By putting here $x_1 = X$, $x_2 = \ldots = x_{k+1} = Y$ we obtain

$$X = F\big(f(X, Y, \ldots, Y), Y, \ldots, Y\big).$$

From the hypothesis, $f(X, Y, \ldots, Y) \in A^{(2,1)}$. If we had $f(X, Y, \ldots, Y) = h(Y)$ with some h, then $X = F\big(h(Y), Y, \ldots, Y\big)$—a contradiction; hence $f(X, Y, \ldots, Y) = h(X)$ with a suitable $h \in A^{(1)}$.

The set $\{f(x_1, \ldots, x_{k+1}), x_1, x_3, x_4, \ldots, x_{k+1}\}$ is independent in the algebra $A^{(k+1)}$; hence, as before, we obtain

$$x_2 = G\big(f(x_1, x_2, \ldots, x_{k+1}), x_1, x_3, \ldots, x_{k+1}\big)$$

with a suitable $G \in A^{(k+1)}$.

By putting here $x_1 = X$, $x_2 = Y$, $x_3 = \ldots = x_{k+1} = Z$ we obtain

$$Y = G\big(f(X, Y, Z, \ldots, Z), X, Z, \ldots, Z\big) \tag{1}$$

From the hypothesis, $f(X, Y, Z, \ldots, Z) \in A^{(3,1)}$. If we had $f(X, Y, Z, \ldots, Z) = H(X)$, or $f(X, Y, Z, \ldots, Z) = H(Z)$ with some H, then $Y = G\big(H(X), X, Z, \ldots, Z\big)$, or $Y = G\big(H(Z), X, Z, \ldots, Z\big)$ and this is impossible. Hence $f(X, Y, Z, \ldots, Z) = H(Y)$ with a suitable $H \in A^{(1)}$. By setting here $Y = Z$, we obtain $f(X, Y, \ldots, Y) = H(Y)$; but we obtained above $f(X, Y, \ldots, Y) = h(X)$, whence H must be a constant function, $H(Y) = c$ for all Y; but now from (1) we have $Y = G\big(H(Y), X, Z, \ldots, Z\big) = G(c, X, Z, \ldots, Z)$—a contradiction, because the right side of the last equation does not depend on Y. Hence every function of $A^{(k+1)}$ depends on at most k variables, and from the hypothesis it follows that every function of $A^{(k+1)}$ depends on at most one variable, q.e.d.

The theorem just proved and theorem III of [3] imply the following

COROLLARY. *If* $\mathfrak{A}$ *is an* v^**-algebra*, $\dim \mathfrak{A} \geqslant 3$, *then* $\mathfrak{A}$ *is decomposable if and only if* $A^{(3)} = A^{(3,1)}$.

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A representation theorem for v^*-algebras

by

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By an *algebra* $\mathfrak{A}$ we mean a pair $(A, \boldsymbol{F})$ where A is a non-empty set and $\boldsymbol{F}$ is a family of A-valued functions of finitely many variables running over A. $\boldsymbol{F}$ is called the class of *fundamental* operations. The class $\boldsymbol{A}$ of *algebraic* operations is, by definition, the smallest class closed with respect to composition, containing all fundamental operations and all trivial operations defined by the formula

$$e_k^{(n)}(x_1, x_2, \ldots, x_n) = x_k \quad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots).$$

Two algebras $(A, \boldsymbol{F}_1)$ and $(A, \boldsymbol{F}_2)$ having the same class of algebraic operations will be treated here as identical. In particular, we have the equation $(A, \boldsymbol{F}) = (A, \boldsymbol{A})$. Further, if the class of algebraic operations on $(A, \boldsymbol{F}_1)$ is contained in the class of algebraic operations on $(A, \boldsymbol{F}_2)$, then we say that $(A, \boldsymbol{F}_1)$ is a *subsystem* of $(A, \boldsymbol{F}_2)$.

The subclass of all algebraic operations of n variables will be denoted by $\boldsymbol{A}^{(n)}$ $(n \geqslant 1)$. Further, by $\boldsymbol{A}^{(0)}$ we shall denote the set of all values of constant algebraic operations. Elements belonging to $\boldsymbol{A}^{(0)}$ will be called *algebraic constants*. If $1 \leqslant k \leqslant n$, then $\boldsymbol{A}^{(n,k)}$ will denote the subclass of $\boldsymbol{A}^{(k)}$ consisting of all operations depending on at most k variables, i.e. $f \in \boldsymbol{A}^{(n,k)}$ if there is an operation $g \in \boldsymbol{A}^{(k)}$ such that $f(x_1, x_2, \ldots, x_n) = g(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ for a system of indices $i_1, i_2, \ldots, i_k$ and for every $x_1, x_2, \ldots, x_n \in A$. By $\boldsymbol{A}^{(n,0)}$ $(n \geqslant 1)$ we shall denote the subclass of $\boldsymbol{A}^{(n)}$ containing all constant operations. The above definitions are given in a more detailed form in [1], [2] and [4].

Following E. Marczewski [1], we say that elements of a set I $(I \subset A)$ are *independent* if for each system of n different elements $a_1, a_2, \ldots, a_n$ from I and for each pair of operations $f, g \in \boldsymbol{A}^{(n)}$ the equation

$$f(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n)$$

implies that f and g are identical in $\mathfrak{A}$. A set whose elements are not independent will be called a set of *dependent* elements. An element $a \in A$ is said to be *self-dependent* if the one-point set containing a is a set of dependent elements.

We say that an element $a \in A$ is *generated* by a set E ($E \subset A$) if it is the result of an algebraic operation applied to some elements in E. We say that a set B ($B \subset A$) is a *basis* of the algebra $\mathfrak{A}$ if it is a set of independent elements and every element from A is generated by B.

Following E. Marczewski we say that an algebra $\mathfrak{A}$ is a *v^*-algebra* if it satisfies the following conditions:

(I) *each self-dependent element is an algebraic constant,*

(II) *if the elements $a_1, a_2, \ldots, a_n$ $(n \geqslant 1)$ are independent and the elements $a_1, a_2, \ldots, a_n, a_{n+1}$ are dependent, then a_{n+1} is generated by $a_1, a_2, \ldots, a_n$.*

Condition (I) may be treated as a degenerated case $(n = 0)$ of (II). The properties of v^*-algebras were discussed by W. Narkiewicz [6], [7]. We note that the notion of v^*-algebra is a generalization of the notion of Marczewski's algebra (called by E. Marczewski *v-algebra*, [3], [8]). It can be proved that every v^*-algebra has a basis and all bases have the same cardinal number, which is called the *dimension* of the algebra (see [6]). In what follows by $\dim \mathfrak{A}$ we shall denote the dimension of a v^*-algebra $\mathfrak{A}$.

The aim of the present paper is to give a complete description of all v^*-algebras of dimension $\geqslant 3$. Namely, we shall prove the following representation theorem, which is a generalization of the representation theorem for Marczewski's algebras [8].

THEOREM. *Let $\mathfrak{A} = (A, \boldsymbol{F})$ be a v^*-algebra of dimension $\geqslant 3$.*

(i) *If $\boldsymbol{A}^{(0)} \neq 0$ and $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$, then there is a field $\mathcal{K}$ such that A is a linear space over $\mathcal{K}$ and, further, there exists a linear subspace A_0 of A such that $\boldsymbol{A}$ is the class of all operations f defined as*

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a ,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{K}$ and $a \in A_0$.

(ii) *If $\boldsymbol{A}^{(0)} = 0$ and $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$, then there is a field $\mathcal{K}$ such that A is a linear space over $\mathcal{K}$ and, further, there exists a linear subspace A_0 of A such that A is the class of all operations f defined as*

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a ,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{K}$, $\sum_{k=1}^{n} \lambda_k = 1$ and $a \in A_0$.

(iii) *If $\boldsymbol{A}^{(3)} = \boldsymbol{A}^{(3,1)}$, then there are a group $\mathcal{G}$ of transformations of the set A and a subset A_0 of A containing all fixed points of transformations that are not the identity and invariant under all transformations from $\mathcal{G}$ such that $\boldsymbol{A}$ is the class of all operations f defined as*

$$f(x_1, x_2, \ldots, x_n) = g(x_j) \qquad (1 \leqslant j \leqslant n)$$

or

$$f(x_1, x_2, \ldots, x_n) = a ,$$

where $g \in \mathcal{G}$ and $a \in A_0$.

We note that every algebra of form (i) or (ii) is a Marczewski algebra. Every algebra of form (iii) is a v^*-algebra. Moreover, an algebra of form (iii) is a Marczewski algebra if and only if every transformation from $\mathcal{G}$ that is not the identity has at most one fixed point in A. In particular, as a direct consequence of the representation theorem, we obtain the following corollary

(*) *Every at least three-dimensional v^*-algebra, with $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$, is a Marczewski algebra.*

It should be noted that the assumption $\dim \mathfrak{A} \geqslant 3$ of the representation theorem is essential. Namely, for any integer k $(k = 0, 1, 2)$ there exists a k-dimensional v^*-algebra which is not of the form stated in the theorem. Now we shall quote a few such counter-examples.

Let T be the set consisting of two elements, 0 and 1. Put $0' = 1$ and $1' = 0$. We define three families of fundamental operations on T. Let $\boldsymbol{F}_0$ be the class of all T-valued operations of finitely many variables defined on T. By $\boldsymbol{F}_1$ we denote the subclass of $\boldsymbol{F}_0$ consisting of all operations f satisfying the condition

$$f(x_1', x_2', \ldots, x_n') = f'(x_1, x_2, \ldots, x_n) .$$

Further, by $\boldsymbol{F}_2$ we denote subclass of $\boldsymbol{F}_1$ containing all operations f for which the equation $f(0, 0, \ldots, 0) = 0$ holds. Put $\mathfrak{A}_k = (T, \boldsymbol{F}_k)$ $(k = 0, 1, 2)$. First we shall show that $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are v^*-algebras. All elements in the algebra $\mathfrak{A}_0$ are self-dependent and are algebraic constants, which implies that $\mathfrak{A}_0$ is a zero-dimensional v^*-algebra. Further, there is no algebraic constant in the algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Thus every one-point set in these algebras is a set of independent elements. Since the operation $f(x) = x'$ is algebraic in $\mathfrak{A}_1$, we see that the elements 0 and 1 are dependent and condition (II) of the definition of v^*-algebras holds. Thus $\mathfrak{A}_1$ is a one-dimensional v^*-algebra. Finally, it is very easy to verify that the only operations of two variables in $\mathfrak{A}_2$ are trivial ones. Thus the elements 0 and 1 are both independent (see [5], p. 291) and, consequently, $\mathfrak{A}_2$ is a two-dimensional v^*-algebra.

