[3] R. Baer, *Gruppentheoretische Eigenschaften*, Math. Ann. 149 (1963), pp. 181-210.
[4] — *Gruppen mit Minimalbedingung*, Math. Ann. 150 (1963), pp. 1-44.
[5] A. M. Duguid, D. H. McLain, *FC-nilpotent and FC-soluble groups*, Proc. Cambridge Phil. Soc. 52 (1956), pp. 391-398.
[6] L. Fuchs, *Abelian groups*, Budapest 1958.
[7] W. Specht, *Gruppentheorie*, Berlin-Göttingen-Heidelberg 1956.

MATHEMATISCHES SEMINAR DER UNIVERSITÄT FRANKFURT AM MAIN

Reçu par la Rédaction le 5. 8. 1963

ACTA ARITHMETICA
IX (1964)

On certain elliptic functions of order three

by

P. Du Val (London)

1. The parametrisation of the general plane cubic curve, in the form

$$y^2 = 4x^3 - g_2x - g_3, \tag{1}$$

where g_2, g_3 are constants, by means of the Weierstrassian elliptic functions

$$x = \wp u, \quad y = \wp' u$$

is familiar. There is, however, another canonical form of the equation of the cubic, in terms of homogeneous coordinates (x, y, z)

$$x^3 + y^3 + z^3 + 6mxyz = 0, \tag{2}$$

which from a geometrical point of view is at least as important as (1); and the elliptic functions by which this equation can be parametrised have not, so far as I know, received attention. A brief study of their outstanding properties is the object of this note.

2. We denote by Ω a lattice of complex numbers $\omega = p\omega_1 + q\omega_2$, where p, q range over all integers, and $I(\omega_2/\omega_1) > 0$. ($I(\tau)$ denoting the imaginary part of any complex number τ.) $n\Omega$ will denote the lattice of numbers $n\omega$ for all ω in Ω. ω_1, ω_2 are a basis for Ω. We define also

$$\omega_3 = -\omega_1 - \omega_2, \quad \omega_4 = \omega_1 - \omega_2.$$

Ω has four sublattices $\Omega^{(i)}$ $(i = 1, 2, 3, 4)$ (i.e. subgroups with respect to addition) of index three, with the bases

$$\begin{cases} \omega_1^{(1)} = \omega_1, \\ \omega_2^{(1)} = 3\omega_2, \end{cases} \quad \begin{cases} \omega_1^{(2)} = 3\omega_1, \\ \omega_2^{(2)} = \omega_2, \end{cases} \quad \begin{cases} \omega_1^{(3)} = 2\omega_1 - \omega_2, \\ \omega_2^{(3)} = -\omega_1 + 2\omega_2, \end{cases} \quad \begin{cases} \omega_1^{(4)} = 2\omega_1 + \omega_2, \\ \omega_2^{(4)} = \omega_1 + 2\omega_2, \end{cases} \tag{3}$$

of which $\Omega^{(i)}$ contains ω_i but none of the other three of $\omega_1, \omega_2, \omega_3, \omega_4$. 3Ω is a sublattice of index three in each of these; in fact, with the convention (3) as to their bases

$$3\Omega = (\Omega^{(1)})^{(2)} = (\Omega^{(2)})^{(1)} = (\Omega^{(3)})^{(4)} = (\Omega^{(4)})^{(3)}. \tag{4}$$

$\Omega^{(i)}$ is generated by adjoining ω_i to 3Ω; in the same way $\frac{1}{3}\Omega^{(i)}$ is generated by adjoining $\frac{1}{3}\omega_i$ to Ω, and Ω by adjoining ω_j $(j \neq i)$ to $\Omega^{(i)}$.

3. A change of basis

$$\omega_1' = a\omega_1 + b\omega_2, \qquad \omega_2' = c\omega_1 + d\omega_2,$$

where a, b, c, d are integers and $ad - bc = 1$, permutes the four sublattices $\Omega^{(i)}$ evenly amongst themselves, and effects on the eight residue classes $\pm\omega_i \pmod{3\Omega}$ $(i = 1, 2, 3, 4)$ the permutations

