(a_1, d_1, c_2) = f(a_1, d_1, f(d_1, a_1, d_2)) = a_1,$$

which contradicts (24).

Now consider the case $f(a_1, d_1, d_2) = c_2$. Since the operation f is alternating, we have, by (22), the equation

$$f(c_2, a_1, d_1) = f(a_1, d_1, c_2) = f(a_1, d_1, f(a_1, d_1, d_2)) = f(a_1, d_1, d_2) = c_2,$$

which also contradicts (24). This completes the proof of the independence of each three-element subset of $[a_1, a_2, a_3]$ which does not contain the element c_2. Since a_1, a_2, a_3 are independent in the algebra $(A; F)$, to prove the Theorem it is sufficient, by Lemma 3, to show that all three-element subsets of $[a_1, a_2, a_3]$ are independent in the algebra $[a_1, a_2, a_3]$. If $c_2 \notin [a_1, a_2, a_3]$, then it is obvious. Further, if $c_2 \in [a_1, a_2, a_3]$, then every three-element subset of $[a_1, a_2, a_3]$ which does not contain c_2 is independent. Moreover, the set $\{c_1, c_2, a_1\}$ containing c_2 is independent and, by (17), is contained in $[a_1, a_2, a_3]$. Thus, by the first part of the proof ($n = 3$, $m = 1$), every three-element subset of $[a_1, a_2, a_3]$ is independent, which completes the proof of Theorem 2.

REFERENCES

[1] E. Marczewski, *A general scheme of the notions of independence in mathematics*, Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques, 6 (1958), p. 731-736.

[2] — *Independence and homomorphisms in abstract algebras*, Fundamenta Mathematicae 50 (1961), p. 45-61.

[3] S. Świerczkowski, *On independent elements in finitely generated algebras*, Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques, 6 (1958), p. 749-752.

[4] — *On algebras which are independently generated by every n elements*, Fundamenta Mathematicae 49 (1960), p. 93-104.

[5] — *A sufficient condition for independence*, Colloquium Mathematicum 9 (1962), p. 39-42.

[6] — *On two numerical constants associated with finite algebras*, Annali di Matematica pura ed applicata, in print.

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ON FREE PRODUCTS OF $\mathfrak{m}$-DISTRIBUTIVE BOOLEAN ALGEBRAS

BY

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1. Introduction. Let $\mathfrak{m}$ be an arbitrary cardinal number and $\{\mathfrak{A}_t\}_{t\in T}$ be an indexed set of non-degenerate $\mathfrak{m}$-complete Boolean algebras. An $\mathfrak{m}$-complete Boolean algebra $\mathfrak{B}$ is said to be a *minimal* $\mathfrak{m}$*-product* of $\{\mathfrak{A}_t\}_{t\in T}$ if there exist $\mathfrak{m}$-isomorphisms

$$i_t : \mathfrak{A} \rightarrow \mathfrak{B} \qquad (t \in T)$$

such that

(a) the union of all the subalgebras $i_t(\mathfrak{A}_t)$ $\mathfrak{m}$-generates $\mathfrak{B}$,

(b) the subalgebras $i_t(\mathfrak{A}_t)$, $t \in T$, are $\mathfrak{m}$-independent in $\mathfrak{B}$,

(c) the set of all meets of the form

$$\bigcap_{t\in T'} i_t(A_t) \quad \text{where} \quad A_t \in \mathfrak{A}_t, \quad T' \subset T,\ \overline{\overline{T'}} \leqslant \mathfrak{m}$$

is dense in $\mathfrak{B}$.

Christensen and Pierce [1] proved the existence of the minimal $\mathfrak{m}$-product of any indexed set of non-degenerate $\mathfrak{m}$-complete Boolean algebras (for $\mathfrak{m} = \aleph_0$ see also Sikorski [5]). They proved also that

1.1. *The minimal $\mathfrak{m}$-product of $\mathfrak{m}$-complete $\mathfrak{m}$-distributive Boolean algebras is a free $\mathfrak{m}$-distributive product of these algebras.*

We recall that an $\mathfrak{m}$-complete $\mathfrak{m}$-distributive Boolean algebra $\mathfrak{B}$ is said to be a *free* $\mathfrak{m}$*-distributive product* of an indexed set $\{\mathfrak{A}_t\}_{t\in T}$ of $\mathfrak{m}$-complete $\mathfrak{m}$-distributive Boolean algebras if there exist isomorphisms

$$i_t : \mathfrak{A}_t \rightarrow \mathfrak{B} \qquad (t \in T)$$

such that

(α) the union of all the subalgebras $i_t(\mathfrak{A}_t)$ $\mathfrak{m}$-generates $\mathfrak{B}$,

(β) if, for every $t \in T$, h_t is a homomorphism of $i_t(\mathfrak{A}_t)$ into any $\mathfrak{m}$-complete $\mathfrak{m}$-distributive Boolean algebra $\mathfrak{C}$, then there is a homomorphism h of $\mathfrak{B}$ into $\mathfrak{C}$ which is a common extension of all the homomorphisms h_t, i. e. $h_t(A) = h(A)$ for $A \in i_t(\mathfrak{A}_t)$ (cf. Sikorski [3], p. 214).