Now we shall prove that the algebras $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are not of the form stated in the theorem. It is very easy to verify that the operation p^* of three variables defined by the formula

$$p^*(x, y, z) = xy + yz + xz \ (\mathrm{mod}\, 2)$$

is algebraic in $\mathfrak{A}_2$ (see [5], p. 292) and, consequently, in $\mathfrak{A}_0$ and $\mathfrak{A}_1$. Since the operation p^* depends on every variable, the algebras $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are not of form (iii). We know that the algebra $\mathfrak{A}_2$ is a subsystem of the algebras $\mathfrak{A}_0$ and $\mathfrak{A}_1$. Since every subsystem of two-element algebras of form (i) or (ii) is a Marczewski algebra, to prove our statement it suffices to show that $\mathfrak{A}_2$ is not a Marczewski algebra. Consider the equation $p^*(x, y, z) = x$ depending on the variable x. Since this equation holds whenever $x = y$ or $x = z$, it cannot be equivalent to any equation of the form $x = f(y, z)$, where f is an algebraic operation in $\mathfrak{A}_2$. Thus $\mathfrak{A}_2$ is not a Marczewski algebra, which implies that the algebras $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are not of form (i) or (ii).

The idea of the proof of the representation theorem is similar to that in [8]. Before proving the theorem we shall prove some lemmas. If $\mathfrak{A} = (A, \boldsymbol{F})$ is an algebra, then by $\mathfrak{A}^{(n)}$ we shall denote the algebra $(\boldsymbol{A}^{(n)}, \boldsymbol{F})$ of all n-ary algebraic operations on $\mathfrak{A}$ (see [4], p. 48). In all further considerations the following statement proved in [6] plays a fundamental rôle:

If $\mathfrak{A}$ is a v^*-algebra and $1 \leqslant n \leqslant \dim\mathfrak{A}$, then $\mathfrak{A}^{(n)}$ is also a v^*-algebra.

In what follows by $\mathfrak{A} = (A, \boldsymbol{F})$ we shall denote a v^*-algebra of dimension $\geqslant 3$. Further, for simplicity in our considerations we shall often write x_k instead of the trivial operation $e_k^{(n)}$, so that $f(x_1, x_2, \ldots, x_n)$ $(f \in \boldsymbol{A}^{(n)})$ treated as an element of $\mathfrak{A}^{(n)}$ will denote the composition $f(e_1^{(n)}, e_2^{(n)}, \ldots, e_n^{(n)})$.

For any $f \in \boldsymbol{A}$ we denote by $\hat{f}$ the operation belonging to $\boldsymbol{A}^{(1)}$ defined as $\hat{f}(x) = f(x, x, \ldots, x)$. $\tilde{\boldsymbol{A}}^{(n)}$ $(n \geqslant 1)$ will denote the subclass of $\boldsymbol{A}^{(n)}$ consisting of all operations f for which $\hat{f}(x) = x$. $\tilde{\boldsymbol{A}}^{(n,k)}$ will denote the intersection $\tilde{\boldsymbol{A}}^{(n)} \cap \boldsymbol{A}^{(n,k)}$.

LEMMA 1. *If $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$, then $\tilde{\boldsymbol{A}}^{(3)} \neq \tilde{\boldsymbol{A}}^{(3,1)}$.*

Proof. First let us suppose that $\boldsymbol{A}^{(2)} \neq \boldsymbol{A}^{(2,1)}$. Let $f \in \boldsymbol{A}^{(2)} \setminus \boldsymbol{A}^{(2,1)}$. Since the operation f depends on both variables, the elements $f(x_1, x_2)$ and x_2 treated as elements of $\mathfrak{A}^{(2)}$ are independent and, consequently, form a basis of the algebra $\mathfrak{A}^{(2)}$. Thus, there exists an operation $g \in \boldsymbol{A}^{(2)}$ such that

(1) $$x_1 = g\big(f(x_1, x_2), x_2\big).$$

Hence we get the equation

$$f\Big(g\big(f(x_1, x_2), x_2\big), x_2\Big) = f(x_1, x_2).$$

Taking into account the independence of $f(x_1, x_2)$ and x_2 we have the equation

(2) $$f\big(g(x_1, x_2), x_2\big) = x_1.$$

Put $h(x_1, x_2, x_3) = f\big(g(x_1, x_2), x_3\big)$. From (1) we obtain the equation

$$h\big(f(x_1, x_2), x_2, x_3\big) = f\Big(g\big(f(x_1, x_2), x_2\big), x_3\Big) = f(x_1, x_3),$$

which shows that the operation h depends on at least two variables. By (2) we have the equation $\hat{h}(x) = x$. Thus $h \in \tilde{\boldsymbol{A}}^{(3)} \setminus \tilde{\boldsymbol{A}}^{(3,1)}$.

Now let us suppose that $\boldsymbol{A}^{(2)} = \boldsymbol{A}^{(2,1)}$. Let $f \in \boldsymbol{A}^{(3)} \setminus \boldsymbol{A}^{(3,1)}$. Of course, the operation f depends on every variable. First we shall prove that $\hat{f}$ is not constant. Contrary to this let us assume that for every x the equation

(3) $$\hat{f}(x) = a$$

holds, where $a \in \boldsymbol{A}^{(0)}$. Since there is no operation in $\boldsymbol{A}^{(2)}$ depending on both variables, we have either $f(x_1, x_1, x_2) = f_1(x_1)$ or $f(x_1, x_1, x_2) = f_2(x_2)$, where f_1 and f_2 are operations from $\boldsymbol{A}^{(1)}$. Putting $x_2 = x_1$ into $f(x_1, x_1, x_2)$ we get, by virtue of (3), $f_1(x_1) = a$ in the first case and $f_2(x_1) = a$ in the second case. Consequently,

(4) $$f(x_1, x_1, x_2) = a \qquad (x_1, x_2 \in A).$$

Further, since the operation f depends on every variable, the elements x_1, x_2 and $f(x_1, x_2, x_3)$ treated as elements of the algebra $\mathfrak{A}^{(3)}$ are independent and, consequently, form a basis of $\mathfrak{A}^{(3)}$. Thus, there exists an operation $g \in \boldsymbol{A}^{(3)}$ such that

$$x_3 = g\big(x_1, x_2, f(x_1, x_2, x_3)\big).$$

Substituting $x_1 = x_2 = a$ into the last equation, we get, in view of (4),

$$x_3 = g\big(a, a, f(a, a, x_3)\big) = g(a, a, a),$$

which gives a contradiction. Thus we have proved the relation $\hat{f} \notin \boldsymbol{A}^{(1,0)}$. It is well known that any operation in $\boldsymbol{A}^{(1)} \setminus \boldsymbol{A}^{(1,0)}$ has an inverse in the sense of composition (see [6], Theorem 1), which is also an algebraic operation. In other words, there exists an operation $g_1 \in \boldsymbol{A}^{(1)}$ such that

(5) $$x = g_1\big(\hat{f}(x)\big) = \hat{f}\big(g_1(x)\big).$$

Put $h(x_1, x_2, x_3) = g_1\big(f(x_1, x_2, x_3)\big)$. From (5) it follows that the operation h depends on every variable and $\hat{h}(x) = x$. Thus, $h \in \tilde{\boldsymbol{A}}^{(3)} \setminus \tilde{\boldsymbol{A}}^{(3,1)}$, which completes the proof.

An operation $s \in \boldsymbol{A}^{(3)}$ is said to be *quasi-symmetric* if

(6) $$s(x_1, x_2, x_1) = s(x_2, x_1, x_1) = x_2$$

for each $x_1, x_2 \in A$.

LEMMA 2. *If* $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$, *then there exists a quasi-symmetric algebraic operation.*

Proof. By Lemma 1 we may assume that $\widetilde{\boldsymbol{A}}^{(3)} \neq \widetilde{\boldsymbol{A}}^{(3,1)}$. First let us suppose that $\widetilde{\boldsymbol{A}}^{(2)} \neq \widetilde{\boldsymbol{A}}^{(2,1)}$. Let $f \in \widetilde{\boldsymbol{A}}^{(2)} \setminus \widetilde{\boldsymbol{A}}^{(2,1)}$. Of course $f(x_1, x_2)$ and x_2 treated as elements of the algebra $\mathfrak{A}^{(2)}$ are independent and, consequently, form a basis of $\mathfrak{A}^{(2)}$. Thus, there exists an operation $g_1 \in \boldsymbol{A}^{(2)}$ such that

$$x_1 = g_1\big(x_2, f(x_1, x_2)\big) . \tag{7}$$

Hence

$$f(x_1, x_2) = f\Big(g_1\big(x_2, f(x_1, x_2)\big), x_2\Big)$$

and, by the independence of $f(x_1, x_2)$ and x_2,

$$x_1 = f\big(g_1(x_2, x_1), x_2\big) . \tag{8}$$

Moreover, from (7) we obtain the equation

$$x_1 = g_1\big(x_1, \hat{f}(x_1)\big) = g_1(x_1, x_1) . \tag{9}$$

Taking into account the independence of $f(x_1, x_2)$ and x_1, we can prove in the same way the existence of an operation $g_2 \in \boldsymbol{A}^{(2)}$ such that

$$x_2 = g_2\big(x_1, f(x_1, x_2)\big) . \tag{10}$$

Hence

$$f(x_1, x_2) = f\Big(x_1, g_2\big(x_1, f(x_1, x_2)\big)\Big)$$

and, by the independence of $f(x_1, x_2)$ and x_1,

$$x_2 = f\big(x_1, g_2(x_1, x_2)\big) . \tag{11}$$

Moreover, by (10),

$$x_1 = g_2\big(x_1, \hat{f}(x_1)\big) = g_2(x_1, x_1) . \tag{12}$$

Setting $s(x_1, x_2, x_3) = f\big(g_1(x_3, x_1), g_2(x_3, x_2)\big)$ we have, according to (8), (9), (11) and (12), the equations

$$s(x_1, x_2, x_1) = f\big(g_1(x_1, x_1), g_2(x_1, x_2)\big) = f\big(x_1, g_2(x_1, x_2)\big) = x_2 ,$$
$$s(x_2, x_1, x_1) = f\big(g_1(x_1, x_2), g_2(x_1, x_1)\big) = f\big(g_1(x_1, x_2), x_1\big) = x_2 .$$

Consequently, s is a quasi-symmetric operation.

Now let us suppose that

$$\widetilde{\boldsymbol{A}}^{(2)} = \widetilde{\boldsymbol{A}}^{(2,1)}. \tag{13}$$

We shall prove that every operation belonging to $\widetilde{\boldsymbol{A}}^{(3)} \setminus \widetilde{\boldsymbol{A}}^{(3,1)}$ is quasi-symmetric. To prove this it suffices to show that for every $f \in \widetilde{\boldsymbol{A}}^{(3)} \setminus \widetilde{\boldsymbol{A}}^{(3,1)}$ and $x_1, x_3 \in A$ the equation

$$f(x_1, x_1, x_3) = x_3 \tag{14}$$

holds. Contrary to this statement, let us suppose that $f(x_1, x_1, x_3) \neq x_3$ for a pair $x_1, x_3 \in A$. Hence and from (13) we get the equation

$$f(x_1, x_1, x_3) = x_1 . \tag{15}$$

Since, by (13), each operation from $\widetilde{\boldsymbol{A}}^{(3)} \setminus \widetilde{\boldsymbol{A}}^{(3,1)}$ depends on every variable, the elements x_1, x_2 and $f(x_1, x_2, x_3)$ treated as elements of the algebra $\mathfrak{A}^{(3)}$ are independent and, consequently, form a basis of $\mathfrak{A}^{(3)}$. Thus, there exists an operation $g \in \boldsymbol{A}^{(3)}$ such that

$$x_3 = g\big(x_1, x_2, f(x_1, x_2, x_3)\big) .$$

Setting $x_2 = x_1$ and taking into account (15) we obtain the equation

$$x_3 = g\big(x_1, x_1, f(x_1, x_1, x_3)\big) = g(x_1, x_1, x_1) ,$$

which gives a contradiction. Formula (14) and, consequently, Lemma 2 are thus proved.