$$(5)\quad \left\{\begin{array}{ll} \pm(\omega_1, \omega_2, \omega_3, \omega_4), & \pm(\omega_2, \omega_3, \omega_1, \omega_4), \\ & \qquad \pm(\omega_3, \omega_1, \omega_2, \omega_4), \\ \pm(\omega_2, -\omega_1, \omega_4, -\omega_3), & \pm(-\omega_1, \omega_4, \omega_2, -\omega_3), \\ & \qquad \pm(\omega_4, \omega_2, -\omega_1, -\omega_3), \\ \pm(\omega_3, -\omega_4, -\omega_1, \omega_2), & \pm(-\omega_4, -\omega_1, \omega_3, \omega_2), \\ & \qquad \pm(-\omega_1, \omega_3, -\omega_4, \omega_2), \\ \pm(\omega_4, \omega_3, -\omega_2, -\omega_1), & \pm(\omega_3, -\omega_2, \omega_4, -\omega_1), \\ & \qquad \pm(-\omega_2, \omega_4, \omega_3, -\omega_1), \end{array}\right.$$

where the ambiguity before the bracket affects the signs of all four elements simultaneously. The changes of basis effecting the identical permutation are those of Klein's subgroup Γ_{12} of the modular group Γ, and each of the twelve permutations (5) is effected by a coset of Γ_{12} in Γ. (Klein-Fricke, *Modulfunktionen I*, pp. 353-354.)

4. In accordance with the usual convention, we take the period lattice of the Weierstrassian elliptic function $\wp u$ to be 2Ω; and we consider the quasi-elliptic function ζu, defined by

$$\zeta' u = -\wp u, \qquad \zeta(-u) = -\zeta u,$$

and satisfying

$$\zeta(u + 2\omega_i) = \zeta u + 2\eta_i \qquad (i=1,2)$$

where η_1, η_2 are constants; from this we define four simply periodic functions

$$\zeta_i u = \zeta u - \frac{\eta_i}{\omega_i} u \qquad (i = 1, 2, 3, 4)$$

(where $\eta_3 = -\eta_1 - \eta_2$, $\eta_4 = \eta_1 - \eta_2$) satisfying

$$\zeta_i(u + 2\omega_i) = \zeta_i u \qquad (i = 1, 2, 3, 4).$$

We define also the constants $c_i = 3\zeta_i(\frac{2}{3}\omega_i)$ $(i = 1, 2, 3, 4)$. From the addition formulae for ζu, $\wp u$ we have

$$(6)\qquad 3\zeta_i(\tfrac{2}{3}\omega_i) = \frac{1}{2}\,\frac{\wp''(\frac{2}{3}\omega_i)}{\wp'(\frac{2}{3}\omega_i)}, \qquad 3\wp(\tfrac{2}{3}\omega_i) = \frac{1}{4}\left(\frac{\wp''(\frac{2}{3}\omega_i)}{\wp'(\frac{2}{3}\omega_i)}\right)^2,$$

so that (as $\wp'' u = 6\wp^2 u - \frac{1}{2}g_2$)

$$(7)\qquad \wp(\tfrac{2}{3}\omega_i) = \tfrac{1}{3}c_i^2, \qquad \wp'(\tfrac{2}{3}\omega_i) = \tfrac{1}{4}g_2/c_i - \tfrac{1}{3}c_i^3.$$

5. Any three of the four quantities c_i $(i = 1, 2, 3, 4)$ satisfy one or other of the eight relations

$$(8)\qquad c_i \pm \varepsilon c_j \pm \varepsilon^2 c_k = 0, \qquad c_i \pm \varepsilon^2 c_j \pm \varepsilon c_k = 0$$

where $\varepsilon = \exp(\frac{2}{3}\pi\mathrm{i})$. (We denote the square root of -1 by i, to distinguish it from i used as a variable suffix.) For the product of the left hand members of (8), substituting $c_i^2 = 3a_i$, and removing the factor 81, is

$$(9)\qquad [(a_i + a_j + a_k)^2 - (a_j a_k + a_k a_i + a_i a_j)]^2 - 12 a_i a_j a_k (a_i + a_j + a_k);$$

but from (6), using the values of $\wp'^2 u$, $\wp'' u$ as polynomials in $\wp u$, a_1, a_2, a_3, a_4 are found to be the roots of

$$(10)\qquad t^4 - \frac{1}{2} g_2 t^2 - g_3 t - \frac{1}{48} g_2^2 = 0;$$

dividing this by $t - a_h$, where a_h is the remaining root, a_i, a_j, a_k are seen to be the roots of

$$t^3 + a_h t^2 + (a_h^2 - \tfrac{1}{2}g_2)t + (a_h^3 - \tfrac{1}{2}g_2 a_h - g_3) = 0;$$

and putting the symmetric functions from this into (9) it reduces to

$$(\tfrac{1}{2}g_2)^2 - 12 a_h (a_h^3 - \tfrac{1}{2}g_2 a_h - g_3),$$

which vanishes, since a_h is a root of (10).

