A theory of propositional types *

by

L. Henkin ** (Berkeley, Calif.)

§ 1. Let $\mathcal{D}_0$ be the set of the two truth values, T and F, and consider the operation of passing from any two sets $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ to the set $\mathcal{D}_{\alpha\beta}$ of all functions which map $\mathcal{D}_\beta$ into $\mathcal{D}_\alpha$. The family $\mathfrak{P}\mathfrak{T}$ of all sets generated from $\mathcal{D}_0$ by repeated application of this operation we call the family of *propositional types*. Thus $\mathfrak{P}\mathfrak{T}$ is the least class of sets, containing $\mathcal{D}_0$ as an element, which is closed under passage from $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ to $\mathcal{D}_{\alpha\beta}$.

In the ordinary propositional (or sentential) logic we have variables which range over $\mathcal{D}_0$, and a constant—the negation sign—which denotes one of the 4 elements of $\mathcal{D}_{00}$. We also have other constants, such as the connective $\wedge$ for conjunction, which denote *binary* operations on $\mathcal{D}_0$. Such operations may be identified with elements of $\mathcal{D}_{(00)0}$ in a familiar way; for example, the operation $\wedge^d$ denoted by the symbol $\wedge$ is described by the equations $(\wedge^d T)T = T$, $(\wedge^d T)F = F$, and $(\wedge^d F)x = F$ for all $x \in \mathcal{D}_0$. By means of formulas built up from propositional variables and connectives we may refer to particular elements of the propositional types $\mathcal{D}_0, \mathcal{D}_{(00)0}, \mathcal{D}_{((00)0)0}, \mathcal{D}_{(((00)0)0)0}, \ldots$

In the present paper we shall construct a theory with a distinct set of variables for *each* propositional type $\mathcal{D}_\alpha$. The theory will be couched in a language which permits these variables to occur bound as well as free.

Theories of this kind were first studied by Leśniewski under the name *protothetic*. An account of Leśniewski's systems is given by Grzegorczyk in [3].

In the systems of protothetic there is incorporated a rule of definition which allows for the introduction of new symbols as names of arbitrary elements of any propositional type. In the present system we start with names for only a relatively few elements, but we allow for the construction of new names by means of variables and the functional abstractor λ. We shall prove that each element possesses a name in our system.

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A theory of types which incorporates the abstractor λ was first given by Church in [1], and our present formulation owes much to his. We depart from Church, however, in taking as our only primitive constants the symbols $Q_{(0\alpha)\alpha}$ which correspond, for each α, to the identity relations over the propositional type $\mathcal{D}_\alpha$. The fact that all of the ordinary propositional connectives can be defined in terms of $Q_{(00)0}$ (the usual biconditional) by means of quantified variables over $\mathcal{D}_{00}$ is due to Tarski in [5]. The fact that the classical quantifiers themselves can be defined in terms of the symbols $Q_{(0\alpha)\alpha}$, with the aid of λ, does not seem to be stated explicitly in the literature (1).

Our theory is provided with a deductive apparatus consisting of axioms and formal rules of inference. We have been at pains *not* merely to translate into our primitive notation one of the usual systems of axioms and rules of inference governing sentential connectives and quantifiers, but to find a deductive basis for our theory which seems to express in a natural manner the fundamental properties of our primitive notions.

Our deductive formalization is *complete*, in the sense that if a formula A has the value T for every assignment of values to its free variables, then A is provable in our system (2). (Conversely, of course, only such valid formulas can be proved.) The completeness of a theory of types in terms of non-standard models was proved in [4], but this result does not seem to imply our present completeness theorem. It is true that by adding suitably to the earlier proof the present result can be obtained, but such a proof would not have the constructive character possessed by the usual completeness proofs for propositional logic, and we have preferred therefore to indicate another method of proof which seems more appropriate for a theory of types each of which is finite.

Our interest was drawn to a theory of propositional types by the problem of constructing non-standard models of a full theory of types. Since many problems of ordinary predicate logic can be reduced to questions about propositional logic (as in Herbrand's theorem, for example), our hope has been that insight into the totality of models for a full theory of types could be obtained from a study of all models of the much simpler propositional type theory. We reserve for a future paper, however, a discussion of the models of our present system other than the standard model $\mathfrak{PT}$ of propositional types.

(1) (Added September, 1962) Mr. Peter Andrews has called my attention to the paper by W. V. Quine, *Unification of universes in set theory*, Journal of Symbolic Logic 21 (1956), pp. 267-279. On page 278 Quine *does* give such a definition of quantifiers, and indeed the same definition which we employ below. I am also greatly indebted to Mr. Andrews for calling my attention to a goodly number of inaccuracies in the original version of my manuscript, and for other suggested improvements. A note by Mr. Andrews is published immediately after this paper, in which it is shown that three of the seven Axioms given in Section 5.1 below can actually be derived from the remaining ones.

(2) It is interesting to note Grzegorczyk's remark in [3] that some of Leśniewski's contemporaries considered that the nature of protothetic made any completeness proof for this system impossible.

§ 2. We now describe the symbolism which underlies our theory.

2.1. The *primitive symbols* of our system consist of three symbols which are called *improper*, namely the left and right paraentheses and the lower case Greek lambda, and an infinite number of other symbols (*proper* symbols), each of which is associated with one of the propositional types $\mathcal{D}_\alpha$. In fact, corresponding to each $\mathcal{D}_\alpha$ we provide a sequence of proper symbols $f_\alpha, g_\alpha, h_\alpha, x_\alpha, y_\alpha, z_\alpha, \ldots$ called *variables of type* $\mathcal{D}_\alpha$ (or simply *of type* α), and a single proper symbol $Q_{(0\alpha)\alpha}$ called a *constant of type* $(0\alpha)\alpha$.

2.2. Certain strings of primitive symbols are called *formulas*, and each formula is associated with a unique type $\mathcal{D}_\alpha$. In any formula each occurrence of a variable is distiguished as *free* or *bound*. The rules for constructing formulas and distinguishing between free and bound occurrences of variables are given inductively below with the aid of the symbols 'X_α', 'Y_α', 'Z_α', which we use henceforth as metamathematical variables ranging over the set of variables of type α, and the symbols 'A_α', 'B_α', 'C_α' which are used to range over arbitrary formulas of type α.

(i) A string consisting only of a proper symbol of type α is a *formula of type* α; and if it is a variable, this occurrence is *free* in the formula.

(ii) If $A_{\alpha\beta}$ and B_β are formulas of type $\alpha\beta$ and β respectively, then the string $(A_{\alpha\beta}B_\beta)$ is a *formula of type* α; and an occurence of a variable in this formula is *free* or *bound* according as it is free or bound in that one of the formulas $A_{\alpha\beta}$ or B_β in which it occurs.

(iii) If A_α is any formula of type α and X_β is any variable of type β then the string $(\lambda X_\beta A_\alpha)$ is a *formula of type* $\alpha\beta$; every occurence of the variable X_β is *bound* in this formula, and an occurrence of any other variables is *free* or *bound* according as it is free or bound in A_α.

A formula containing no free occurrence of a variable will be called *closed*.

In practice we shall often omit parentheses, in describing formulas, in accordance with the following conventions. (a) Parentheses at the beginning or end of a formula may be omitted. (b) A dot may replace a left parenthesis, and its mate will be suppressed, if the mate comes at the end of the formula. (c) Parentheses may be omitted if their use is to indicate association to the left in a sequence of three or more expressions. For example,

$$\lambda x_\alpha . \lambda y_\beta . Q_{0(\alpha\beta)(\alpha\beta)} f_{\alpha\beta}(\lambda z_\beta(g_{\alpha\gamma}x_\gamma))$$

will stand for

$$\Big(\lambda x_\alpha\Big(\lambda y_\beta\Big(\big(Q_{(0(\alpha\beta))(\alpha\beta)}f_{\alpha\beta}\big)\big(\lambda z_\beta(g_{\alpha\gamma}x_\gamma)\big)\Big)\Big)\Big).$$

§ 3. We now describe an *interpretation* of our symbolism to indicate how the formulas may be used to refer to elements of the types $\mathcal{D}_\alpha$. Under this interpretation each closed formula of type α will denote a unique element of $\mathcal{D}_\alpha$, but a formula of type α which contains free occurrences of variables will only refer to a specific element of $\mathcal{D}_\alpha$ when values (of appropriate type) are assigned to these variables.

3.1. By an *assignment* we mean a function φ, having as its domain the set of all variables of our system, such that to each variable X_α of type α the function φ assigns as value an element (φX_α) of $\mathcal{D}_\alpha$. With each formula A_α and assignment φ we associate an element $V(A_\alpha,\varphi)$ of $\mathcal{D}_\alpha$, as follows. (i) If X_α is any variable then $V(X_\alpha,\varphi)=(\varphi X_\alpha)$. (ii) Independent of φ, $V(Q_{(0\alpha)\alpha},\varphi)$ is the element f of $\mathcal{D}_{(0\alpha)\alpha}$ such that for any $x,y\in\mathcal{D}_\alpha$ we have $(fx)y=T$ if $x=y$ and $(fx)y=F$ if $x\neq y$. (iii) If $A_{\alpha\beta}$ and B_β are any formulas of type $\alpha\beta$ and β respectively, then $V\big((A_{\alpha\beta}B_\beta),\varphi\big)$ is the element of $\mathcal{D}_\alpha$ obtained by operating on the element $V(B_\beta,\varphi)$ of $\mathcal{D}_\beta$ with the function $V(A_{\alpha\beta},\varphi)$ of $\mathcal{D}_{\alpha\beta}$. (iv) If A_α is any formula of type α and X_β is any variable of type β then $V\big((\lambda X_\beta A_\alpha),\varphi\big)$ is the function of $\mathcal{D}_{\alpha\beta}$ whose value, for any $x\in\mathcal{D}_\beta$, is the element $V(A_\alpha,\varphi_x)$ of $\mathcal{D}_\alpha$, where φ_x is the assignment such that $(\varphi_x X_\beta)=x$ and $(\varphi_x Y_\gamma)=(\varphi Y_\gamma)$ for all $Y_\gamma\neq X_\beta$.

3.2. It is easy to show by induction that if A_α is any formula, and if φ and ψ are any assignments such that $(\varphi X_\beta)=(\psi X_\beta)$ for every variable X_β which occurs free in A_α, then $V(A_\alpha,\varphi)=V(A_\alpha,\psi)$. In particular, if A_α is closed then $V(A_\delta,\varphi)$ is independent of φ; this element of $\mathcal{D}_\alpha$ is then called the *denotation* of A_α and we shall put $(A_\alpha)^d=V(A_\alpha,\varphi)$ in this case. For example, $(\lambda x_\alpha x_\alpha)^d$ is the identity function of $\mathcal{D}_{\alpha\alpha}$ such that $\big((\lambda x_\alpha x_\alpha)^d y\big)=y$ for all $y\in\mathcal{D}_\alpha$.