LEMMA 3. *Let* $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$, $3 \leqslant n \leqslant \dim \mathfrak{A}$ *and* $f, g \in \boldsymbol{A}^{(n)}$. *If the equation* $f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n)$ *holds whenever* $x_1 = x_2$ *or* $x_1 = x_3$, *then* $f = g$ *in* $\mathfrak{A}$.

Proof. If operations f and g are both independent of the variable x_1, then the assertion of the Lemma is obvious. Therefore we may suppose without loss of generality that the operation g depends on the variable x_1.

First we shall prove that the operations f and g are dependent in the algebra $\mathfrak{A}^{(n)}$. Contrary to this, let us suppose that they are independent in $\mathfrak{A}^{(n)}$. Put

$$f_2(x_2, x_3, \ldots, x_n) = f(x_2, x_2, x_3, \ldots, x_n) , \tag{16}$$

$$f_3(x_2, x_3, \ldots, x_n) = f(x_3, x_2, x_3, \ldots, x_n) . \tag{17}$$

Consider first the following case.

I. *At least one of the systems* $f_2, x_3, x_4, \ldots, x_n$ *and* $f_3, x_2, x_4, \ldots, x_n$ *is independent in* $\mathfrak{A}^{(n)}$.

Without loss of generality we may assume that the system $f_2, x_3, x_4, \ldots, x_n$ is independent. If $f, g, x_3, x_4, \ldots, x_n$ are also independent in $\mathfrak{A}^{(n)}$, then they form a basis of the algebra $\mathfrak{A}^{(n)}$. Since, by Lemma 2, the class $\boldsymbol{A}^{(3)}$ contains a quasi-symmetric operation s, we can find an operation $h \in \boldsymbol{A}^{(n)}$ such that

$$s(x_2, x_3, x_1) = h\big(f(x_1, x_2, \ldots, x_n), g(x_1, x_2, \ldots, x_n), x_3, x_4, \ldots, x_n\big) . \tag{18}$$

Substituting $x_1 = x_2$ into the last equation we get, in view of (6) and (16), the formula

$$x_3 = h\big(f_2(x_2, x_3, \ldots, x_n), f_2(x_2, x_3, \ldots, x_n), x_3, x_4, \ldots, x_n\big) .$$

Since $f_2, x_3, x_4, \ldots, x_n$ are independent in $\mathfrak{A}^{(n)}$, we have the equation

(19) $$x_3 = h\big(f_3(x_2, x_3, \ldots, x_n), f_3(x_2, x_3, \ldots, x_n), x_3, x_4, \ldots, x_n\big).$$

Now substituting $x_1 = x_3$ into (18) we obtain, in view of (6) and (17), the equation

$$x_2 = h\big(f_3(x_2, x_3, \ldots, x_n), f_3(x_2, x_3, \ldots, x_n), x_3, x_4, \ldots, x_n\big),$$

which contradicts (19). Thus the operations $f, g, x_3, x_4, \ldots, x_n$ are dependent in $\mathfrak{A}^{(n)}$. Let k be the least integer $\geqslant 3$ such that $f, g, x_3, x_4, \ldots, x_k$ are dependent and $f, g, x_3, x_4, \ldots, x_{k-1}$ are independent. Such an integer exists because of the independence of f and g. Since $\mathfrak{A}^{(n)}$ is a v^*-algebra whenever $n \leqslant \dim\mathfrak{A}$, we can find an operation $h_1 \in A^{(k-1)}$ such that

$$x_k = h_1\big(f(x_1, x_2, \ldots, x_n), g(x_1, x_2, \ldots, x_n), x_3, x_4, \ldots, x_{k-1}\big).$$

Setting $x_1 = x_2$ into the last equation we obtain the following one:

$$x = h_1\big(f_2(x_2, x_3, \ldots, x_n), f_2(x_2, x_3, \ldots, x_n), x_3, x_4, \ldots, x_{k-1}\big),$$

which contradicts the independence of $f_2, x_3, x_4, \ldots, x_n$. Thus we have proved that case I is impossible.

Now consider the following case:

II. *The systems $f_2, x_3, x_4, \ldots, x_n$ and $f_3, x_2, x_4, \ldots, x_n$ both consist of dependent elements of $\mathfrak{A}^{(n)}$.*

If $f, g, x_3, x_4, \ldots, x_n$ are independent, then they form a basis of $\mathfrak{A}^{(n)}$. There is then an operation $h_2 \in A^{(n)}$ such that

$$x_2 = h_2\big(f(x_1, x_2, \ldots, x_n), g(x_1, x_2, \ldots, x_n), x_3, x_4, \ldots, x_n\big).$$

Substituting $x_1 = x_2$ into this equation we obtain the following one:

(20) $$x_2 = h_2\big(f_2(x_2, x_3, \ldots, x_n), f_2(x_2, x_3, \ldots, x_n), x_3, x_4, \ldots, x_n\big).$$

By II, $f_2, x_3, x_4, \ldots, x_n$ are dependent, which implies that the operation f_2 does not depend on the variable x_2. Hence and from (20) we get a contradiction.

Now let us assume that the operations $f, g, x_3, x_4, \ldots, x_n$ are dependent. Since the operation g depends on the variable x_1, the operations $g, x_3, x_4, \ldots, x_n$ are independent. Hence it follows that there exists an operation $h_3 \in A^{(n-1)}$ such that

(21) $$f(x_1, x_2, \ldots, x_n) = h_3\big(g(x_1, x_2, \ldots, x_n), x_3, x_4, \ldots, x_n\big).$$

Substituting $x_1 = x_3$ into this equation we get the formula

$$f_3(x_2, x_3, \ldots, x_n) = h_3\big(f_3(x_2, x_3, \ldots, x_n), x_3, x_4, \ldots, x_n\big).$$

If $f_3, x_3, x_4, \ldots, x_n$ are independent, then the last equation implies the following one:

$$g(x_1, x_2, \ldots, x_n) = h_3\big(g(x_1, x_2, \ldots, x_n), x_3, x_4, \ldots, x_n\big),$$

which, by virtue of (21), contradicts the independence of f and g. Thus, $f_3, x_3, x_4, \ldots, x_n$ are dependent. On the other hand, by II, $f_3, x_2, x_4, \ldots, x_n$ are also dependent. Hence, by a simple reasoning we infer that the operation f_3 does not depend on the variables x_2 and x_3. If $g, x_1, x_3, x_4, \ldots, x_n$ are dependent, then by virtue of the independence of $g, x_3, x_4, \ldots, x_n$, there is an operation $h_4 \in A^{(n-1)}$ such that

$$x_1 = h_4\big(g(x_1, x_2, \ldots, x_n), x_3, x_4, \ldots, x_n\big).$$

Substituting $x_1 = x_2$ into this equation we get the formula

(22) $$x_2 = h_4\big(f_2(x_2, x_3, \ldots, x_n), x_3, x_4, \ldots, x_n\big).$$

By II the operation f_2 does not depend on the variable x_2. Thus, the right-hand side of (22) does not depend on x_2, which gives a contradiction. Consequently, the operations $g, x_1, x_3, x_4, \ldots, x_n$ are independent and, consequently, form a basis of the algebra $\mathfrak{A}^{(n)}$. Let h_5 be an operation from $A^{(n)}$ such that

$$x_2 = h_5\big(g(x_1, x_2, \ldots, x_n), x_1, x_3, x_4, \ldots, x_n\big).$$

Substituting $x_1 = x_3$ into this equation we get the following one:

(23) $$x_2 = h_5\big(f_3(x_2, x_3, \ldots, x_n), x_3, x_3, x_4, \ldots, x_n\big).$$

Since f_3 does not depend on the variable x_2, the right-hand side of (23) is also independent on x_2, which is impossible. This completes the proof of the dependence of the operations f and g.

We have assumed that the operation g depends on the variable x_1. Consequently, $g \in A^{(n,0)}$ and we can find an operation $h_6 \in A^{(1)}$ such that

(24) $$f(x_1, x_2, \ldots, x_n) = h_6\big(g(x_1, x_2, \ldots, x_n)\big).$$

Setting $x_1 = x_2$ or $x_1 = x_3$ into the last equation we obtain the relation

$$f_j(x_2, x_3, \ldots, x_n) = h_6\big(f_j(x_2, x_3, \ldots, x_n)\big) \quad (j = 2 \text{ or } 3),$$

which implies $h_6(x) = x$ provided at least one of the operations f_2 and f_3 is not constant. Hence and from (24) we obtain the assertion of our Lemma in the case where at least one of the operations f_2 and f_3 is not constant.

Finally consider the case where the operations f_2 and f_3 are both constant. Since $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$ by Lemma 2, the class $\boldsymbol{A}^{(3)}$ contains a quasi-symmetric operation s. Putting

$$f_0(x_1, x_2, \ldots, x_n) = s\big(f(x_1, x_2, \ldots, x_n), x_2, x_3\big), \tag{25}$$

$$g_0(x_1, x_2, \ldots, x_n) = s\big(g(x_1, x_2, \ldots, x_n), x_2, x_3\big) \tag{26}$$

we have the equation $f_0(x_1, x_2, \ldots, x_n) = g_0(x_1, x_2, \ldots, x_n)$ whenever $x_1 = x_2$ or $x_1 = x_3$. Moreover, $f_0(x_2, x_2, x_3, \ldots, x_n) = s(a, x_2, x_3)$, where $a = f_2(x_2, x_3, \ldots, x_n)$ is an algebraic constant. Hence, by (6), $f_0(x_2, x_2, a, x_4, \ldots, x_n) = x_2$, which shows that $f_0(x_2, x_2, x_3, \ldots, x_n)$ is not an algebraic constant. Thus, by the first part of the proof, $f_0 = g_0$. By virtue of (24), (25) and (26) this equation can be rewritten in the form

$$s\Big(h_6\big(g(x_1, x_2, \ldots, x_n), x_2, x_3\big)\Big) = s\big(g(x_1, x_2, \ldots, x_n), x_2, x_3\big). \tag{27}$$

Since $g(x_1, x_2, \ldots, x_n)$ depends on the variable x_1, the operations g, x_2 and x_3 are independent and, consequently, equation (27) yields

$$s\big(h_6(x_1), x_2, x_2\big) = s(x_1, x_2, x_3).$$

Substituting $x_3 = x_2$ into this equation we obtain $h_6(x_1) = x_1$, which together with (24) completes the proof of the Lemma.