A formula A_0 is called *valid* if and only if $V(A_0,\varphi)=T$ for *all* assignments φ. In particular, a *closed* formula A_0 is valid if and only if $(A_0)^d=T$.

§ 4. We now apply the rules of the preceding section to examine the meaning of certain special formulas under our interpretation. In particular, we shall associate, with each element x of an arbitrary type $\mathcal{D}_\alpha$, a closed formula x^n of type α such that $(x^n)^d=x$.

4.1. For any formulas A_α and B_α of type α we let $A_\alpha\equiv B_\alpha$ be the formula $(Q_{(0\alpha)\alpha}A_\alpha)B_\alpha$ of type 0. Clearly $V(A_\alpha\equiv B_\alpha,\varphi)$ is T or F according as $V(A_\alpha,\varphi)=V(B_\alpha,\varphi)$ or $V(A_\alpha,\varphi)\neq V(B_\alpha,\varphi)$.

4.2. We put $T^n=\big((\lambda x_0x_0)\equiv(\lambda x_0x_0)\big)$ and $F^n=\big((\lambda x_0x_0)\equiv(\lambda x_0T^n)\big)$. Clearly T^n and F^n are closed formulas of type 0, and we have $(T^n)^d=T$ and $(F^n)^d=F$.

4.3. Let $\neg=\big(\lambda x_0(F^n\equiv x_0)\big)$. Clearly $\neg$ is a closed formula of type (00) and denotes the negation operation: $(\neg)^dT=F$ and $(\neg)^dF=T$.

4.4. We set $\wedge=\lambda x_0\cdot\lambda y_0\cdot\big(\lambda f_{00}(f_{00}x_0\equiv y_0)\big)\equiv\big(\lambda f_{00}(f_{00}T^n)\big)$. Clearly $\wedge$ is a closed formula of type (00)0. We shall show that $(\wedge)^d$ is the operation of conjunction, i.e., that $(\wedge^dT)T=T$, $(\wedge^dT)F=F$, $(\wedge^dF)T=F$, and $(\wedge^dF)F=F$.

Consider the four elements f^1,f^2,f^3, and f^4 of $\mathcal{D}_{00}$, where f^1 is the identity function on $\mathcal{D}_0$, f^2 is the negation function $(\neg)^d$ of 4.3 above, f^3 is the function on $\mathcal{D}_0$ with constant value T, and f^4 is the function with constant value F. We see at once that $\big(\lambda f_{00}(f_{00}T^n)\big)^d$ is the element g of $\mathcal{D}_{0(00)}$ which produces the value T when acting on f^1 or f^3 and produces F when acting on f^2 or f^4.

Now turn to the part $\big(\lambda f_{00}(f_{00}x_0\equiv y_0)\big)$ of the formula $\wedge$. For any assignment φ, let g^φ be the function $V\Big(\big(\lambda f_{00}(f_{00}x_0\equiv y_0)\big),\varphi\Big)$ of $\mathcal{D}_{00}$. From the definition of $\wedge$, 4.1, and the rules of section 3.1, we see that $V(\wedge x_0y_0,\varphi)$ will be T or F according as $g^\varphi=g$ or $g^\varphi\neq g$. So now we compute.

We see that if $(\varphi_1x_0)=(\varphi_1y_0)=T$ then $(g^{\varphi_1}f^1)=(g^{\varphi_1}f^3)=T$ and $(g^{\varphi_1}f^2)=(g^{\varphi_1}f^4)=F$, so that $g^{\varphi_1}=g$ and hence $(\wedge^dT)T=T$. If $(\varphi_2x_0)=T$ and $(\varphi_2y_0)=F$ then $(g^{\varphi_2}f^2)=(g^{\varphi_2}f^4)=T$ and $(g^{\varphi_2}f^1)=(g^{\varphi_2}f^3)=F$, so that $g^{\varphi_2}\neq g$ and hence $(\wedge^dT)F=F$. If $(\varphi_3x_0)=F$ and $(\varphi_3y_0)=T$ then $(g^{\varphi_3}f^2)=(g^{\varphi_3}f^3)=T$ and $(g^{\varphi_3}f^1)=(g^{\varphi_3}f^4)=F$, so that $g^{\varphi_3}\neq g$ and hence $(\wedge^dF)T=F$. Finally, if $(\varphi_4x_0)=(\varphi_4y_0)=F$, then $(g^{\varphi_4}f^1)=(g^{\varphi_4}f^4)=T$ and $(g^{\varphi_4}f^2)=(g^{\varphi_4}f^3)=F$, so that $g^{\varphi_4}\neq g$ and hence $(\wedge^dF)F=F$.

For any formulas A_0 and B_0 of type 0 we shall write $A_0\wedge B_0$ instead of $\big((\wedge A_0)B_0\big)$.

4.5. We let $\rightarrow=\Big(\lambda x_0\Big(\lambda y_0\big((x_0\vee y_0)\equiv x_0\big)\Big)\Big)$ and $\vee=\Big(\lambda x_0\Big(\lambda y_0\big((\neg x_0)\rightarrow y_0\big)\Big)\Big)$. Clearly these are closed formulas of type (00)0, and from 4.4 we compute $(\rightarrow^dT)T=T$, $(\rightarrow^dT)F=F$, $(\rightarrow^dF)T=T$, $(\rightarrow^dF)F=T$, and $(\vee^dT)T=T$, $(\vee^dT)F=T$, $(\vee^dF)T=T$, $(\vee^dF)F=F$.

For any formulas A_0 and B_0 we write $(A_0\rightarrow B_0)$ and $(A_0\vee B_0)$ in place of $\big((\rightarrow A_0)B_0\big)$ and $\big((\vee A_0)B_0\big)$ respectively.

4.6. For any formula A_0 and variable X_α of indicated types, we put $(\forall X_\alpha A_0)=\big((\lambda X_\alpha A_0)\equiv(\lambda X_\alpha T^n)\big)$ and $(\exists X_\alpha A_0)=\neg\big(\forall X_\alpha(\neg A_0)\big)$. All occurrences of X_α are bounded in the formulas $(\forall X_\alpha A_0)$ and $(\exists X_\alpha A_0)$ (which are of type 0), and if $Y_\beta\neq X_\alpha$ then an occurrence of Y_β is free or bound in either of these formulas according as it is free or bound in A_0. By 3.1 and 4.1 we see that for any assignment φ we have $V\big((\forall X_\alpha A_0),\varphi\big)=T$ (resp. $V\big((\exists X_\alpha A_0),\varphi\big)=T$) if and only if $V(A_0,\psi)=T$) for all assignments ψ (resp., some assignment ψ) such that $(\psi Y_\beta)=(\varphi Y_\beta)$ for every variable $Y_\beta\neq X_\beta$.

4.7. Let A_0 be any formula of type 0 and X_α any variable of type α, and let Y_α be the first variable of type α not occurring in A_0. We put $(\exists ! X_\alpha A_0) = \big(\exists Y_\alpha \big(\forall X_\alpha \big(A_0 \equiv (X_\alpha \equiv Y_\alpha)\big)\big)\big)$. All occurrences of X_α or Y_α are bound in the formula $(\exists ! X_\alpha A_0)$ (which is of type 0), and an occurrence of any *other* variable is bound or free in this formula according as it is bound or free in A_0. By 3.1, 4.1, and 4.6 we see that for any assignment φ we have $V\big((\exists ! X_\alpha A_0), \varphi\big) = T$ if and only if there is exactly one assignment ψ such that $V(A_0, \psi) = T$ and $(\psi Z_\beta) = (\varphi Z_\beta)$ for every variable $Z_\beta \neq X_\alpha$.

4.8. In order to obtain a neat treatment of the description operator it is desirable first to fix one element of each type. This we do inductively by setting $a_0 = F$ and, for any α and β, taking $a_{\alpha\beta}$ to be the function of $\mathcal{D}_{\alpha\beta}$ such that $(a_{\alpha\beta} x) = a_\alpha$ for all $x \in \mathcal{D}_\beta$.

4.9. For an arbitrary type α let $t^{(\alpha)}$ be the function of $\mathcal{D}_{\alpha(0\alpha)}$ such that, for any $f \in \mathcal{D}_{0\alpha}$, $(t^{(\alpha)} f)$ is the unique $x \in \mathcal{D}_\alpha$ for which $(fx) = T$, in case there is such a unique x, or else $(t^{(\alpha)} f) = a_\alpha$ if there is no x, or if there are more than one x, such that $(fx) = T$. We shall show inductively that for each α there is a closed formula $\iota_{\alpha(0\alpha)}$ such that $(\iota_{\alpha(0\alpha)})^d = t^{(\alpha)}$. Then, for any formula A_0 and variable X_α we shall set $(\imath X_\alpha A_0) = \big(\iota_{\alpha(0\alpha}(\lambda X_\alpha A_0)\big)$. From the fact that $(\iota_{\alpha(0\alpha)})^d = t^{(\alpha)}$ we easily infer that $(\imath X_\alpha A_0)$ is a formula of type α such that, for any assignment φ, $V\big((\imath X_\alpha A_0), \varphi\big)$ is either the unique $x \in \mathcal{D}_\alpha$ such that $V(A_0, \varphi_x) = T$ [where $(\varphi_x X_\alpha) = x$ and $(\varphi_x Y_\beta) = (\varphi Y_\beta)$ for all $Y_\beta \neq X_\alpha$] if there *is* such a unique x, or else $V\big((\imath X_\alpha A_0), \varphi\big) = a_\alpha$ in the contrary case.

We begin by taking $\iota_{0(00)} = \big(\lambda f_{00} \big(f_{00} \equiv (\lambda x_0 x_0)\big)\big)$. Clearly $\iota_{0(00)}$ is a closed formula of type $0(00)$, and referring to the functions $f^1, \ldots, f^4$ of $\mathcal{D}_{00}$ (section 4.4), we see that $(\iota_{0(00)})^d (f^1) = T$ and $(\iota_{0(00)})^d (f^i) = F$ for $i = 2, 3, 4$. Now there is exactly one $x \in \mathcal{D}_0$ such that $(f^1 x) = T$ (resp., $(f^2 x) = T$), namely $x = T$ (resp., $x = F$); hence $(t^{(0)} f^1) = T$ and $(t^{(0)} f^2) = F$. On the other hand there are two $x \in \mathcal{D}_0$ for which $(f^3 x) = T$ and no $x \in \mathcal{D}_0$ for which $(f^4 x) = T$ so that $(t^{(0)} f^3) = (t^{(0)} f^4) = a_0 = F$. Hence $(\iota_{0(00)})^d = t^{(0)}$, as claimed.