Lemma 4. *If $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$ then for any quasi-symmetric operation s and for all $x_1, x_2, x_3, x_4 \in A$ the following equations are true:*

$$s(x_1, x_2, x_3) = s(x_2, x_1, x_3), \tag{28}$$

$$f\big(s(x_1, x_2, x_3), x_3\big) = s\big(f(x_1, x_3), f(x_2, x_3), x_3\big) \quad \textit{for any} \quad f \in \tilde{\boldsymbol{A}}^{(2)}, \tag{29}$$

$$f(x_1, x_2, x_3) = s\big(f(x_1, x_1, x_3), f(x_1, x_2, x_1), x_1\big) \quad \textit{for any} \quad f \in \tilde{\boldsymbol{A}}^{(3)}, \tag{30}$$

$$s\big(s(x_1, x_2, x_3), x_4, x_3\big) = s\big(x_1, s(x_2, x_4, x_3), x_3\big). \tag{31}$$

Proof. From formula (6) it follows that equation (28) holds whenever $x_3 = x_1$ or $x_3 = x_2$. Thus, by Lemma 3, it holds for all $x_1, x_2, x_3 \in A$. Further, from the equations

$$f\big(s(x_1, x_2, x_1), x_1\big) = f(x_2, x_1),$$

$$s\big(f(x_1, x_1), f(x_2, x_1), x_1\big) = s\big(x_1, f(x_2, x_1), x_1\big) = f(x_2, x_1),$$

$$f\big(s(x_1, x_2, x_2), x_2\big) = f(x_1, x_2),$$

$$s\big(f(x_1, x_2), f(x_2, x_2), x_2\big) = s\big(f(x_1, x_2), x_2, x_2\big) = f(x_1, x_2),$$

where $f \in \tilde{\boldsymbol{A}}^{(2)}$, it follows that equation (29) holds for $x_3 = x_1$ and $x_3 = x_2$, which implies, in view of Lemma 3, that it holds for all x_1, x_2 and x_3.

Further, taking into account formula (6), we have, for every $f \in \tilde{\boldsymbol{A}}^{(3)}$, the equations

$$s\big(f(x_2, x_2, x_3), f(x_2, x_2, x_2), x_2\big) = f(x_2, x_2, x_3),$$

$$s\big(f(x_3, x_3, x_3), f(x_3, x_2, x_3), x_3\big) = f(x_3, x_2, x_3),$$

which show that equation (30) holds for $x_1 = x_2$ and $x_1 = x_3$. Thus, by Lemma 3, equation (30) holds for all x_1, x_2 and x_3.

In the proof of equation (31) we distinguish two cases. First let us assume that $\dim \mathfrak{A} \geqslant 4$. Taking into account formula (6) we have the equations

$$s\big(s(x_1, x_2, x_2), x_4, x_2\big) = s(x_1, x_4, x_2),$$

$$s\big(x_1, s(x_2, x_4, x_2), x_2\big) = s(x_1, x_4, x_2),$$

$$s\big(s(x_1, x_2, x_4), x_4, x_4\big) = s(x_1, x_2, x_4),$$

$$s\big(x_1, s(x_2, x_4, x_4), x_4\big) = s(x_1, x_2, x_4),$$

which imply that equation (31) holds for $x_3 = x_2$ and $x_3 = x_4$. Hence, by Lemma 3, we get equation (31) for all x_1, x_2, x_3 and x_4.

Finally let us suppose that $\dim \mathfrak{A} = 3$. If x_3 is not an algebraic constant in $\mathfrak{A}$, then we have one of the cases

$$x_1 = f_1(x_2, x_3, x_4), \qquad x_2 = f_2(x_1, x_3, x_4), \qquad x_4 = f_4(x_1, x_2, x_3),$$

where $f_1, f_2, f_4 \in \boldsymbol{A}^{(3)}$ and equation (31) can be written in one of the following forms:

$$s\Big(s\big(f_1(x_2, x_3, x_4), x_2, x_3\big), x_4, x_3\Big) = s\big(f_1(x_2, x_3, x_4), s(x_2, x_4, x_3), x_3\big), \tag{32}$$

$$s\Big(s\big(x_1, f_2(x_1, x_3, x_4), x_3\big), x_4, x_3\Big) = s\Big(x_1, s\big(f_2(x_1, x_3, x_4), x_4, x_3\big), x_3\Big), \tag{33}$$

$$s\big(s(x_1, x_2, x_3), f_4(x_1, x_2, x_3), x_3\big) = s\Big(x_1, s\big(x_2, f_4(x_1, x_2, x_3), x_3\big), x_3\Big). \tag{34}$$

From (6) we get the equations

$$s\Big(s\big(f_1(x_2, x_2, x_4), x_2, x_2\big), x_4, x_2\Big) = s\big(f_1(x_2, x_2, x_4), x_4, x_2\big),$$

$$s\big(f_1(x_2, x_2, x_4), s(x_2, x_4, x_2), x_2\big) = s\big(f_1(x_2, x_2, x_4), x_4, x_2\big),$$

$$s\Big(s\big(f_1(x_2, x_4, x_4), x_2, x_4\big), x_4, x_4\Big) = s\big(f_1(x_2, x_4, x_4), x_2, x_4\big),$$

$$s\big(f_1(x_2, x_4, x_4), s(x_2, x_4, x_4), x_4\big) = s\big(f_1(x_2, x_4, x_4), x_2, x_4\big),$$

which show that (32) holds for $x_3 = x_2$ and $x_3 = x_4$. Applying Lemma 3 we infer that (32) holds for all x_2, x_3 and x_4.

Further, from (6) we obtain the equalities

$$s\big(s(x_1, f_2(x_1, x_4, x_4), x_4), x_4, x_4\big) = s\big(x_1, f_2(x_1, x_4, x_4), x_4\big)\,,$$
$$s\big(x_1, s(f_2(x_1, x_4, x_4), x_4, x_4), x_4\big) = s\big(x_1, f_2(x_1, x_4, x_4), x_4\big)\,,$$
$$s\big(s(x_1, f_2(x_1, x_1, x_4), x_1), x_4, x_1\big) = s\big(f_2(x_1, x_1, x_4), x_4, x_1\big)\,,$$
$$s\big(x_1, s(f_2(x_1, x_1, x_4), x_4, x_1), x_1\big) = s\big(f_2(x_1, x_1, x_4), x_4, x_1\big)\,,$$

which show that (33) holds for $x_3 = x_4$ and $x_3 = x_1$. Consequently, by Lemma 3, it holds for all x_1, x_3 and x_4.

Finally, by (6), we have the equations

$$s\big(s(x_1, x_2, x_2), f_4(x_1, x_2, x_2), x_2\big) = s\big(x_1, f_4(x_1, x_2, x_2), x_2\big)\,,$$
$$s\big(x_1, s(x_2, f_4(x_1, x_2, x_2), x_2), x_2\big) = s\big(x_1, f_4(x_1, x_2, x_2), x_2\big)\,,$$
$$s\big(s(x_1, x_2, x_1), f_4(x_1, x_2, x_1), x_1\big) = s\big(x_2, f_4(x_1, x_2, x_1), x_1\big)\,,$$
$$s\big(x_1, s(x_2, f_4(x_1, x_2, x_1), x_1), x_1\big) = s\big(x_2, f_4(x_1, x_2, x_1), x_1\big)\,,$$

which show that (34) holds for $x_3 = x_2$ and $x_3 = x_1$. Consequently, by Lemma 3, it holds for all x_1, x_2 and x_3. Thus we have proved equation (31) if x_3 is not an algebraic constant.

Now let us suppose that x_3 is an algebraic constant. Then equation (31) can be written in the form

$$s\big(s(x_1, x_2, c), x_4, c\big) = s\big(x_1, s(x_2, x_4, c), c\big)\,, \tag{35}$$

where $c \in A^{(0)}$. Let us consider an auxilliary equation

$$s\big(s(x_1, x_2, x_3), x_2, x_3\big) = s\big(x_1, s(x_2, x_2, x_3), x_3\big)\,. \tag{36}$$

From (6) we get the formulas

$$s\big(s(x_1, x_2, x_1), x_2, x_1\big) = s(x_2, x_2, x_1)\,,$$
$$s\big(x_1, s(x_2, x_2, x_1), x_1\big) = s(x_2, x_2, x_1)\,,$$
$$s\big(s(x_1, x_2, x_2), x_2, x_2\big) = x_1\,,$$
$$s\big(x_1, s(x_2, x_2, x_2), x_2\big) = x_1\,,$$

which show that (36) holds for $x_3 = x_1$ and $x_3 = x_2$. Hence, by Lemma 3, we infer that (36) holds for all x_1, x_2 and x_3. From (28) and (36) we obtain the equations

$$s\big(s(x_1, x_2, c), x_1, c\big) = s\big(x_1, s(x_2, x_1, c), c\big)\,,$$
$$s\big(s(x_1, x_2, c), x_2, c\big) = s\big(x_1, s(x_2, x_2, c), c\big)\,,$$

which imply that (35) holds for $x_4 = x_1$ and $x_4 = x_2$. Consequently, by Lemma 3, equation (35) holds for all x_1, x_2 and x_4. Thus (31) holds also for three-dimensional algebras, which completes the proof of the Lemma.

In the sequel we shall denote by $\mathcal{K}$ the class $\tilde{A}^{(2)}$. Elements of $\mathcal{K}$ will be denoted by small Greek letters: $\lambda, \mu, \nu, \ldots$

Lemma 5. *If $A^{(3)} \neq A^{(3,1)}$, then $\mathcal{K}$ is a field with respect to the operations*

$$(\lambda+\mu)(x_1, x_2) = s\big(\lambda(x_1, x_2), \mu(x_1, x_2), x_2\big)\,, \tag{37}$$

$$(\lambda\cdot\mu)(x_1, x_2) = \lambda\big(\mu(x_1, x_2), x_2\big)\,, \tag{38}$$

where s is a quasi-symmetric algebraic operation.

Proof. First of all we remark that the existence of an algebraic quasi-symmetric operation follows from Lemma 2.

We define the zero-element and the unit element by the following formulas

$$0(x_1, x_2) = x_2\,, \qquad 1(x_1, x_2) = x_1\,.$$

Obviously, $0 \neq 1$. From (6) and (37) it follows for every $\lambda \in \mathcal{K}$ that

$$(\lambda+0)(x_1, x_2) = s\big(\lambda(x_1, x_2), x_2, x_2\big) = \lambda(x_1, x_2)\,.$$

Thus $\lambda+0 = \lambda$ for every $\lambda \in \mathcal{K}$. Further,

$$(\lambda\cdot 1)(x_1, x_2) = \lambda(x_1, x_2) = (1\cdot\lambda)(x_1, x_2)\,,$$

which implies $\lambda\cdot 1 = 1\cdot\lambda = \lambda$ for every $\lambda \in \mathcal{K}$.

The following formula is a direct consequence of (38)

$$\lambda\cdot(\mu\cdot\nu) = (\lambda\cdot\mu)\cdot\nu \qquad (\lambda, \mu, \nu \in \mathcal{K})\,.$$

Since, by (6), the operation $s(x_1, x_3, x_2)$ depends on the variable x_3, the operations x_1, x_2 and $s(x_1, x_3, x_2)$ treated as elements of $\mathfrak{A}^{(3)}$ are independent and, consequently, form a basis of $\mathfrak{A}^{(3)}$. Thus there exists an operation $g \in A^{(3)}$ such that

$$x_3 = g\big(x_1, x_2, s(x_1, x_3, x_2)\big)\,. \tag{39}$$

Hence we get the equation

$$s\big(x_1, g(x_1, x_2, s(x_1, x_3, x_2)), x_2\big) = s(x_1, x_3, x_2)\,,$$

which, by the independence of x_1, x_2 and $s(x_1, x_3, x_2)$, implies

$$s\big(x_1, g(x_1, x_2, x_2), x_2\big) = x_2\,. \tag{40}$$

By (39) $\hat{g}(x) = x$, i.e. $g \in \tilde{A}^{(3)}$. Given $\lambda \in \mathcal{K}$, we put $-\lambda(x_1, x_2) = g(\lambda(x_1, x_2), x_2, x_2)$. Taking into account (40) we have the equation

$$s(\lambda(x_1, x_2), -\lambda(x_1, x_2), x_2) = x_2,$$

which, by (37), implies $\lambda + (-\lambda) = 0$ for every $\lambda \in \mathcal{K}$.

Let $\lambda \neq 0$, i.e. let $\lambda(x_1, x_2)$ depend on the variable x_1. Then the operations $\lambda(x_1, x_2)$ and x_2 treated as elements of the algebra $\mathfrak{A}^{(2)}$ are independent and, consequently, form a basis of $\mathfrak{A}^{(2)}$. Thus there is an operation $\lambda^{-1} \in A^{(2)}$ such that

$$x_1 = \lambda^{-1}(\lambda(x_1, x_2), x_2). \tag{41}$$

Setting $x_2 = x_1$ into the last equation we obtain the formula $x_1 = \lambda^{-1}(x_1, x_1)$ which shows that $\lambda^{-1} \in \mathcal{K}$. Further, from (41) we get the equation

$$\lambda(x_1, x_2) = \lambda(\lambda^{-1}(\lambda(x_1, x_2), x_2), x_2)$$

which, by the independence of $\lambda(x_1, x_2)$ and x_2 implies

$$x_1 = \lambda(\lambda^{-1}(x_1, x_2), x_2).$$

This equation and (41) can be written in the form $\lambda^{-1} \cdot \lambda = \lambda \cdot \lambda^{-1} = 1$. Taking into account assertions (28), (29) and (31) of Lemma 4, we have the equations

$$\begin{aligned}
(\lambda+\mu)(x_1, x_2) &= s(\lambda(x_1, x_2), \mu(x_1, x_2), x_2) \\
&= s(\mu(x_1, x_2), \lambda(x_1, x_2), x_2) = (\mu+\lambda)(x_1, x_2), \\
((\lambda+\mu)+\nu)(x_1, x_2) &= s(s(\lambda(x_1, x_2), \mu(x_1, x_2), x_2), \nu(x_1, x_2), x_2) \\
&= s(\lambda(x_1, x_2), s(\mu(x_1, x_2), \nu(x_1, x_2), x_2), x_2) = (\lambda+(\mu+\nu))(x_1, x_2), \\
(\lambda\cdot(\mu+\nu))(x_1, x_2) &= \lambda(s(\mu(x_1, x_2), \nu(x_1, x_2), x_2), x_2) \\
&= s(\lambda(\mu(x_1, x_2), x_2), \lambda(\nu(x_1, x_2), x_2), x_2) = (\lambda\cdot\mu+\mu+\lambda\cdot\nu)(x_1, x_2),
\end{aligned}$$

which imply

$$\lambda+\mu = \mu+\lambda, \quad (\lambda+\mu)+\nu = \lambda+(\mu+\nu), \quad \lambda\cdot(\mu+\nu) = \lambda\cdot\mu+\lambda\cdot\nu$$

for every $\lambda, \mu, \nu \in \mathcal{K}$.

Finally the following equations are a direct consequence of definitions (37) and (38)

$$\begin{aligned}
((\mu+\nu)\cdot\lambda)(x_1, x_2) &= s(\mu(\lambda(x_1, x_2), x_2), \nu(\lambda(x_1, x_2), x_2), x_2) \\
&= (\mu\cdot\lambda+\nu\cdot\lambda)(x_1, x_2).
\end{aligned}$$

Thus $(\mu+\nu)\cdot\lambda = \mu\cdot\lambda+\nu\cdot\lambda$ for every $\lambda, \mu, \nu \in \mathcal{K}$, which completes the proof.

LEMMA 6. *If $A^{(3)} \neq A^{(3,1)}$, then A is a linear space over $\mathcal{K}$ with respect to the operations*

$$\begin{aligned}
x+y &= s(x, y, \Theta) \qquad (x, y \in A), \\
\lambda\cdot x &= \lambda(x, \Theta) \qquad (\lambda \in \mathcal{K}, x \in A),
\end{aligned}$$

where Θ is an element of $A^{(0)}$ if $A^{(0)} \neq 0$ and is an arbitrary element of A if $A^{(0)} = 0$.

Proof. The element Θ is the zero-element of A. In fact, according to (6), $x+\Theta = s(x, \Theta, \Theta) = x$ for every $x \in A$. Further, we have, in virtue of Lemma 4, the equations

$$\begin{aligned}
x+y &= s(x, y, \Theta) = s(y, x, \Theta) = y+x, \\
(x+y)+z &= s(s(x, y, \Theta), z, \Theta) = s(x, s(y, z, \Theta), \Theta) = x+(y+z), \\
\lambda\cdot(x+y) &= \lambda(s(x, y, \Theta), \Theta) = s(\lambda(x, \Theta), \lambda(y, \Theta), \Theta) = \lambda\cdot x+\lambda\cdot y
\end{aligned}$$

for any $x, y, z \in A$ and $\lambda \in \mathcal{K}$. Moreover, we have the equations

$$\begin{aligned}
\lambda\cdot(\mu\cdot x) &= \lambda(\mu(x, \Theta), \Theta) = (\lambda\cdot\mu)\cdot x, \\
1\cdot x &= x, \\
(\lambda+\mu)x &= s(\lambda(x, \Theta), \mu(x, \Theta), \Theta) = \lambda\cdot x+\mu\cdot x
\end{aligned}$$

for any $x \in A$ and $\lambda, \mu \in \mathcal{K}$. Hence, setting $-x = (-1)x$, we get the equation $x+(-x) = 0\cdot x = \Theta$. The Lemma is thus proved.

LEMMA 7. *Let $A^{(3)} \neq A^{(3,1)}$ and the addition in $\mathcal{K}$ be defined by an operation s. If the field $\mathcal{K}$ has the characteristic 2, then*

$$s(x_1, s(x_2, x_3, x_4), x_4) = s(x_1, x_2, x_3) \tag{42}$$

for all $x_1, x_2, x_3, x_4 \in A$.

Proof. First of all we shall prove that the operation s is symmetric, i.e. that

$$s(x_1, x_2, x_3) = s(x_{i_1}, x_{i_2}, x_{i_3}) \tag{43}$$

for every permutation i_1, i_2, i_3 of indices $1, 2, 3$. To prove this, in view of formula (28) of Lemma 4, it suffices to show that for every system $x_1, x_2, x_3 \in A$ the equation $s(x_1, x_2, x_3) = s(x_1, x_3, x_2)$ holds. In other words, according to Lemma 4, it sufficies to show that the operation $s_0(x_1, x_2, x_3) = s(x_3, x_1, x_2)$ is quasi-symmetric. We have, according to the definition of addition in $\mathcal{K}$, the equation

$$s_0(x_1, x_2, x_1) = s(x_1, x_1, x_2) = (1+1)(x_1, x_2) = 0(x_1, x_2) = x_2$$

and, accordnig to (6), the equation

$$s_0(x_2, x_1, x_1) = s(x_1, x_2, x_1) = x_2,$$

which imply the quasi-symmetry of s_0 and, consequently, the symmetry of the operation s.

Now let us suppose that $\dim\mathfrak{A} \geq 4$. From (6) and (43) we get the equations

$$s\big(x_1, s(x_2, x_3, x_2), x_2\big) = s(x_1, x_3, x_2) = s(x_1, x_2, x_3),$$
$$s\big(x_1, s(x_2, x_3, x_3), x_3\big) = s(x_1, x_2, x_3),$$

which imply that equation (42) holds for $x_4 = x_2$ and $x_4 = x_3$. Applying Lemma 3 we obtain (42) for all $x_1, x_2, x_3, x_4 \in A$.

Finally let us suppose that $\dim\mathfrak{A} = 3$. If x_4 is not an algebraic constant, then we have one of the cases

$$x_1 = f_1(x_2, x_3, x_4), \qquad x_2 = f_2(x_1, x_3, x_4), \qquad x_3 = f_3(x_1, x_2, x_4),$$

where $f_1, f_2, f_3 \in A^{(3)}$ and equation (42) can be written in one of the following forms:

$$s\big(f_1(x_2, x_3, x_4), s(x_2, x_3, x_4), x_4\big) = s\big(f_1(x_2, x_3, x_4), x_2, x_3\big), \tag{44}$$

$$s\Big(x_1, s\big(f_2(x_1, x_3, x_4), x_3, x_4\big), x_4\Big) = s\big(x_1, f_2(x_1, x_3, x_4), x_3\big), \tag{45}$$

$$s\Big(x_1, s\big(x_2, f_3(x_1, x_2, x_4), x_4\big), x_4\Big) = s\big(x_1, x_2, f_3(x_1, x_2, x_4)\big). \tag{46}$$

From (6) and (43) we get the equations

$$\begin{aligned} s\big(f_1(x_2, x_3, x_2), s(x_2, x_3, x_2), x_2\big) &= s\big(f_1(x_2, x_3, x_2), x_3, x_2\big) \\ &= s\big(f_1(x_2, x_3, x_2), x_2, x_3\big), \end{aligned}$$

$$s\big(f_1(x_2, x_3, x_3), s(x_2, x_3, x_3), x_3\big) = s\big(f_1(x_2, x_3, x_3), x_2, x_3\big),$$

which show that (44) holds for $x_4 = x_2$ and $x_4 = x_3$. Thus, by Lemma 3, (44) holds for all x_2, x_3 and x_4.

Further, according to (6) and (43), we have the equations

$$\begin{aligned} s\Big(x_1, s\big(f_2(x_1, x_3, x_1), x_3, x_1\big), x_1\Big) &= s\big(f_2(x_1, x_3, x_1), x_3, x_1\big) \\ &= s\big(x_1, f_2(x_1, x_3, x_1), x_3\big), \end{aligned}$$

$$s\Big(x_1, s\big(f_2(x_1, x_3, x_3), x_3, x_3\big), x_3\Big) = s\big(x_1, f_2(x_1, x_3, x_3), x_3\big),$$

which show that (45) holds for $x_4 = x_1$ and $x_4 = x_3$. Thus, by Lemma 3, equation (45) holds for all x_1, x_3, and x_4.