Proceeding by induction we now assume that $\iota_{\alpha(0\alpha)}$, and hence $(\imath X_\alpha A_0)$, have been defined and have the required properties, and we set

$$\iota_{(\alpha\beta)(0(\alpha\beta))} = \Big(\lambda f_{0(\alpha\beta)} \cdot \lambda x_\beta \cdot \imath y_\alpha \cdot \big(\exists ! z_{\alpha\beta} (f_{0(\alpha\beta)}) z_\beta\big)\Big) \wedge \Big(\forall z_{\alpha\beta} \big(f_{0(\alpha\beta)} z_{\alpha\beta} \to (z_{\alpha\beta} x_\beta \equiv y_\alpha)\big)\Big).$$

Clearly $\iota_{(\alpha\beta)(0(\alpha\beta))}$ is a closed formula of the indicated type, and it remains to show that $(\iota_{(\alpha\beta)(0(\alpha\beta))})^d = t^{(\alpha\beta)}$.

To this end, suppose first that g is any element of $\mathcal{D}_{0(\alpha\beta)}$ such that there is exactly one $h \in \mathcal{D}_{\alpha\beta}$—say h^*—such that $(gh) = T$. Now if φ is any assignment such that $(\varphi f_{0(\alpha\beta)}) = g$, then

$$V\Big(\Big[[\exists ! z_{\alpha\beta} (f_{0(\alpha\beta)} z_\beta)] \wedge \Big[\forall z_{\alpha\beta} \Big((f_{0(\alpha\beta)} z_{\alpha\beta}) \to \big((z_{\alpha\beta} x_\beta) \equiv y_\alpha\big)\Big)\Big]\Big], \varphi\Big)$$

is T or F according as $(\varphi y_\alpha) = h^*(\varphi x_\beta)$ or $(\varphi y_\alpha) \neq h^*(\varphi x_\beta)$. Hence

$$V\Big(\Big[\imath y_\alpha \cdot [\exists ! z_{\alpha\beta} (f_{0(\alpha\beta)} z_\beta)] \wedge \Big[\forall z_{\alpha\beta} \Big((f_{0(\alpha\beta)} z_{\alpha\beta}) \to \big((z_{\alpha\beta} x_\beta) \equiv y_\beta\big)\Big)\Big]\Big], \varphi\Big) = h^*(\varphi x_\beta),$$

and therefore $\big((\iota_{(\alpha\beta)(0(\alpha\beta))})^d g\big) = h^*$ for this g.

On the other hand, suppose that g is any element of $\mathcal{D}_{0(\alpha\beta)}$ for which there is no $h \in \mathcal{D}_{\alpha\beta}$, or more than one $h \in \mathcal{D}_{\alpha\beta}$, such that $(gh) = T$. Then for any assignment φ such that $(\varphi f_{0(\alpha\beta)}) = g$ we have

$$V\Big(\Big[[\exists ! z_{\alpha\beta} (f_{0(\alpha\beta)} z_\beta)] \wedge \Big[\forall z_{\alpha\beta} \Big((f_{0(\alpha\beta)} z_{\alpha\beta}) \to \big((z_{\alpha\beta} x_\beta) \equiv y_\beta\big)\Big)\Big]\Big], \varphi\Big) = F.$$

Hence

$$V\Big(\Big[\imath y_\alpha \cdot [\exists ! z_{\alpha\beta} (f_{0(\alpha\beta)} z_\beta)] \wedge \Big[\forall z_{\alpha\beta} \Big((f_{0(\alpha\beta)} z_{\alpha\beta}) \to \big((z_{\alpha\beta} x_\beta) \equiv y_\alpha\big)\Big)\Big]\Big], \varphi\Big) = a_\alpha.$$

Using 4.8 we obtain $\big((\iota_{(\alpha\beta)(0(\alpha\beta))})^d g\big) = a_{\alpha\beta}$ for this g.

Thus for every $g \in \mathcal{D}_{0(\alpha\beta)}$ we see that $\big((\iota_{(\alpha\beta)(0(\alpha\beta))})^d g\big) = (t^{(\alpha\beta)} g)$, which completes the demonstration that $\iota_{(\alpha\beta)(0(\alpha\beta))}$ has the required property.

4.10. We are now ready to assign to each element x of any type $\mathcal{D}_\beta$ a *name*, i.e., a closed formula x^n of type α such that $(x^n)^d = x$. Indeed, if x is either of the two elements of $\mathcal{D}_0$ this has already been done in 4.2. Hence we may proceed by induction.

Suppose that $y_1, \ldots, y_q$ are distinct and are all of the elements of $\mathcal{D}_\beta$, and let us make the induction hypothesis that to every x of $\mathcal{D}_\alpha$ or of $\mathcal{D}_\beta$ we have already assigned a name x^n. Let f be any element of $\mathcal{D}_{\alpha\beta}$. Then we take

$$f^n = \Big[\lambda x_\beta \cdot \imath z_\alpha \cdot \big[(x_\beta \equiv y_1^n) \wedge \big(z_\alpha \equiv (f y_1)^n\big)\big] \wedge \ldots \wedge \big[(x_\beta \equiv y_q^n) \wedge \big(z_\alpha \equiv (f y_q)^n\big)\big]\Big].$$

Now consider any assignment φ, and say $(\varphi x_\beta) = y_i$. Clearly $V\big((x_\beta \equiv y_j^n), \varphi\big)$ will be T or F according as $j = i$ or $j \neq i$. Hence

$$V\Big(\Big[\imath z_\alpha \cdot \big[x_\beta \equiv y_1^n) \wedge \big(z_\alpha \equiv (f y_1)_1^n\big)\big] \wedge \ldots \wedge \big[x_\beta \equiv y_q^n) \wedge \big(z_\alpha \equiv (f y_0)^n\big)\big]\Big], \varphi\Big) = (f y_i),$$

so that $\big((f^n)^d y_i\big) = (f y_i)$. Since this is true for each $i = 1, \ldots, q$ we get $(f^n)^d = f$, as claimed (³).

§ 5. We turn now to the formulation of a formal deductive system, based upon the symbolism of § 3 above, by providing axioms and rules of inference.

5.1. Certain formulas of type 0 are called *axioms*. These are described under seven headings, below, some of which comprise single axioms and

(³) In place of f^n we could have used the simpler formula $A_{\alpha\beta} = \big(\imath h_{\alpha\beta} (h_{\alpha\beta} y_1^n \equiv (f y_1)^n) \wedge \ldots \wedge (h_{\alpha\beta} y_q^n \equiv (f y_q)^n)\big)$, but this would not be convenient for use in § 8 below.

others infinitely many axioms grouped in a single schema. In formulating schemata we use 'α', 'β', and 'γ' to refer to arbitrary types, 'A', 'B', 'C' for arbitrary formulas whose type is indicated by means of a subscript, and 'X', 'Y', and 'Z' for arbitrary variables (with a similar indication of type).

5.1.1. AXIOM SCHEMA 1. $A_\alpha \equiv A_\alpha$.

5.1.2. AXIOM SCHEMA 2. $(A_0 \equiv T^m) \equiv A_0$.

5.1.3. AXIOM 3. $(T^m \wedge F^n) \equiv F^n$.

5.1.4. AXIOM SCHEMA 4. $(g_{00}T^m \wedge g_{00}F^n) \equiv \big(\forall X_0(g_{00}X_0)\big)$.

5.1.5. AXIOM 5.

$$(x_\beta \equiv y_\beta) \rightarrow .\,(f_{\alpha\beta} \equiv g_{\alpha\beta}) \rightarrow .\,(f_{\alpha\beta}x_\beta) \equiv (g_{\alpha\beta}y_\beta)\,.$$

5.1.6. AXIOM SCHEMA 6.

$$\big(\forall X_\beta(f_{\alpha\beta}X_\beta \equiv g_{\alpha\beta}X_\beta)\big) \rightarrow (f_{\alpha\beta} \equiv g_{\alpha\beta})\,.$$

5.1.7. AXIOM SCHEMA 7. $\big((\lambda X_\beta B_\alpha)A_\beta\big) \equiv C_\alpha$, where C_α is obtained from B_α by replacing each free occurrence of X_β in B_α by an occurrence of A_β, *providing* no such occurrence of X_β is within a part of B_α which is a formula beginning '$(\lambda Y_\gamma$' where Y_γ is a variable free in A_β.

5.2. By the *Rule of Replacement* we refer to the ternary relation on formulas of type 0 which holds for $\langle A_0', C_0, D_0\rangle$ if and only if $A_0' = (A_\alpha \equiv B_\alpha)$ for some formulas A_α and B_α and D_0 is obtained from C_0 by replacing one occurrence of A_α by an occurrence of B_α. When this relation holds for $\langle A_\alpha \equiv B_\alpha, C_0, D_0\rangle$ we shall say that *D_0 is obtained by Rule R from $A_\alpha \equiv B_\alpha$ and C_0*.

5.3. By a *formal* proof we mean a finite column of formulas each of which is either an axiom or else is obtained by Rule R from two earlier formulas of the column. By a *formal theorem* we mean a formula which is the last line of some formal proof. We put $\vdash A_0$ if and only if A_0 is a formal theorem.

5.4. Without altering the class of formal theorems, we may replace the above list of axioms and Rule R by a longer list of axioms and rules having a somewhat simpler character. These possibilities are described below without proof of their equivalence.

5.4.1. Axiom 5 may be replaced by:

AXIOM 5.1. $(x_\beta \equiv y_\beta) \rightarrow (f_{\alpha\beta}x_\beta \equiv f_{\alpha\beta}y_\beta)$; and

AXIOM 5.2. $(f_{\alpha\beta} \equiv g_{\alpha\beta}) \rightarrow (f_{\alpha\beta}x_\beta \equiv g_{\alpha\beta}x_\beta)$.