Finally, from (6) and (43) we get the equations

$$\begin{aligned} s\Big(x_1, s\big(x_2, f_3(x_1, x_2, x_1), x_1\big), x_1\Big) &= s\big(x_2, f_3(x_1, x_2, x_1), x_1\big) \\ &= s\big(x_1, x_2, f_3(x_1, x_2, x_1)\big), \end{aligned}$$

$$\begin{aligned} s\Big(x_1, s\big(x_2, f_3(x_1, x_2, x_2), x_2\big), x_2\Big) &= s\big(x_1, f_3(x_1, x_2, x_2), x_2\big) \\ &= s\big(x_1, x_2, f_3(x_1, x_2, x_2)\big), \end{aligned}$$

which show that (46) holds for $x_4 = x_1$ and $x_4 = x_2$. Consequently, by Lemma 3, it holds for all x_1, x_2 and x_4. Thus we have proved equation (42) if x_4 is not an algebraic constant.

Now let us suppose that x_4 is an algebraic constant. Then equation (42) can be written in the form

$$s\big(x_1, s(x_2, x_3, c), c\big) = s(x_1, x_2, x_3), \tag{47}$$

where $c \in A^{(0)}$. From (6), (31) and (43) we obtain the equations

$$s\big(x_1, s(x_1, x_3, c), c\big) = s\big(s(x_1, x_1, c), x_3, c\big) = s(c, x_3, c) = x_3 = s(x_1, x_1, x_3),$$
$$s\big(x_1, s(x_3, x_3, c), c\big) = s(x_1, c, c) = x_1 = s(x_1, x_3, x_3),$$

which show that (47) holds for $x_2 = x_1$ and $x_2 = x_3$. Consequently, by Lemma 3, it holds for all x_1, x_2 and x_3. Thus (42) holds also for three-dimensional algebras, which completes the proof of the Lemma.

LEMMA 8. *If $A^{(3)} \neq A^{(3,1)}$, then all operations f defined as*

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{K}$ and $\sum_{k=1}^{n} \lambda_k = 1$ belong to $\widetilde{A}^{(n)}$ $(n = 1, 2, \ldots)$. Moreover, each operation λ from $A^{(2)}$ is of the form

$$\lambda(x_1, x_2) = \lambda x_1 + (1-\lambda)x_2. \tag{48}$$

Proof. We prove our Lemma by induction with respect to n. For $n = 1$ the assertion is obvious. To prove our assertion for $n = 2$ it suffices to prove formula (48). Setting $f(x_1, x_2, x_3) = \lambda(x_2, x_3)$ into formula (30) of Lemma 4 we infer that

$$\lambda(x_2, x_3) = s\big(\lambda(x_1, x_3), \lambda(x_2, x_1), x_1\big) \tag{49}$$

for every $x_1, x_2, x_3 \in A$. Replacing in the last formula x_2 and x_3 by x_1, x_1 by x_2 we obtain the equation

$$x_1 = s\big(\lambda(x_2, x_1), \lambda(x_1, x_2), x_2\big).$$

Hence, according to the definition of the unit element and addition in $\mathcal{K}$, we have the equation

$$\lambda(x_2, x_1) = (1-\lambda)(x_1, x_2).$$

Setting $x_1 = \Theta$ into (49) and replacing x_2 by x_1 and x_3 by x_2 we infer that

$$\lambda(x_1, x_2) = s\big(\lambda(\Theta, x_2), \lambda(x_1, \Theta), \Theta\big) = \lambda\cdot x_1 + (1-\lambda)x_2,$$

which completes the proof of (48).

Now let us suppose that $n \geqslant 3$ and that the assertion of our Lemma is true for indices less than n. Let us consider an operation

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k,$$

where $\sum_{k=1}^{n} \lambda_k = 1$.

First we assume that there is an index k_0 $(1 \leqslant k_0 \leqslant n)$ for which $\lambda_{k_0} \neq 1$. Of course, without loss of generality we may suppose that $k_0 = 1$. Put

$$g(x_1, x_2) = (1-\lambda_1)x_1 + \lambda_1 x_2,$$

$$h(x_2, x_3, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k (1-\lambda_1)^{-1} x_k.$$

By the induction assumption $g \in \tilde{A}^{(2)}$ and $h \in \tilde{A}^{(n-1)}$. It is easy to verify that $f(x_1, x_2, \ldots, x_n) = g\big(h(x_2, x_3, \ldots, x_n), x_1\big)$, which implies $f \in \tilde{A}^{(n)}$.

Now let us assume that $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$ and that the field $\mathcal{K}$ has a characteristic different from 2. Since $1 \neq 0$ and $n \cdot 1 = \sum_{k=1}^{n} \lambda_k = 1$, we have the inequality $(n-2)\cdot 1 \neq 0$. Put

$$g_1(x_1, x_2) = 2x_1 + (n-2)x_2,$$
$$g_2(x_1, x_2) = 2^{-1}x_1 + 2^{-1}x_2,$$
$$g_3(x_3, x_4, \ldots, x_n) = \sum_{k=1}^{n} (n-2)^{-1} x_k.$$

By the induction assumption, $g_1, g_2 \in \tilde{A}^{(2)}$ and $g_3 \in \tilde{A}^{(n-2)}$. Since

$$f(x_1, x_2, \ldots, x_n) = g_1\big(g_2(x_1, x_2), g_3(x_3, x_4, \ldots, x_n)\big),$$

we have $f \in \tilde{A}^{(n)}$.

Finally let us assume that $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$ and that the field $\mathcal{K}$ has the characteristic 2. Since $(n-2)\cdot 1 = n \cdot 1 = 1$, by the induction assumption the operation

$$f_0(x_1, x_2, \ldots, x_{n-2}) = \sum_{k=1}^{n-2} x_k$$

belongs to $\tilde{A}^{(n-2)}$. Using Lemma 7 we infer that

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= f_0(x_1, x_2, \ldots, x_{n-2}) + x_{n-1} + x_n \\ &= s\big(f_0(x_1, x_2, \ldots, x_{n-2}), s(x_{n-1}, x_n, \Theta), \Theta\big) \\ &= s\big(f_0(x_1, x_2, \ldots, x_{n-2}), x_{n-1}, x_n\big), \end{aligned}$$

which implies $f \in \tilde{A}^{(n)}$. The Lemma is thus proved.

LEMMA 9. *If $A^{(3)} \neq A^{(3,1)}$, then the set*

$$A_0 = \{f(\Theta)\colon f \in A^{(1)}\} \tag{50}$$

is a linear subspace of A. Moreover, for every $f \in A^{(1)}$ there exists an element $\lambda \in \mathcal{K}$ such that

$$f(x) = \lambda x + f(\Theta) \tag{51}$$

for any $x \in A$.

Proof. First we shall prove formula (51). By Lemma 8 the operation g defined by the formula

$$g(x_1, x_2, x_3) = x_1 - x_2 + x_3$$

belongs to $\tilde{A}^{(3)}$. Given $f \in A^{(1)}$, we put

$$\lambda(x_1, x_2) = g\big(f(x_1), f(x_2), x_2\big) = f(x_1) - f(x_2) + x_2. \tag{52}$$

Obviously, $\lambda(x, x) = x$ and, consequently, $\lambda \in \tilde{A}^{(2)}$. By the definition of scalar-multiplication in A we have

$$\lambda(x, \Theta) = \lambda x.$$

On the other hand, from (52) we get the equation

$$\lambda(x, \Theta) = f(x) - f(\Theta).$$

Hence equation (51) follows.

Consider an arbitrary pair f_1, f_2 of operations from $A^{(1)}$ and an arbitrary pair λ_1, λ_2 of elements of $\mathcal{K}$. By Lemma 8, the operation

$$h(x_1, x_2, x_3) = \lambda_1 x_1 + \lambda_2 x_2 + (1-\lambda_1-\lambda_2)x_3 \tag{53}$$

belongs to $\tilde{A}^{(3)}$. Consequently, the operation $f_3(x) = h\big(f_1(x), f_2(x), x\big)$ belongs to $A^{(1)}$. From (53) we get the equation

$$f_3(\Theta) = \lambda_1 f_1(\Theta) + \lambda_2 f_2(\Theta),$$

which shows that A_0 is a linear subspace of A.

LEMMA 10. *Let $A^{(3)} \neq A^{(3,1)}$ and let $f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a$, where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{K}$ and $a \in A_0$. If $\hat{f} \in A^{(1)}$, then $f \in A^{(n)}$.*

Proof. First we assume that $\hat{f} \in A^{(1)} \setminus A^{(1,0)}$ and, consequently, $\hat{f}$ generates all operations in $A^{(1)}$. Thus there is an operation $g \in A^{(1)}$ such that

$$g\big(\hat{f}(x)\big) = x. \tag{54}$$

Hence we get the equation $\hat{f}(x) = \hat{f}\big(g(\hat{f}(x))\big)$, which implies the formula

$$\hat{f}\big(g(x)\big) = x. \tag{55}$$

By Lemma 9, there are elements λ and b belonging to $\mathcal{K}$ and A_0 respectively such that $g(x) = \lambda x + b$. Put

$$h(x_1, x_2, \ldots, x_n) = g(f(x_1, x_2, \ldots, x_n)) = \sum_{k=1}^{n} \lambda\lambda_k x_k + \lambda a + b .$$

Taking into account formula (54) we have the equation

$$\Big(\sum_{k=1}^{n} \lambda\lambda_k\Big) x + \lambda a + b = h(x) = g(\hat{f}(x)) = x ,$$

which implies $\sum_{k=1}^{n}\lambda\lambda_k = 1$ and $\lambda a + b = \Theta$. Thus, by Lemma 8, $h \in \tilde{A}^{(n)}$. Further, from (55) it follows that

$$f(x_1, x_2, \ldots, x_n) = f\big(g(f(x_1, x_2, \ldots, x_n))\big) = f(h(x_1, x_2, \ldots, x_n)) ,$$

which implies $f \in A^{(n)}$.

Now let us suppose that f is an algebraic constant. Of course, in this case $\Theta \in A^{(0)}$ and $a = \hat{f}(\Theta) \in A^{(0)}$. Put

$$f_0(x_1, x_2, \ldots, x_n, x_{n+1}) = \sum_{k=1}^{n} \lambda_k x_k + \Big(1 - \sum_{k=1}^{n} \lambda_k\Big) x_{n+1} .$$

By Lemma 8, the operation f_0 belongs to $\tilde{A}^{(n+1)}$. Since, by the definition of addition in A,

$$f(x_1, x_2, \ldots, x_n) = s\big(f_0(x_1, x_2, \ldots, x_n, \Theta), a, \Theta\big) ,$$

the operation f belongs also to $A^{(n)}$. The Lemma is thus proved.

LEMMA 11. *If $A^{(3)} \neq A^{(3,1)}$, then each operation g in $\tilde{A}^{(3)}$ is of the form*

$$g(x_1, x_2, x_3) = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 ,$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{K}$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

Proof. First we shall prove the equation

(56) $$s(x_1, x_2, x_3) = x_1 + x_2 - x_3 .$$

By Lemma 8, the operation $x_1 + x_2 - x_3$ belongs to $\tilde{A}^{(3)}$. Taking into account formula (6) it is very easy to verify that equation (56) holds for $x_3 = x_1$ and $x_3 = x_2$. Thus, by Lemma 3, it holds for all $x_1, x_2, x_3 \in A$. Let $g \in \tilde{A}^{(3)}$. By Lemma 8 we have the equations

$$g(x_1, x_1, x_3) = \lambda x_1 + (1-\lambda) x_3 ,$$
$$g(x_1, x_2, x_1) = \mu x_1 + (1-\mu) x_2 .$$

Hence and from (30) and (56) we obtain the equation

$$\begin{aligned} g(x_1, x_2, x_3) &= s\big(g(x_1, x_1, x_3), g(x_1, x_2, x_1), x_1\big) \\ &= \lambda x_1 + (1-\lambda) x_3 + \mu x_1 + (1-\mu) x_2 - x_1 \\ &= (\lambda + \mu - 1) x_1 + (1-\mu) x_2 + (1-\lambda) x_3 , \end{aligned}$$

which completes the proof of the Lemma.

LEMMA 12. *Let $\dim \mathfrak{A} = n$ $(n \geqslant 3)$ and $A^{(3)} \neq A^{(3,1)}$. If all operations from $A^{(n)}$ are of the form $\sum_{k=1}^{n} \lambda_k x_k + a$, where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{K}$ and $a \in A_0$, then all operations from $A^{(n+1)}$ are also of the form $\sum_{k=1}^{n+1} \lambda_k x_k + a$, where $\lambda_1, \lambda_2, \ldots, \lambda_{n+1} \in \mathcal{K}$ and $a \in A_0$.*

Proof. Let $f \in A^{(n+1)}$. For each pair $i \neq j$ $(i, j = 1, 2, \ldots, n+1)$ setting $x_i = x_j$ into $f(x_1, x_2, \ldots, x_{n+1})$ we obtain the operation $f_{ij} \in A^{(n)}$ of the form

$$\sum_{\substack{k=1 \\ k \neq 1}}^{n+1} \lambda_k(i, j) x_k + a(i, j) .$$

First we shall prove that there exist a system $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ of elements of $\mathcal{K}$ and an element a of A_0 such that

(57) $$\lambda_k(i, j) = \lambda_k \quad (k \neq i, j;\ i, j, k = 1, 2, \ldots, n+1) ,$$

(58) $$\lambda_j(i, j) = \lambda_i + \lambda_j \quad (i, j = 1, 2, \ldots, n+1)$$

and

(59) $$a(i, j) = a \quad (i, j = 1, 2, \ldots, n+1) .$$

To prove (57) it suffices to show that $\lambda_k(i, j) = \lambda_k(r, s)$ whenever $k \neq i, j, r, s$. Setting $x_m = \Theta$ $(m \neq k;\ m = 1, 2, \ldots, n+1)$ into f_{ij} and f_{rs} we obtain identical expressions $\lambda_k(i, j) x_k + a(i, j)$ and $\lambda_k(r, s) x_k + a(r, s)$ respectively. Hence we get the equation $\lambda_k(i, j) = \lambda(r, s)$ and, consequently, formula (57). Thus each operation f_{ij} is of the form

$$\sum_{\substack{k=1 \\ k \neq i,j}}^{n+1} \lambda_k x_k + \lambda_j(i, j) x_j + a(i, j) .$$

Since $n \geqslant 3$, we can find a pair of indices i_0, j_0 in such a way that $i_0 \neq i, j$ and $j_0 \neq i, j$. Setting $x_i = x_j$ into $f_{i_0 j_0}$ and $x_{i_0} = x_{j_0}$ into f_{ij} we get identical expressions

$$\sum_{\substack{k=1 \\ k \neq i,j,i_0,j_0}}^{n+1} \lambda_k x_k + (\lambda_i + \lambda_j) x_j + \lambda_{j_0}(i_0, j_0) x_{j_0} + a(i_0, j_0)$$

and

$$\sum_{\substack{k=1\\ k\neq i,j,i_0,j_0}}^{n+1} \lambda_k x_k + (\lambda_{i_0}+\lambda_{j_0})x_{j_0} + \lambda_j(i,j)x_j + a(i,j) .$$

Hence equation (58) follows.

Finally, setting $x_1 = x_2 = \ldots = x_{n+1} = \Theta$ into f_{ij} we obtain identical expressions $a(i,j)$, which completes the proof of (59). Thus each operation f_{ij} is of the form

(60) $$f_{ij}(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) = \sum_{\substack{k=1\\ k\neq i,j}}^{n+1} \lambda_k x_k + (\lambda_i+\lambda_j)x_j + a ,$$

where, of course, $a \in A_0$.

Consider an arbitrary system $x_1, x_2, \ldots, x_{n+1}$ of elements of A. Since $\dim\mathfrak{A} = n$, at least one of these elements is generated by the remaining ones. Without loss of generality we may suppose that x_{n+1} is generated by $x_1, x_2, \ldots, x_n$, i.e.

(61) $$x_{n+1} = \sum_{r=1}^{n} \mu_r x_r + b ,$$

where $\mu_1, \mu_2, \ldots, \mu_n \in \mathcal{K}$ and $b \in A_0$. To prove our Lemma it suffices to show that

(62) $$f(x_1, x_2, \ldots, x_{n+1}) = \sum_{k=1}^{n+1} \lambda_k x_k + a ,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ and a are defined by formula (60) and x_{n+1} satisfies condition (61). We note that operation $\sum_{k=1}^{n+1}\lambda_k x + a$ is equal to $\hat{f}_{ij}(x)$. Thus, by Lemma 10, the operation $\sum_{k=1}^{n+1}\lambda_k x_k + a$ belongs to $A^{(n+1)}$.

Put

(63) $$f_0(x_1, x_2, \ldots, x_n) = f\Big(x_1, x_2, \ldots, x_n, \sum_{r=1}^{n} \mu_r x_r + b\Big) .$$

To prove (62) it is sufficient to prove the equation of two algebraic operations of n variables

(64) $$f_0(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} (\lambda_k + \lambda_{n+1}\mu_k) x_k + a + \lambda_{n+1} b .$$

From (60) and (63) we get the equations

$$f_0(x_2, x_2, x_3, \ldots, x_n) = (\lambda_1+\lambda_2+\lambda_{n+1}\mu_1+\lambda_{n+1}\mu_2)x_2 + \\ + \sum_{k=3}^{n} (\lambda_k+\lambda_{n+1}\mu_k)x_k + a + \lambda_{n+1} b ,$$

$$f_0(x_3, x_2, x_3, \ldots, x_n) = (\lambda_1+\lambda_3+\lambda_{n+1}\mu_1+\lambda_{n+1}\mu_3)x_3 + \\ + \sum_{\substack{k=2\\ k\neq 3}}^{n} (\lambda_k+\lambda_{n+1}\mu_k)x_k + a + \lambda_{n+1} b ,$$

which show that (64) holds for $x_1 = x_2$ and $x_1 = x_3$. Hence, by Lemma 3, we obtain equation (64) for all $x_1, x_2, \ldots, x_n$. Equation (62) is thus proved.

LEMMA 13. *Given two ordered pairs of positive integers $\langle i,j\rangle$, $\langle r,s\rangle$ and an integer s_0 satisfying the conditions $i \neq j$, $r \neq s$, $s \neq s_0$ and $j \neq s_0$, there exists a chain of pairs of positive integers $\langle i_1,j_1\rangle, \langle i_2,j_2\rangle, \ldots, \langle i_n,j_n\rangle$ $(1 \leqslant n \leqslant 4)$ such that*

$$\langle i_1,j_1\rangle = \langle i,j\rangle , \quad \langle i_n,j_n\rangle = \langle r,s\rangle ,$$

$$i_k \leqslant \max(4,i,j,r,s) , \quad j_k \leqslant \max(4,i,j,r,s) \quad (k = 1,2,\ldots,n) ,$$

$$i_k \neq j_k \quad (k = 1,2,\ldots,n) ,$$

$$i_{k+1} \neq j_k , \quad j_{k+1} \neq i_k, j_k, s_0 \quad (k = 1,2,\ldots,n-1) .$$

Proof. Put $N = \max(4,i,j,r,s)$. Without loss of generality we may assume that $\langle i,j\rangle = \langle 1,2\rangle$ and, consequently, $s_0 \neq 2$.

If $r \neq 2$ and $s \neq 1,2$, then the chain $\langle 1,2\rangle$, $\langle r,s\rangle$ satisfies the assertion of the lemma.

Let $r = 2$ and $s \neq 1,2$. If $s_0 \neq 1$, then we denote by p an integer satisfying the conditions $p \neq s_0$, $3 \leqslant p \leqslant N$. It is easy to verify that the chain

(65) $$\langle 1,2\rangle , \langle s_0,p\rangle , \langle 2,1\rangle , \langle 2,s\rangle$$

satisfies the assertion of the lemma. If $s_0 = 1$, then by q we denote an integer satisfying the conditions $q \neq s$, $3 \leqslant q \leqslant N$. The chain $\langle 1,2\rangle$, $\langle 1,q\rangle$, $\langle 2,s\rangle$ satisfies the assertion of the lemma.

Further, let $r = 2$ and $s = 1$. Then of course $s_0 \neq 1,2$ and the subchain $\langle 1,2\rangle$, $\langle s_0,p\rangle$, $\langle 2,1\rangle$ of chain (65) satisfies the assertion of the lemma.

Now let us suppose that $r \neq 2$ and $s = 1$. If $s_0 \neq r$, then we take chain $\langle 1,2\rangle$, $\langle 1,r\rangle$, $\langle s_0,2\rangle$, $\langle r,1\rangle$. If $s_0 = r$, then we take the chain $\langle 1,2\rangle$, $\langle r,t\rangle$, $\langle r,1\rangle$, where t is an integer satisfying the conditions $t \neq r$, $3 \leqslant t \leqslant N$.

Finally, let $r \neq 2$ and $s = 2$. If $r = 1$, then the chain $\langle 1,2\rangle$ satisfies the assertion of the lemma. Therefore we may assume that $r \neq 1,2$. If $s_0 = 1$ or $s_0 = r$, then we take the chain $\langle 1,2\rangle$, $\langle r,t\rangle$, $\langle r,2\rangle$, where t is an integer satisfying the conditions $t \neq r$, $3 \leqslant t \leqslant N$. If $s_0 \neq 1$, r, then the chain $\langle 1,2\rangle$, $\langle s_0,r\rangle$, $\langle s_0,1\rangle$, $\langle r,2\rangle$ satisfies the assertion of the lemma, which completes the proof.