5.4.2. Axiom Schema 7 may be replaced by:

AXIOM SCHEMA 7.1. $\big((\lambda X_\beta X_\beta)A_\beta\big) \equiv A_\beta$;

AXIOM SCHEMA 7.2. $\big((\lambda X_\beta Y_\alpha)A_\beta\big) \equiv Y_\alpha$ if $X_\beta \neq Y_\alpha$;

AXIOM SCHEMA 7.3. $\big((\lambda X_\beta(B_{\alpha\gamma}D_\gamma))A_\beta\big) \equiv \big(((\lambda X_\beta B_{\alpha\gamma})A_\beta)((\lambda X_\beta D_\gamma)A_\beta)\big)$;

AXIOM SCHEMA 7.4. $\big((\lambda X_\beta(\lambda X_\beta B_\alpha))A_\beta\big) \equiv (\lambda X_\beta B_\alpha)$; and

AXIOM SCHEMA 7.5. $\big((\lambda X_\beta(\lambda Y_\gamma B_\alpha))A_\beta\big) \equiv \big(\lambda Y_\gamma((\lambda X_\beta B_\alpha)A_\beta)\big)$ if $Y_\gamma \neq X_\alpha$ and Y_γ does not occur free in A_β.

5.4.3. Rule R may be replaced by:

RULE R1. *From $A_0 \equiv B_0$ and A_0 to obtain B_0*;

RULE R2. *From $A_{\alpha\beta} \equiv B_{\alpha\beta}$ to obtain $(A_{\alpha\beta}C_\beta) \equiv (B_{\alpha\beta}C_\beta)$*;

RULE R3. *From $A_\beta \equiv B_\beta$ to obtain $(C_{\alpha\beta}A_\beta) \equiv (C_{\alpha\beta}B_\beta)$*; and

RULE R4. *From $A_\alpha \equiv B_\alpha$ to obtain $(\lambda X_\beta A_\alpha) \equiv (\lambda X_\beta B_\alpha)$*.

§ 6. To jusfity consideration of the system of axioms and rules of § 5 we wish to show that every formal theorem is valid. By 5.3 a simple inductive argument reduces the problem to that of showing that every axiom is valid, and that Rule R preserves validity.

6.1. With the aid of 3.1, 4.4, 4.5, and 4.6 it is a trivial matter to verify the validity of all axioms falling under headings 1 through 6. In particular it will be observed that: Axiom Schema 1 expresses the most basic law of equality; Axiom Schema 2 is a simple identity involving the biconditional operation on truth values; Axiom 3 is an entry from the usual table of values for $\wedge$; Axiom 4 is a way of expressing that $\mathcal{D}_0$ contains the elements T, F, and no others; Axiom 5 expresses the substitutivity property of the identity relation; and Axiom 6 states the principle of extensionality.

Axiom Schema 7 gives the fundamental property of the functional abstractor, λ. Because of the relative complexity of its formulation, verification of the validity of its axioms by 3.1 is not as simple as in the preceding cases. The simplest way to proceed is to show (by induction on the length of B_α) that any instance of Axiom Schema 7 can be obtained by a succession of applications of Rule R to instances of Schemata 7.1-7.5 in 5.4.2 above. The validity of each instance of these schemata is a simple matter to establish by 3.1 and 3.2.

6.2. The preservation of validity by Rule R is most easily established by first showing that any application of Rule R can be effected by a succession of applications of Rules R1-R4. (This is shown by induction on the length of C_0 in the application of Rule R.) That Rules R1-R4 preserve validity may be shown directly from 3.1 and 3.2. These rules (and indeed Rule R itself) express a form of the well-known substitutivity principle for the identity relation. Despite the fact that Rule R and Axiom 5 largely overlap in their intuitive meanings, neither one seems to be dispensable in our deductive system.

§ 7. We now show how the usual theorems and rules involving propositional connectives and quantifiers may be derived within our deductive system. As in the preceding section, A_α, B_α, and C_α (resp., X_α, Y_α, Z_α) are understood to be arbitrary formulas (resp., variables) of indicated type.

7.1. RULE OF BICONDITIONAL (RULE B): *If* $\vdash A_0$ *and* $\vdash A_0 \equiv B_0$ *then* $\vdash B_0$. This is immediate by Rule R (5.2) and the definition of $\vdash$ (5.3).

7.2. EQUIVALENCE RULES (E-RULES): (i) *If* $\vdash A_\alpha \neq B_\alpha$ *then* $\vdash B_\alpha \equiv A_\alpha$, *and* (ii) *If* $\vdash A_\alpha \equiv B_\alpha$ *and* $\vdash B_\alpha \equiv C_\alpha$ *then* $\vdash A_\alpha \equiv C_\alpha$. These are derived in a familiar way by Axiom 1 (5.1.1) and Rule R.

7.3. RULE T: $\vdash A_0$ *if and only if* $\vdash A_0 \equiv T^n$. The proof is by Axiom 2 (5.1.2), Rule B, and the E-rules.

7.4. RULE OF GENERALIZATION (RULE G): *If* $\vdash A_0$ *then* $\vdash (\forall X_\alpha A_0)$.

Proof.

1. Suppose $\vdash A_0$.
2. $\vdash A_0 \equiv T^n$; by T-Rule (7.3).
3. $\vdash (\lambda X_\alpha A_0) \equiv (\lambda X_\alpha A_0)$; Axiom 1.
4. $\vdash (\lambda X_\alpha A_0) \equiv (\lambda X_\alpha T^n)$; by Rule R applied to lines 2 and 3.
5. $\vdash (\forall X_\alpha A_0)$; by line 4 and the definition of $\forall$ (4.6).

7.5. RULE OF SPECIALIZATION (RULE S): *If* $\vdash (\forall X_\alpha B_0)$ *then* $\vdash C_0$, *where* C_0 *results from substituting some formula* A_α *for all free occurrences of* X_α *in* B_0, *providing no such occurrence of* X_α *is in a part of* B_0 *which is a formula beginning with the symbols* $(\lambda Y_\gamma$, *where* Y_γ *is a variable occuring free in* A_α.

Proof.

1. Suppose that B_0, X_α, A_α, and C_0 are related as above, and that $\vdash (\forall X_\alpha B_0)$.
2. $\vdash (\lambda X_\alpha B_0) \equiv (\lambda X_\alpha T^n)$; by line 1 and definition of $\forall$ (4.6).
3. $\vdash (\lambda X_\alpha B_0) A_\alpha \equiv (\lambda X_\alpha B_0) A_\alpha$; Axiom 1.
4. $\vdash (\lambda X_\alpha B_0) A_\alpha \equiv (\lambda X_\alpha T^n) A_\alpha$; by Rule R from lines 2 and 3.
5. $\vdash (\lambda X_\alpha T^n) A_\alpha \equiv T^n$; by Axiom 7 and definition of T^n (4.2).
6. $\vdash (\lambda X_\alpha B_0) A_\alpha \equiv T^n$; by E-Rules from lines 4 and 5.
7. $\vdash (\lambda X_\alpha B_0) A_\alpha$; by T-Rule from line 6.
8. $\vdash (\lambda X_\alpha B_0) A_\alpha \equiv C_0$; by line 1 and Axiom 7.
9. $\vdash C_0$; by Rule B (7.1) from lines 7 and 8.

7.6. RULE OF SUBSTITUTION FOR FREE VARIABLES (RULE Sub): *If* B_0, X_α, A_α, *and* C_0 *are related as in the hypothesis of Rule* S (7.5), *and if* $\vdash B_0$, *then* $\vdash C_0$. This is proved by combining Rule G with Rule S.

7.7. THEOREM. $\vdash (T^n \wedge T^n) \equiv T^n$.

Proof.

1. $\vdash (T^n \wedge T^n) \equiv \big((\lambda f_{00}(f_{00}T^n \equiv T^n)) \equiv (\lambda f_{00}(f_{00}T^n))\big)$; by definitions of T^n (4.2) and $\wedge$ (4.4), Axioms 1 and 7 and Rule R.
2. $\vdash (f_{00}T^n \equiv T^n) \equiv (f_{00}T^n)$; Axiom 2.
3. $\vdash (T^n \wedge T^n) \equiv \big((\lambda f_{00}(f_{00}T^n)) \equiv (\lambda f_{00}(f_{00}T^n))\big)$; by E-Rules (7.2) from lines 1 and 2.
4. $\vdash \big((\lambda f_{00}(f_{00}T^n)) \equiv (\lambda f_{00}(f_{00}T^n))\big) \equiv T^n$; by Axiom 1 and Rule T.
5. $\vdash (T^n \wedge T^n) \equiv T^n$; by E-Rules from lines 3 and 4.

7.8. RULE OF CONJUNCTION (RULE C): *If* $\vdash A_0$ *and* $\vdash B_0$ *then* $\vdash (A_0 \wedge B_0)$.

Proof.

1. Suppose $\vdash A_0$ and $\vdash B_0$.
2. $\vdash A_0 \equiv T^n$ and $\vdash B_0 \equiv T^n$; by Rule T from line 1.
3. $\vdash (A_0 \wedge B_0) \equiv (A_0 \wedge B_0)$; Axiom 1.
4. $\vdash (A_0 \wedge B_0) \equiv (T^n \wedge T^n)$; by Rule R from lines 2 and 3.
5. $\vdash (A_0 \wedge B_0) \equiv T^n$; by E-Rules from line 4 and Theorem 7.7.
6. $\vdash (A_0 \wedge B_0)$; by Rule T from line 5.

7.9. RULE OF (PROPOSITIONAL) CASES: *If* A_0 *is any formula of type* 0, *if* X_0 *is any variable of type* 0, *if* A_0' *and* A_0'' *are obtained from* A_0 *by substituting* T^n *and* F^n *respectively for all free occurrences of* X_0 *in* A_0, *and if* $\vdash A_0'$ *and* $\vdash A_0''$, *then also* $\vdash A_0$.

Proof.

1. Suppose that A_0, X_0, A_0', and A_0'' are related as above, and that $\vdash A_0'$ and $\vdash A_0''$.
2. $\vdash \big((\lambda X_0 A_0) T^n\big) \equiv A_0'$ and $\vdash \big((\lambda X_0 A_0) F^n\big) \equiv A_0''$; Axiom Schema 7, by line 1.
3. $\vdash \big((\lambda X_0 A_0) T^n\big) \wedge \big((\lambda X_0 A_0) F^n\big)$; by Rule C (7.8) and line 1, then Rule R and line 2.
4. $\vdash \Big(\big((\lambda X_0 A_0) T^n\big) \wedge \big((\lambda X_0 A_0) F^n\big)\Big) \equiv \Big(\forall X_0\big((\lambda X_0 A_0) X_0\big)\Big)$; by Rule Sub (7.6) from Axiom Schema 4.
5. $\vdash \forall X_0 \cdot (\lambda X_0 A_0) X_0$; by Rule M (7.1) from lines 4 and 3.
6. $\vdash \big((\lambda X_0 A_0) X_0\big) \equiv A_0$; Axiom 7.
7. $\vdash (\forall X_0 A_0)$; by Rule R from lines 6 and 5.
8. $\vdash A_0$; by Rule S (7.5) from line 7.