LEMMA 14. *Let* $\dim\mathfrak{A} = n$ $(n \geqslant 3)$ *and* $\boldsymbol{A}^{(n)} = \boldsymbol{A}^{(n,1)}$. *Then* $\boldsymbol{A}^{(n+1)} = \boldsymbol{A}^{(n+1,1)}$.

Proof. Let $f \in \boldsymbol{A}^{(n+1)}$. For each pair $\langle i, j\rangle$ $(i \neq j;\ i, j = 1, 2, \ldots, n+1)$ and each operation $h \in \boldsymbol{A}^{(1)}$ setting $x_j = h(x_i)$ into $f(x_1, x_2, \ldots, x_{n+1})$ we obtain an operation f_{ij}^h of one variable.

First let us suppose that for every operation $h \in \boldsymbol{A}^{(1)}$ and for every pair $\langle i, j\rangle, f_{ij}^h$ is an operation of the variable x_i. Let $\langle i_1, j_1\rangle$, $\langle i_2, j_2\rangle$ be pairs satisfying the conditions $i_2 \neq j_1$ and $j_2 \neq i_1, j_1$. Setting $x_{j_2} = h_2(x_{i_2})$ into $f_{i_1j_1}^{h_1}$ and $x_{j_1} = h_1(x_{i_1})$ into $f_{i_2j_2}^{h_2}$ we get identical expressions. Thus

$$f_{i_1j_1}^{h_1}(x_{i_1}) = f_{i_2j_2}^{h_2}(x_{i_2}) \,. \tag{66}$$

Since $n+1 \geqslant 4$, by Lemma 13, every two pairs $\langle i, j\rangle$, $\langle r, s\rangle$ $(i \neq j,\ r \neq s;\ i, j, r, s = 1, 2, \ldots, n+1)$ can be connected by a chain $\langle i_1, j_1\rangle, \langle i_2, j_2\rangle, \ldots$ $\ldots, \langle i_m, j_m\rangle$ satisfying the inequalities $i_k \neq j_k$, $i_{k+1} \neq j_k$, $j_{k+1} \neq i_k, j_k$, $1 \leqslant i_k, j_k \leqslant n+1$ $(k = 1, 2, \ldots, m)$. Consequently, equation (66) holds for all operations $h_1, h_2 \in \boldsymbol{A}^{(1)}$ and all pairs $\langle i_1, j_1\rangle$, $\langle i_2, j_2\rangle$. Hence we infer that there exists an algebraic constant c such that

$$f_{ij}^h(x) = c \tag{67}$$

for any $h \in \boldsymbol{A}^{(1)}$ and any pair $\langle i, j\rangle$.

Consider a system $x_1, x_2, \ldots, x_{n+1}$ of elements of A. Since $\dim\mathfrak{A} = n$, certainly one of these elements is generated by the remaining ones. If x_j is such an element, then $x_j = h(x_i)$ for an index i different from j because of the equation $\boldsymbol{A}^{(n)} = \boldsymbol{A}^{(n,1)}$. Thus, by virtue of (67),

$$f(x_1, x_2, \ldots, x_{n+1}) = f_{ij}^h(x_i) = c \,,$$

which shows that f is a constant operation.

Now let us suppose that there exist an operation h_0 and a pair $\langle i_0, j_0\rangle$ such that $f_{i_0j_0}^{h_0}$ depends on a variable x_{s_0}, where $s_0 \neq i_0$. Let $\langle i, j\rangle$ be a pair satisfying the condition $i \neq j_0$, $j \neq i_0, j_0, s_0$. Setting $x_{j_0} = h_0(x_{i_0})$ into f_{ij}^h and $x_j = h(x_i)$ into $f_{i_0j_0}^{h_0}$ we get identical expressions. Thus f_{ij}^h depends only on the variable x_{s_0} and, consequently, $f_{ij}^h = f_{i_0j_0}^{h_0}$. Since $n+1 \geqslant 4$, by Lemma 13, every two pairs $\langle i, j\rangle$, $\langle r, s\rangle$ satisfying the inequalities $i \neq j$, $r \neq s$, $j, s \neq s_0$ can be connected by a chain $\langle i_1, j_1\rangle, \langle i_2, j_2\rangle, \ldots$ $\ldots, \langle i_m, j_m\rangle$ with following properties $i_k \neq j_k$, $i_{k+1} \neq j_k$, $j_{k+1} \neq i_k, j_k$, $j_k \neq s_0$, $1 \leqslant i_k, j_k \leqslant n+1$ $(k = 1, 2, \ldots, m)$. Consequently, for all operations $h \in \boldsymbol{A}^{(1)}$ and all pairs $\langle i, j\rangle$, with $j \neq s_0$, the operation f_{ij}^h depends only on the variable x_{s_0} and

$$f_{ij}^h(x) = g(x) \,, \tag{68}$$

where $g(x) = f_{i_0j_0}^{h_0}(x)$.

Now consider the operations $f_{is_0}^h$. Let $f_{is_0}^h$ be an operation of the variable x_k, where of course $k \neq s_0$. Since $n+1 \geqslant 4$, we can find a pair $\langle i, j\rangle$, with $j \neq i, k, s_0$. Setting $x_j = h(x_i)$ into $f_{is_0}^h$ and $x_{s_0} = h(x_i)$ into f_{ij}^h we get identical expressions. But this substitution does not change the operation $f_{is_0}^h$. Thus, by (68), $f_{is_0}^h(x_k) = g\big(h(x_i)\big)$ and, consequently, $f_{is_0}^h$ is an operation of the variable x_i satisfying the equation

$$f_{is_0}^h(x) = g\big(h(x)\big) \,. \tag{69}$$

Now we shall prove that

$$f(x_1, x_2, \ldots, x_{n+1}) = g(x_{s_0}) \,. \tag{70}$$

In the same way as in the first part of the proof we show that for any system $x_1, x_2, \ldots, x_{n+1}$ of elements of A there are indices i, j $(i \neq j)$ and an operation $h \in \boldsymbol{A}^{(1)}$ such that $x_j = h(x_i)$. Hence and from (68) and (69) by simple reasoning we get equation (70). The lemma is thus proved.

Proof of the Theorem.

(i) *Let* $\boldsymbol{A}^{(0)} \neq 0$ and $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$. By Lemma 5 and 6, there is a field $\mathcal{K}$ such that A is a linear space over $\mathcal{K}$. Now we shall prove by induction with respect to n that the class $\boldsymbol{A}^{(n)}$ consists of all operations f of the form

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a \,, \tag{71}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{K}$, $a \in A_0$, A_0 is defined by formula (50) of Lemma 9.

It should be noted that in this case we have $\Theta \in \boldsymbol{A}^{(0)}$ and, consequently, by the definition of addition and scalar-multiplication in A, all operations of form (71) are algebraic.

Let $n = 3$ and $f \in \boldsymbol{A}^{(3)}$. By Lemma 9, $\hat{f}(x) = \lambda x + a$, where $\lambda \in \mathcal{K}$ and $a \in A_0$. Since each operation of form (71) is algebraic, the operation

$$g(x_1, x_2, x_3) = f(x_1, x_2, x_3) + (1-\lambda)x_1 - a \tag{72}$$

is also algebraic. Moreover, $\hat{g}(x) = \hat{f}(x) + (1-\lambda)x - a = x$ and, consequently, $g \in \widetilde{\boldsymbol{A}}^{(3)}$. Thus, by Lemma 11 and formula (72), the operation f is of the form (71).

Now let us suppose that $n \geqslant 3$ and all operations in $\boldsymbol{A}^{(n)}$ are of the form (71). Consider the algebra $\mathfrak{A}^{(n)}$ of all n-ary algebraic operations. Taking into account (71) we infer that $\mathfrak{A}^{(n)}$ is a v^*-algebra of dimension n. Thus, by Lemma 12, all operations from $\boldsymbol{A}^{(n+1)}$ are of form (71), which completes the proof.

(ii) Let $\boldsymbol{A}^{(0)} = 0$ and $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$. By Lemmas 5 and 6, there is a field $\mathcal{K}$ such that A is a linear space over $\mathcal{K}$. Now we shall prove by induction with respect to n that the class $A^{(n)}$ consists of all operations f of the form

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a, \tag{73}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{K}$, $\sum_{k=1}^{n} \lambda_k = 1$, $a \in A_0$, A_0 is defined by formula (50) of Lemma 9.

First we shall prove that each operation $h \in A^{(1)}$ is of form (73), i.e.

$$h(x) = x + a, \quad \text{where} \quad a \in A_0. \tag{74}$$

From Lemma 9 we obtain the formula $h(x) = \lambda x + a$, where $\lambda \in \mathcal{K}$ and $a \in A_0$. If $\lambda \neq 1$, then, by Lemma 8, the operation

$$h_0(x_1, x_2) = (1-\lambda)^{-1} x_1 - \lambda(1-\lambda)^{-1} x_2$$

belongs to $\tilde{A}^{(2)}$. Thus the composition $h_0(h(x), x)$ is an algebraic operation. But this composition is equal to $(1-\lambda)^{-1} a$, which contradicts equation $A^{(0)} = 0$. Consequently, each operation $h \in A^{(1)}$ is of form (74). Hence and from Lemma 8 we infer that all operations of form (73) are algebraic.

Let $n = 3$ and $f \in A^{(3)}$. Since $a = \hat{f}(\Theta) \in A_0$ and A_0 is a linear subspace of A, the operation $h(x) = x - a$ belongs to $A^{(1)}$. Consequently, the operation

$$g(x_1, x_2, x_3) = f(x_1, x_2, x_3) - a \tag{75}$$

belongs to $A^{(3)}$. Moreover, $\hat{g}(\Theta) = \hat{f}(\Theta) - a = \Theta$ and, consequently, by (74), $\hat{g}(x) = x$, which implies $g \in \tilde{A}^{(3)}$. Thus, from Lemma 11 and formula (75) we infer that operation f is of form (73).

Now let us suppose that $n \geqslant 3$ and all operations in $A^{(n)}$ are of form (73). Considering as in part (i) of the proof the algebra $\mathfrak{A}^{(n)}$ of n-ary algebraic operations and applying Lemma 12 we infer that all operations in $A^{(n+1)}$ are of the form

$$f(x_1, x_2, \ldots, x_{n+1}) = \sum_{k=1}^{n+1} \lambda_k x_k + a,$$

where $a \in A_0$. Since, by (74), $\hat{f}(x) = x + a$, we have $\sum_{k=1}^{n+1} \lambda_k = 1$, which completes the proof of assertion (ii).

(iii) Let $A^{(3)} = A^{(3,1)}$. First we shall prove by induction with respect to n that

$$A^{(n)} = A^{(n,1)}. \tag{76}$$

For $n = 3$ it is supposed. Let $n \geqslant 3$ and let equation (76) be fulfilled. Considering, as previously, the algebra $\mathfrak{A}^{(n)}$ and applying Lemma 14, we get the equation $A^{(n+1)} = A^{(n+1,1)}$. Thus (76) holds for all integers n.

Now our assertion is a direct consequence of (76) and a theorem of Narkiewicz [7]. The Theorem is thus proved.

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