7.10. THEOREM: $\vdash (T^n \wedge x_0) \equiv x_0$.

Proof. By Rule of Cases (7.9) from Theorem 7.7 and Axiom 3.

7.11. THEOREM SCHEMA: $\vdash (T^n \rightarrow B_0) \equiv B_0$.

Proof.

1. $\vdash (T^n \rightarrow B_0) \equiv (T^n \rightarrow B_0)$; Axiom 1.

2. $\vdash (T^n \to B_0) \equiv \big(\lambda x_0(\lambda y_0(x_0 \wedge y_0 \equiv x_0))\big) T^n B_0$; from line 1 by definition of $\to$ (4.5).
3. $\vdash (T^n \to B_0) \equiv \big((T^n \wedge B_0) \equiv T^n\big)$; by E-Rules from line 2 and Axiom Schema 7.
4. $\vdash (T^n \to B_0) \equiv B_0$; by Rule T and E-Rules from line 3 and Theorem 7.10.

7.12. Rule of Modus Ponens (Rule MP). *If* $\vdash A_0$ *and* $\vdash (A_0 \to B_0)$ *then* $\vdash B_0$.

Proof.

1. Suppose $\vdash A_0$ and $\vdash (A_0 \to B_0)$.
2. $\vdash (A_0 \equiv T^n)$; by Rule T from line 1.
3. $\vdash (T^n \to B_0)$; by Rule R from lines 2 and 1.
4. $\vdash (T^n \to B_0) \equiv B_0$; Theorem Schema 7.11.
5. $\vdash B_0$; by Rule B from lines 4 and 3.

7.13. Theorem Schema. $\vdash (F^n \to A_0)$.

Proof.

1. $\vdash (A_0 \equiv A_0) \to . \big((\lambda x_0 x_0) \equiv (\lambda x_0 T^n)\big) \to . \big((\lambda x_0 x_0) A_0\big) \equiv \big((\lambda x_0 T^n) A_0\big)$; by Rule Sub from Axiom 5.
2. $\vdash \big((\lambda x_0 x_0) \equiv (\lambda x_0 T^n)\big) \to . \big((\lambda x_0 x_0) A_0\big) \equiv \big((\lambda x_0 T^n) A_0\big)$; by Rule MP (7.11) from Axiom 1 and line 1.
3. $\vdash F^n \to (A_0 \equiv T^n)$; from line 2 by definition of F^n (4.2), Axiom 7, and Rule R.
4. $\vdash F^n \to A_0$; by Axiom 2 and Rule R from line 3.

7.14. Theorem. $\vdash (T^n \to T^n) \equiv T^n$, $\vdash (T^n \to F^n) \equiv F^n$, $\vdash (F^n \to T^n) \equiv T^n$, *and* $\vdash (F^n \to F^n) \equiv T^n$.

Proof. By Theorems 7.11 and 7.13 and Rule T.

7.15. Theorem. $\vdash (\neg T^n) \equiv F^n$ *and* $\vdash (\neg F^n) \equiv T^n$; *also* $\vdash (T^n \vee T^n) \equiv T^n$, $\vdash (T^n \vee F^n) \equiv T^n$, $\vdash (F^n \vee T^n) \equiv T^n$, *and* $\vdash (F^n \vee F^n) \equiv F^n$.

Proof. Using the definitions of $\neg$ (4.3) and of $\vee$ (4.5), these results follow easily by Axioms 1 and 7, Rules R and T, and Theorem 7.14.

7.16. Theorem. $\vdash (F^n \wedge T^n) \equiv F^n$, $\vdash (F^n \wedge F^n) \equiv F^n$, $\vdash (T^n \wedge T^n) \equiv T^n$, *and* $\vdash (T^n \wedge F^n) \equiv F^n$.

Proof. We have $\vdash (F^n \to T^n) \equiv \big((F^n \wedge T^n) \equiv F^n\big)$; by Axioms 1 and 7 and definition of $\to$ (4.5). Hence $\vdash (F^n \wedge T^n) \equiv F^n$ by Theorem 7.14, E-Rules, and Rule T. Similarly $\vdash (F^n \to F^n) \equiv \big((F^n \wedge F^n) \equiv F^n\big)$, and hence $\vdash (F^n \wedge F^n) \equiv F^n$. The remaining parts of the theorem come from 7.10.

7.17. Theorem. $\vdash (T^n \equiv T^n) \equiv T^n$, $\vdash (F^n \equiv T^n) \equiv F^n$, $\vdash (F^n \equiv F^n) \equiv T^n$, *and* $\vdash (T^n \equiv F^n) \equiv F^n$.

Proof. The first two of these results are instances of Axiom Schema 2, and the third is obtained from Axiom 1 by Rule T. To obtain the last, we first notice that

$$\vdash (T^n \equiv F^n) \to . \Big(\big(\lambda x_0(F^n \equiv x_0)\big) \equiv \big(\lambda x_0(F^n \equiv x_0)\big)\Big) \to . \big(\lambda x_0(F^n \equiv x_0)\big) T^n \equiv \big(\lambda x_0(F^n \equiv x_0)\big) F^n.$$

by Axiom 5 and Rule Sub. Using Axiom 1, Rule R, and Theorem 7.11 we can simplify this to

$$\vdash (T^n \equiv F^n) \to . \big(\lambda x_0(F^n \equiv x_0)\big) T^n \equiv \big(\lambda x_0(F^n \equiv x_0)\big) F^n.$$

Then, by Axiom 7 and Rule R are obtain from this:

$$\vdash (T^n \equiv F^n) \to . (F^n \equiv T^n) \equiv (F^n \equiv F^n).$$

Using the parts of Theorem 7.17 already established, and Rule R, we get from this, in turn:

$$\vdash (T^n \equiv F^n) \to F^n.$$

Upon applying the definition of $\to$ (4.5), Axiom 7, and Rule B, this leads to

$$\vdash \big((T^n \equiv F^n) \wedge F^n\big) \equiv (T^n \equiv F^n).$$

But $\vdash (A_0 \wedge F^n) \equiv F^n$ as we see from Theorem 7.16 and the Rule of Cases (7.9), so that, taking A_0 to be $(T^n \equiv F^n)$ and using Rule R we obtain

$$\vdash F^n \equiv (T^n \equiv F^n).$$

The desired result now follows by E-Rules.

7.18. We now consider those formulas of our system which correpond to formulas of the ordinary propositional logic. We define the class of P-*formulas* to be least class of formulas containing T^n, F^n, and each variable X_0 of type 0 as members, which is such that whenever A_0 and B_0 are in the class then so also are $\neg A_0$, $A_0 \wedge B_0$, $A_0 \vee B_0$, $A_0 \to B_0$, and $A_0 \equiv B_0$. A P-formula which is *valid* (3.2) is called a *tautology*.

7.19. Theorem. *Every tautology is a formal theorem.*

The proof is by induction on the number of free variables in the given tautology. If A_0 is a tautology containing no free variables then we easily show $\vdash A_0 \equiv T^n$ by Axioms 1 and 2 and Theorems 7.14-7.17; and then $\vdash A_0$ by Rule T. If A_0 contains some free variable, say X_0, we let A_0' and A_0'' be the formulas obtained from A_0 by substituting T^n and F^n respectively for all free occurrences of X_0 in A_0. It is easily seen that A_0' and A_0'' are themselves tautologies, since for any assignment φ we have $V(A_0', \varphi) = V(A_0, \varphi') = T$ and $V(A_0'', \varphi) = V(A_0, \varphi'') = T$, where $(\varphi' X_0) = T$, $(\varphi'' X_0) = F$, and $(\varphi' Y_\beta) = (\varphi'' Y_\beta) = (\varphi Y_\beta)$ for all variables $Y_\beta \neq X_0$.

By induction hypothesis, therefore, we obtain $\vdash A_0'$ and $\vdash A_0''$. But then $\vdash A_0$ by the Rule of Cases (7.9).

7.20. THEOREM. *If A_0 is a tautology and B_0 is obtained from A_0 by simultaneous substitution of arbitrary formulas of type 0 for the free variables of A_0, then B_0 is called a tautological formula. Clearly by Rule* Sub *and* 7.19 *we have $\vdash B_0$ for each such B_0.*

Having shown how the ordinary theorems of propositional logic are included among ours we turn to theorems of predicate logic (with equality).

7.21. THEOREM ON CHANGE OF BOUND VARIABLES. *Suppose that A_α and B_α are formulas, and X_β and Y_β are variables, such that B_α is obtained by replacing all free occurrences of X_β in A_α by Y_β, and A_α is obtained by replacing all free occurrences of Y_β in B_α by X_β. Then $\vdash(\lambda X_\beta A_\alpha)\equiv(\lambda Y_\beta B_\alpha)$.*

Proof. By Axiom 7 and our hypothesis we have $\vdash\big((\lambda X_\beta A_\alpha)X_\beta\big)\equiv A_\alpha$ and $\vdash\big((\lambda Y_\beta B_\alpha)X_\beta\big)\equiv A_\alpha$, so that $\vdash\forall X_\beta.\big((\lambda X_\beta A_\alpha)X_\beta\big)\equiv\big((\lambda Y_\beta B_\alpha)X_\beta\big)$ by E-Rules and Rule G (7.4). We then obtain $\vdash(\lambda X_\beta A_\alpha)\equiv(\lambda Y_\beta B_\alpha)$ by applying Rule Sub to Axiom 6 and using Rule MP.

7.22. *Suppose that C_0 results from substituting some formula A_α for all free occurrences of X_α in B_0, and that no such occurrence of X_α is in a part of B_0 which is a formula beginning with the symbols $(\lambda Y_\gamma$, where Y_γ is a variable occurring free in A_α. Then $\vdash(\forall X_\alpha B_0)\to C_0$.*

Proof. We have

$$\vdash\big((\lambda X_\alpha B_0)\equiv(\lambda X_\alpha T^n)\big)\to.\big((\lambda X_\alpha B_0)A_\alpha\big)\equiv\big((\lambda X_\alpha T^n)A_\alpha\big)$$

by Rule Sub applied to Axiom 5, followed by Rule MP with Axiom 1. From this we obtain $\vdash(\forall X_\alpha B_0)\to C_0$ by definition of $\forall$ (4.6), Axiom 7, E-Rules, and Rule R.

7.23. THEOREM. *If the variable X_α does not occur free in B_0, then $\vdash\Big(\big(\forall X_\alpha(B_0\to C_0)\big)\to\big(B_0\to(\forall X_0 C_0)\big)\Big)$.*

Proof. We have $\vdash\big(\forall X_\alpha(F^n\to C_0)\big)\to\big(F^n\to(\forall X_n C_0)\big)$ since this formula is tautological (7.20). We also have $\vdash\big(\forall X_\alpha(T^n\to C_0)\to\big(T^n\to(\forall X_\alpha C_0)\big)$, since this can be obtained by Rule R and Theorem 7.11 from the tautological formula $(\forall X_\alpha C_0)\to(\forall X_\alpha C_0)$. Hence by Rule of Cases (7.9) we get $\vdash\big(\forall X_\alpha(Y_0\to C_0)\big)\to\big(Y_0\to(X_\alpha C_0)\big)$, where Y_0 is a variable other than X_α which does not occur in C_0. The desired result now follows by Rule Sub.

7.24. THEOREM. *Suppose that B_α is obtained from A_α by replacing one free occurrence of X_β by a free occurrence of Y_β. Then $\vdash\big((X_\beta\equiv Y_\beta)\to(A_\alpha\equiv B_\alpha)\big)$.*

The proof is by induction on the length of A_α. If A_α is a variable (namely, X_β), the formula involved is tautological. If A_α has the form $(A_{\alpha\gamma}'A_\gamma'')$ we carry through the induction step with the aid of Axiom 5 and Rules Sub and MP. Thus it remains only to consider the case where

$A_\alpha=(\lambda Z_\gamma C_\delta)$, with $X_\beta\neq Z_\gamma\neq Y_\beta$. From the induction hypothesis $\vdash\big((X_\beta\equiv Y_\beta)\to(C_\delta\equiv D_\delta)\big)$, where D_δ arises from C_δ by replacing one free occurrence of X_β with a free occurrence of Y_β, and we then obtain $\vdash(X_\beta\equiv Y_\beta)\to\big(\forall Z_\gamma(C_\delta\equiv D_\delta)\big)$ by Rule G and Theorem 7.23. But $\vdash\big((\lambda Z_\gamma C_\delta)Z_\gamma\big)\equiv C_\delta$ and $\vdash\big((\lambda Z_\gamma D_\delta)Z_\gamma\big)\equiv D_\delta$ by Axiom 7. Hence by Rule R we obtain $\vdash\big(\forall Z_\delta(C_\gamma\equiv D_\delta)\big)\to\big((\lambda Z_\gamma C_\delta)\equiv(\lambda Z_\gamma D_\delta)\big)$ from Axiom Schema 6 (using Rule Sub). By use of Rule MP and a suitable tautological formula we now combine the above results to obtain $\vdash(X_\beta\equiv Y_\beta)\to\big((\lambda Z_\gamma C_\delta)\equiv(\lambda Z_\gamma D_\delta)\big)$, which completes our inductive proof.

7.25. Theorems 7.20-7.24, together with Rules G and MP, show that all of the usual formal theorems involving propositional connectives, quantifiers, and the equality sign are available in our system ([4]). In the sequel we shall make free use of such theorems with the simple reference "*by predicate logic*".

7.26. By induction we define for our system a relation of formal consequence which holds between a finite set Γ of formulas of type 0, and a single formula A_0, as follows. If Γ is empty then $\Gamma\vdash A_0$ if and only if $\vdash A_0$. If Γ is non-empty then $\Gamma\vdash A_0$ if and only if, for every $B_0\in\Gamma$, we have $\Gamma\sim\{B_0\}\vdash(B_0\to A_0)$, where $\Gamma\sim\{B_0\}$ is the set obtained from Γ by removing the element B_0. We list below some basic properties of this relation, all of which are easily proved by induction on the size of Γ (with the aid of the observation of 7.25).

(i) If $\Gamma\vdash A_0$ and $\Gamma\subseteq\Delta$ then $\Delta\vdash A_0$.

(ii) If $\Gamma\vdash A_0$ and $\Gamma\vdash(A_0\to B_0)$ then $\Gamma\vdash B_0$.

(iii) If $\Gamma\vdash A_0$ and X_α is not free in any formula of Γ then $\Gamma\vdash(\forall X_\alpha A_0)$.

(iv) If $\Gamma\vdash(A_\alpha\equiv B_\alpha)$, if $\Gamma\vdash C_0$, if D_0 is obtained by replacing one occurrence of A_α in C_0 by an occurrence of B_α, and if no part of C_0 containing this occurrence of A_α is a formula beginning with the symbols $(\lambda Y_\beta$, where Y_β is a variable free in $(A_\alpha\equiv B_\alpha)$ and in some formula of Γ, then $\Gamma\vdash D_0$.

We can paraphrase the content of (ii)-(iv) by saying that Rule MP, Rule G, and Rule R are valid for deductions from a set Γ, providing certain restrictions on the binding of variables are observed. The property of the consequence relation that whenever $\Gamma\vdash A_0$ we have also $\Gamma\sim\{B_0\}\vdash(B_0\to A_0)$, which is immediate from our definition of this relation, is known in the literature as the *Deduction Theorem*.

§ 8. In this section we develop our completeness result for the theory of propositional types by generalizing our method of proof of Theorem 7.19.

([4]) Compare Sections 30 and 48 of [2].

8.1. Consider the following four closed formulas of type (00): $E^1 = (\lambda x_0 x_0)$, $E^2 = \neg = (\lambda x_0 (F^n \equiv x_0))$, $E^3 = (\lambda x_0 T^n)$, and $E^4 = (\lambda x_0 F^n)$. We first establish some simple formal theorems about these.

8.1.1. THEOREM. *We have* $\vdash (E^1 T^n) \equiv T^n$, $\vdash (E^1 F^n) \equiv F^n, \ldots, \vdash (E^4 F^n) \equiv F^n$. *More precisely, for each* $i = 1, \ldots, 4$ *and each* $x \in \mathcal{D}_0$ *we have* $\vdash (E^i x^n) \equiv (f^i x)^n$, *where* $f^1, \ldots, f^4$ *are the elements of* $\mathcal{D}_{00}$ *specified in* 4.4.

Each part of this theorem is established with the aid of Axiom 7. In the case of E_2 an additional step is needed, using Axiom 1 or Axiom 2.

8.1.2. THEOREM. *For* $i \neq j$ *we have* $\vdash \neg(E^i \equiv E^j)$, $1 \leq i, j \leq 4$.

To indicate the proof for one part of this theorem, take the case $i = 1$, $j = 4$. From 8.1.1 we have $\vdash (E^1 T^n) \equiv T^n$ and $\vdash (E^4 T^n) \equiv F^n$. Hence by predicate logic (in fact, by propositional logic), we get $\vdash \neg((E^1 T^n) \equiv (E^4 T^n))$. The desired result, $\vdash \neg(E^1 \equiv E^4)$, now follows by predicate logic from Axiom 6.

8.1.3. THEOREM. $\vdash \forall g_{00}.(g_{00} \equiv E^1) \vee (g_{00} \equiv E^2) \vee (g_{00} \equiv E^3) \vee (g_{00} \equiv E^4)$.

Proof. The formula

$$[(g_{00}T^n \equiv T^n) \wedge (g_{00}F^n \equiv T^n)] \vee [(g_{00}T^n \equiv T^n) \wedge (g_{00}F^n \equiv F^n)] \vee \vee [(g_{00}T^n \equiv F^n) \wedge (g_{00}F^n \equiv T^n)] \vee [(g_{00}T^n \equiv F^n) \wedge (g_{00}F^n \equiv F^n)]$$

is tautologous, and hence is a formal theorem (7.20). Using 8.1.1 and Rule R we get

$$\vdash [(g_{00}T^n \equiv E^3T^n) \wedge (g_{00}F^n \equiv E^3F^n)] \vee \ldots \vee [(g_{00}T^n \equiv E^4T^n) \wedge (g_{00}F^n \equiv E^4F^n)].$$

Using Axiom 7 and Rule R this leads to

$$\vdash \Big[\big((\lambda x_0(g_{00}x_0 \equiv E^3x_0))T^n\big) \wedge \big((\lambda x_0(g_{00}x_0 \equiv E^3x_0))F^n\big)\Big] \vee \ldots \vee \Big[\big((\lambda x_0(g_{00}x_0 \equiv E^4x_0))T^n\big) \wedge \big((\lambda x_0(g_{00}x_0 \equiv E^4x_0))F^n\big)\Big].$$

Then, by Axiom 4 and Rule R, followed by Axiom 7 and Rule R again, we obtain

$$[\forall x_0(g_{00}x_0 \equiv E^3x_0)] \vee \ldots \vee [\forall x_0(g_{00}x_0 \equiv E^4x_0)].$$

Finally, using Axiom 6 with Rule Sub and Rule R we get $\vdash (g_{00} \equiv E^4) \vee \ldots \vee (g_{00} \equiv E^4)$ and the theorem follows by Rule G.

8.1.4. THEOREM. $\vdash \big((\exists! x_0(E^1x_0)) \wedge (E^1T^n)\big)$, $\vdash \big((\exists! x_0(E^2x_0)) \wedge E^2F^n\big)$, $\vdash \neg(\exists! x_0(E^3x_0))$, *and* $\vdash \neg(\exists! x_0(E^4x_0))$.

Each of these formal theorems is established by combining 8.1.1, Axiom 4, and predicate logic.

8.2. THEOREM. *We turn now to the formulas* $\iota_{\alpha(0\alpha)}$, *introduced in* 4.9, *and shall show by induction on* α *that*

(I) $$\vdash \forall g_{0\alpha}.(\exists! x_\alpha(g_{0\alpha}x_\alpha)) \to (g_{0\alpha}(\iota_{\alpha(0\alpha)}g_{0\alpha})).$$

We remark that once we have established this theorem for some α we may employ predicate logic, the definition (4.10) of $(\imath X_\alpha A_0)$, and Axiom 7 to obtain

(II) $$(\exists! X_\alpha A_0) \to ((\lambda X_\alpha A_0)(\imath X_\alpha A_0))$$

for any formula A_0.

8.2.1. Since $\iota_{0(00)} = (\lambda f_{00}(f_{00} \equiv (\lambda x_0 x_0)))$, we see by definition (8.1) of E^1 and Axiom 7 that for $i = 1, 2$ we have $\vdash (\iota_{0(00)}E^i) \equiv (E^i \equiv E^1)$. Using 8.1.2 we then obtain $\vdash (\iota_{0(00)}E^1) \equiv T^n$ and $\vdash (\iota_{0(00)}E^2) \equiv F^n$. We can then infer from Theorems 8.1.4, 8.1.3, and 8.1.1 again, by predicate logic, that $\vdash \forall g_{00}.(\exists! x_0(g_{00}x_0)) \to (g_{00}(\iota_{0(00)}g_{00}))$, as required.

8.2.2. We now make the induction hypothesis that

(I) $$\vdash \forall g_{0\alpha}.(\exists! x_\alpha(g_{0\alpha}x_\alpha)) \to (g_{0\alpha}(\iota_{\alpha(0\alpha)}g_{0\alpha})),$$

so that, by the remark following 8.2 (I), we have

(II) $$\vdash (\exists! X_\alpha A_0) \to ((\lambda X_\alpha A_0)(\imath X_\alpha A_0))$$

for any formula A_0. And we seek to prove that

(III) $$\vdash \forall g_{0(\alpha\beta)}.(\exists! z_{\alpha\beta}(g_{0(\alpha\beta)}z_{\alpha\beta})) \to (g_{0(\alpha\beta)}(\iota_{(\alpha\beta)(0(\alpha\beta))}g_{0(\alpha\beta)})).$$

From the definition (4.9) of $\iota_{(\alpha\beta)(0(\alpha\beta))}$ and Axiom 7 we obtain

$$\vdash (\iota_{(\alpha\beta)(0(\alpha\beta))}g_{0(\alpha\beta)}) \equiv [\lambda x_\beta.\imath y_\alpha.(\exists! z_{\alpha\beta}(g_{0(\alpha\beta)}z_{\alpha\beta})) \wedge (\forall z_{\alpha\beta}.g_{0(\alpha\beta)}z_{\alpha\beta} \to (z_{\alpha\beta}x_\beta \equiv y_\alpha))].$$

Hence, by predicate logic,

(IV) $$\vdash (\exists! z_{\alpha\beta}(g_{0(\alpha\beta)}z_{\alpha\beta})) \to. ((\iota_{(\alpha\beta)(0(\alpha\beta))}g_{0(\alpha\beta)})) \equiv (\lambda x_\beta.\imath y_\alpha.\forall z_{\alpha\beta}.g_{0(\alpha\beta)}z_{\alpha\beta} \to (z_{\alpha\beta}x_\beta \equiv y_\alpha)).$$

However, again by predicate logic,

$$\vdash (\exists! z_{\alpha\beta}(g_{0(\alpha\beta)}z_{\alpha\beta})) \to .\exists! y_\alpha.\forall z_{\alpha\beta}.g_{0(\alpha\beta)}z_{\alpha\beta} \to .z_{\alpha\beta}x_\alpha \equiv y_\alpha.$$

From this, using (II) above, and Axiom 7, we get

$$\vdash (\exists! z_{\alpha\beta}(g_{0(\alpha\beta)}z_{\alpha\beta})) \to .\forall z_{\alpha\beta}.g_{0(\alpha\beta)}z_{\alpha\beta} \to .z_{\alpha\beta}x_\beta \equiv (\imath y_\alpha.\forall z_{\alpha\beta}.g_{0(\alpha\beta)}z_{\alpha\beta} \to .z_{\alpha\beta}x_\beta \equiv y_\alpha).$$

By predicate logic, Axiom 6, and Axiom 7 this, in turn, leads to

$$\vdash (\exists! z_{\alpha\beta}(g_{0(\alpha\beta)}z_{\alpha\beta})) \to .\forall z_{\alpha\beta}.g_{0(\alpha\beta)}z_{\alpha\beta} \to .z_{\alpha\beta} \equiv (\lambda x_\beta.\imath y_\alpha.\forall z_{\alpha\beta}.g_{0(\alpha\beta)}z_{\alpha\beta} \to .z_{\alpha\beta}x_\beta \equiv y_\alpha).$$

When this is combined with (IV) we get

$$\vdash (\exists!\, z_{\alpha\beta}(g_{0(\alpha\beta)} z_{\alpha\beta})) \to . \forall z_{\alpha\beta} . g_{0(\alpha\beta)} z_{\alpha\beta} \to . z_{\alpha\beta} \equiv \iota_{(\alpha\beta)(0(\alpha\beta))} g_{0(\alpha\beta)}$$

from which the desired (III)

$$\vdash \forall g_{0(\alpha\beta)} . (\exists!\, z_{\alpha\beta}(g_{0(\alpha\beta)} z_{\alpha\beta})) \to . g_{0(\alpha\beta)}(\iota_{(\alpha\beta)(0(\alpha\beta))} g_{0(\alpha\beta)})$$

follows by elementary predicate logic.

This completes our inductive proof of Theorem 8.2.

8.3. Using the result of the preceding section that

(II) $$\vdash (\exists!\, X_\alpha A_0) \to ((\lambda X_\alpha A_0)(\imath X_\alpha A_0)),$$

we turn now to some formal theorems involving the formulas x^n (for every element x of any propositional type $\mathcal{D}_\gamma$) which were defined in 4.10. In fact, if $\mathcal{D}_\gamma$ is any propositional type and $z_1, \ldots, z_p$ are distinct and include all elements of $\mathcal{D}_\gamma$, we shall show (by induction on γ) that:

(1) $$\vdash \neg(z_i^n \equiv z_j^n) \quad \textit{if } i \neq j,,$$

(2) $$\vdash \forall x_\gamma . (x_\gamma \equiv z_1^n) \vee \ldots \vee (x_\gamma \equiv z_p^n),$$

(3) *if* $\gamma = (\alpha\beta)$ *then for any* $y \in \mathcal{D}_\alpha$ *we have* $\vdash (z_i^n y^n) \equiv (z_i y)^n$.

For the case where $\gamma = 0$ we have $p = 2$ so there is just one formula (1) and this is a tautology; the formula (2) is obtained by Rule G From another tautology; and (3) holds vacuously.

Proceeding by induction, let us now assume that $\gamma = (\alpha\beta)$, and that $f_1, \ldots, f_p$ is a list of the distinct elements of $\mathcal{D}_\gamma$ while $y_1, \ldots, y_q$ is a list of the distinct elements of $\mathcal{D}_\beta$.

We first give a proof of (3). From the definition (4.10) of f_i^n and Axiom 7 we have

$$\vdash (f_i^n y_j^n) \equiv \imath z_\alpha . (y_j^n \equiv y_1^n \wedge z_\alpha \equiv (f_i y_1)^n) \vee \ldots \vee (y_j^n \equiv y_q^n \wedge z_\alpha \equiv (f_i y_q)^n).$$

Then, using part (1) of our induction hypothesis concerning $\mathcal{D}_\beta$, we have $\vdash \neg(y_j^n \equiv y_k^n)$ for $j \neq k$, so we obtain

(*) $$\vdash (f_i^n y_j^n) \equiv \imath z_\alpha . z_\alpha \equiv (f_i y_j)^n.$$

But by predicate logic we know $\vdash \exists!\, z_\alpha . z_\alpha \equiv (f_i y_j)^n$, and hence by (II) above

$$\vdash ((\lambda z_\alpha . z_\alpha \equiv (f_i y_j)^n)(\imath z_\alpha . z_\alpha \equiv (f_i y_j)^n),$$

which in turn yields

$$\vdash (\imath z_\alpha . z_\alpha = (f_i y_j)^n) \equiv (f_i y_j)^n$$

by Axiom 7. Combining this with (*) above we get the desired

(3) $$\vdash (f_i^n y_j^n) \equiv (f_i y_j)^n.$$

Turning next to a proof of (1), we observe that if $i \neq j$ $(1 \leqslant i, j \leqslant p)$ then for some $k = 1, \ldots, q$ we have $(f_i y_k) \neq (f_j y_k)$. Hence by part 1 of our induction hypothesis (concerning $\mathcal{D}_\alpha$) we know

$$\vdash \neg((f_i y_k)^n \equiv (f_j y_k)^n).$$

Combining this with (3), which has just been proved, we obtain

$$\vdash \neg((f_i^n y_k^n) \equiv (f_j^n y_k^n)).$$

Finally, using this with Axiom 5, we get the desired $\vdash \neg(f_i^n \equiv f_j^n)$ by predicate logic.

It remains only to establish (2). For this purpose we note that part (2) of our induction hypothesis (concerning the types $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$) implies by predicate logic that the disjunction of all formulas

(**) $$((x_\gamma y_1^n) \equiv w_{k_1}^n) \wedge \ldots \wedge ((x_\gamma y_q^n) \equiv w_{k_q}^n),$$

where $w_1, \ldots, w_r$ are all elements of $\mathcal{D}_\alpha$ and $k_1, \ldots, k_q$ are arbitrary integers between 1 and r inclusive, is a formal theorem. But for each sequence $k = \langle k_1, \ldots, k_r \rangle$ there is a unique $f_{i_k} \in \mathcal{D}_{\alpha\beta}$ such that $(f_{i_k} y_j) = w_{k_j}$ for each $j = 1, \ldots, q$, and hence by (3), which was proved above, we have

$$\vdash (f_{i_k}^n y_1^n \equiv w_{k_1}^n) \wedge \ldots \wedge (f_{i_k}^n y_q^n \equiv w_{k_q}^n).$$

From this and (**) we see that the disjunction of all formulas

$$((x_\gamma y_1^n) \equiv f_{i_k}^n y_1^n) \wedge \ldots \wedge ((x_\gamma y_q^n) \equiv f_{i_k}^n y_q^n),$$

for all $f_{i_k} \in \mathcal{D}_{\alpha\beta}$, is a formal theorem, and so by another use of Part (2) of our induction hypothesis (concerning $\mathcal{D}_\beta$) we see that

$$\vdash (\forall y_\beta(x_\gamma y_\beta \equiv f_1^n y_\beta)) \vee \ldots \vee (\forall y_\beta(x_\gamma y_\beta \equiv f_p^n y_\beta)).$$

When this is combined with Axiom 6 we obtain the desired

(2) $$\vdash \forall x_\gamma . x_\gamma \equiv f_1^n \vee \ldots \vee x_\gamma \equiv f_p^n$$

by predicate logic.

8.4. We have now developed all of the machinery needed for our completeness proof. In this section we shall establish the following result.

LEMMA. *Let* A_α *be any formula and* φ *any assignment. Let* $A_\alpha^{(\varphi)}$ *be the formula obtained from* A_α *by substituting, for each free occurrence of any variable* X_β *in* A_α, *the formula* $(\varphi X_\beta)^n$. *Then* $\vdash A_\alpha^{(\varphi)} \equiv (V(A_\alpha, \varphi))^n$.

In particular, if A_α is closed we have $A_\alpha^{(\varphi)} = A_\alpha$, and if A_0 is valid we have $V(A_0, \varphi) = T$ (for any φ), so for closed, valid formulas A_0 the lemma gives $\vdash A_0 \equiv T^n$ from which $\vdash A_0$ is obtained at once by Axiom 2 and Rule R. For valid A_0 which are not closed we then infer easily that

$\vdash A_0$ must also hold by considering the closure $\forall X_{\beta_1} \ldots \forall X_{\beta_n}. A_\alpha$, where $X_{\beta_1}, \ldots, X_{\beta_n}$ are all of the variables occurring freely in A_α. Thus the completeness proof is completed.

The lemma is proved by induction on the length of A_α. Indeed, if A_α is a variable then $A_\alpha^{(\varphi)} = \big(V(A_\alpha, \varphi)\big)^n$, so $A_\alpha^{(\varphi)} \equiv \big(V(A_\alpha, \varphi)\big)^n$ is an instance of Axiom 1.

Next consider the case where A_α is $Q_{(0\gamma)\gamma}$, and suppose that $y_1, \ldots, y_q$ are all of the elements of $\mathcal{D}_\gamma$. For $1 \leqslant i,\ j \leqslant q$ we have $\vdash (Q_{(0\gamma)\gamma} y_i^n y_j^n) \equiv F^n$ if $i \neq j$, by 8.3(1) and definitions of $\equiv$ and $\neg$, and we have $\vdash (Q_{(0\gamma)\gamma} y_i^n y_i^n) \equiv T^n$ by Axioms 1 and 2 and definition of $\equiv$. But if f is the element $(Q_{(0\gamma)\gamma})^d$ of $\mathcal{D}_{(0\gamma)\gamma}$ we also have $\vdash (f^n y_i^n y_j^n) \equiv F^n$ if $i \neq j$ and $\vdash (f_i^n y_i^n y_i^n) \equiv T^n$ by 8.3(3). Thus we obtain, for each $i = 1, \ldots, q$,

$$\vdash \big((Q_{(0\gamma)\gamma} y_i^n y_1^n) \equiv (f^n y_i^n y_1^n)\big) \wedge \ldots \wedge \big((Q_{(0\gamma)\gamma} y_i^n y_q^n) \equiv (f^n y_i^n y_q^n)\big),$$

and then

$$\vdash \forall x_\gamma . (Q_{(0\gamma)\gamma} y_i^n x_\gamma) \equiv (f^n y_i^n x_\gamma),$$

by using 8.3(2), and finally $\vdash (Q_{(0\gamma)\gamma} y_i^n) \equiv (f^n y_i^n)$ by Axiom 6. Since this holds for each $i = 1, \ldots, q$, we repeat the same pattern of argument to arrive at the conclusion $\vdash Q_{(0\gamma)\gamma} \equiv f^n$. Since $Q_{(0\gamma)\gamma}^{(\varphi)} = Q_{(0\gamma)\gamma}$ and $V(Q_{(0\gamma)\gamma}, \varphi) = f$, the lemma is seen to hold in this case.

Turing to the case where A_α has the form $(B_{\alpha\beta} C_\beta)$, we make the induction hypothesis that

$$\vdash B_{\alpha\beta}^{(\varphi)} \equiv \big(V(B_{\alpha\beta}, \varphi)\big)^n$$

and

$$\vdash C_\beta^{(\varphi)} \equiv \big(V(C_\beta, \varphi)\big)^n.$$

Using 8.3(3) we then conclude that

$$\vdash (B_{\alpha\beta}^{(\varphi)} C_\beta^{(\varphi)}) \equiv \big(V(B_{\alpha\beta}, \varphi) V(C_\beta, \varphi)\big)^n.$$

But by definition of $A_\alpha^{(\varphi)}$ we have $(B_{\alpha\beta}^{(\varphi)} C_\beta^{(\varphi)}) = (B_{\alpha\beta} C_\beta)^{(\varphi)}$, and by definition of V we have $\big(V(B_{\alpha\beta}, \varphi) V(C_\beta, \varphi)\big) = V\big((B_{\alpha\beta} C_\beta), \varphi\big)$, so that $\vdash (B_{\alpha\beta} C_\beta)^{(\varphi)} \equiv \big(V\big((B_{\alpha\beta} C_\beta), \varphi\big)\big)^n$, as needed to establish the lemma in this case.

Finally we take up proof of the lemma for the case where A_α has the form $(\lambda X_\gamma C_\beta)$, so that $\alpha = (\beta\gamma)$. Let us suppose that $y_1, \ldots, y_q$ are all of the elements of $\mathcal{D}_\gamma$. By induction hypothesis we have $\vdash C_\beta^{(\psi)} \equiv \big(V(C_\beta, \psi)\big)^n$ for *every* assignment ψ.

Now given any assignment φ and any $y_i \in \mathcal{D}_\gamma$, we known from Axiom 7 and the definition of $(\lambda X_\gamma C_\beta)^{(\varphi)}$ that $\vdash (\lambda X_\gamma C_\beta)^{(\varphi)} y_i^n \equiv C_\beta^{(\varphi_i)}$, where $\varphi_i(X_\gamma) = y_i$ and $\varphi_i(Z_\delta) = \varphi(Z_\delta)$ for every variable $Z_\delta \neq X_\gamma$. But $V(C_\beta, \varphi_i) = \big(V\big((\lambda X_\gamma C_\beta), \varphi\big) y_i\big)$ by definition of V and the fact (Section 4) that $V(y_i^n, \varphi) = (y_i^n)^d = y_i$. Hence our induction hypothesis yields

$$\vdash (\lambda X_\gamma C_\beta)^{(\varphi)} y_i^n \equiv \big(V\big((\lambda X_\gamma C_\beta), \varphi\big) y_i\big)^n.$$

Using 8.3(3) we therefore get

$$\vdash (\lambda X_\gamma C_\beta)^{(\varphi)} y_i^n \equiv \big(V\big((\lambda X_\gamma C_\beta), \varphi\big)\big)^n y_i^n.$$

Since this is true for each $i = 1, \ldots, q$ we can use 8.3(2) and Axiom 6 to obtain

$$\vdash (\lambda X_\gamma C_\beta)^{(\varphi)} \equiv \big(V\big((\lambda X_\gamma C_\beta), \varphi\big)\big)^n$$

which completes the inductive proof of our lemma.

Remark. It is easy to see that while our completeness proof (§ 8) depends in an essential way on the restriction of our system to the class of propositional types, $\mathcal{PT}$, the essential features of our semantical and syntactical development of propositional and predicate logic (§4—§7) can be carried through for a system of the full theory of types based on the same primitive notions. To obtain such a system, following Church [1], we deal with a class $\mathcal{FT}$ of finite types obtained by starting from $\mathcal{D}_0$ and from an arbitrary set $\mathcal{D}_1$ (whose elements are called *individuals*), and generating all further types obtained by passing from any $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ to $\mathcal{D}_{\alpha\beta}$.

Now consider a symbolism $\mathcal{S}$ obtained from our present symbolism (§ 2) simply by adding variables X_α and constants $Q_{(0\alpha)\alpha}$ of the new types to those of the old. The definitions of propositional connectives and quantifiers (4.1 — 4.7) obviously remain valid for this symbolism $\mathcal{S}$. Furthermore, if we adopt the same axiom system (§ 5), it is clear that the derivation of the basic rules of predicate logic, consisting of sections 7.20 — 7.24, 5.1.1, 7.4, and 7.12, will all continue to hold in the new system.

However, the system obtained in this way does not seem to be a really adequate formulation of type-theoretic predicate logic, since it does not seem possible to prove such a formula as

$$(**) \qquad \neg(x_1 \equiv u_1) \to (\exists f_{11})\,(f_{11} x_1 \equiv y_1 \wedge f_{11} u_1 \equiv v_1).$$

To remedy this defect it is necessary to add to the system $\mathcal{S}$ a new primitive constant $\iota_{1(01)}$, to enable us to extend 4.9 by introducing description-formulas $\iota_{\alpha(0\alpha)}$ for all $\alpha \in \mathcal{FT}$. We must then add to our axiom system an

Axiom 8.

$$(\exists! x_1)\,(f_{01} x_1) \to f_{01}(\iota_{1(01)} f_{01}),$$

which will enable us to extend Theorem 8.2 to the new system. With the help of this we can easily prove such formulas as (**).

Added October 15, 1962. The referee has just called to my attention the paper *St. Leśniewski's protothetics* by Jerzy Słupecki, *Studia Logica*, vol 1 (1953), pp. 44-111. This paper, constituting a reconstruction of Leśniewski's work based upon notes which he left, presents three formulations of the theory of propositional types, one of which is based upon equivalence. A proof of completeness is given. However, the systems differ from ours in various ways, principally in a rule of definition allowing the introduction of names for arbitrary elements of the hierarchy of propositional types.

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A reduction of the axioms for the theory of propositional types

by

P. Andrews * (Princeton, N. J.)

Throughout this paper we shall follow the notation used by Henkin in his paper *A theory of propositional types* (this Volume, pp. 323-342), hereafter referred to as [H]. Reference numbers followed by 'H' refer to sections of that paper. (1)

Henkin's paper is of particular interest in that it takes symbols for the identity relation as the sole primitive constants. That there is ample historical precedent for special interest in such a system is attested by the following passage from Ramsey's article, *The Foundations of Mathematics*:

"The preceding and other considerations led Wittgenstein to the view that mathematics does not consist of tautologies, but of what he called 'equations', for which I should prefer to substitute 'identities'.... (It) is interesting to see whether a theory of mathematics could not be constructed with identities for its foundation. I have spent a lot of time developing such a theory, and found that it was faced with what seemed to me insuperable difficulties." (2)

The full beauty of Henkin's theory of propositional types can perhaps best be appreciated when the system of axioms in section 5.1H is simplified somewhat. Therefore let us replace this system of axioms by the following

AXIOMS.

(1) $(g_{00}T^{n} \wedge g_{00}F^{n}) \equiv \forall x_0(g_{00}x_0)$.

($2^{\alpha 0}$) $(f_{\alpha 0} \equiv g_{\alpha 0}) \rightarrow (h_{0(\alpha 0)}f_{\alpha 0} \equiv h_{0(\alpha 0)}g_{\alpha 0})$.

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(1) I wish to express my appreciation to Professor Henkin for the privilege of reading his paper before publication, and for his many helpful suggestions, comments, and criticisms during the writing of the present paper.

(2) F. P. Ramsey, *The Foundations of Mathematics*, Proceedings of the London Mathematical Society, series 2, 25 (1926), p. 350.