6. For any given choice of i, j, k from $1, 2, 3, 4$, the same one of the relations (8) must hold for all lattices 2Ω. The homogeneity of ζu shows that it must be the same for $2n\Omega$ as for 2Ω; and each c_i, for the lattice $(2/\omega_i)\Omega$, whose basis is $(2, 2\tau)$, where $\tau = \omega_2/\omega_1$, is easily seen to be a function of τ, analytic throughout the open upper half of the τ plane, $I(\tau) > 0$, so that the relation in question is an identity between analytic functions of τ. But for the triangular lattice, with $\omega_1 : \omega_2 : \omega_3 = 1 : \varepsilon : \varepsilon^2$, we have $c_1 : c_2 : c_3 = 1 : \varepsilon^2 : \varepsilon$, again from the homogeneity of ζu, so that $c_1 + \varepsilon^2 c_2 + \varepsilon c_3 = 0$. Moreover, a change of basis permutes the eight quantities $\pm c_i$ $(i=1,2,3,4)$ in the same way as it permutes the eight residue classes $\pm\omega_i$ $(\bmod 3\Omega)$ in (5). Thus the four relations are

$$\begin{aligned} c_2-\varepsilon c_3-\varepsilon^2 c_4 &= 0,\\ -c_1\qquad +\varepsilon^2 c_3-\varepsilon c_4 &= 0,\\ \varepsilon c_1-\varepsilon^2 c_2 \qquad -c_4 &= 0,\\ \varepsilon^2 c_1+\varepsilon c_2+\ c_3 \qquad &= 0 \end{aligned}\tag{11}$$

of which any three are linearly dependent.

7. We now define the four functions

$$\mathfrak{g}_i(u)=\zeta_i u+\varepsilon\zeta_i(u-\tfrac{2}{3}\omega_i)+\varepsilon^2\zeta_i(u+\tfrac{2}{3}\omega_i)\qquad (i=1,2,3,4).\tag{12}$$

These are elliptic functions of order 3, with period lattice 2Ω, having simple poles with residues 1, ε, ε^2 in points congruent to $0,\ \frac{2}{3}\omega_i,\ -\frac{2}{3}\omega_i$ $(\bmod 2\Omega)$, and satisfying

$$\mathfrak{g}_i(u+\tfrac{2}{3}\omega_i)=\varepsilon\mathfrak{g}_i(u)\qquad (i=1,2,3,4).\tag{13}$$

Their values in the eight residue classes $\pm\frac{2}{3}\omega_i$ $(\bmod 2\Omega)$ are seen to be

$$\begin{array}{lcccccccc} u= & \frac{2}{3}\omega_1 & -\frac{2}{3}\omega_1 & \frac{2}{3}\omega_2 & -\frac{2}{3}\omega_2 & \frac{2}{3}\omega_3 & -\frac{2}{3}\omega_3 & \frac{2}{3}\omega_4 & -\frac{2}{3}\omega_4\\ \mathfrak{g}_1(u)= & \infty & \infty & b_1 & 0 & 0 & \varepsilon b_1 & 0 & \varepsilon^2 b_1\\ \mathfrak{g}_2(u)= & 0 & b_2 & \infty & \infty & \varepsilon^2 b_2 & 0 & 0 & \varepsilon b_2\\ \mathfrak{g}_3(u)= & \varepsilon b_3 & 0 & 0 & \varepsilon^2 b_3 & \infty & \infty & 0 & b_3\\ \mathfrak{g}_4(u)= & \varepsilon^2 b_4 & 0 & b_4 & 0 & b_4 & 0 & \infty & \infty \end{array}\tag{14}$$

the zeros being given directly by (11); where for instance

$$b_4=\tfrac{1}{3}(\varepsilon c_1+\varepsilon^2 c_2+c_3)=\frac{\mathrm{i}}{\sqrt{3}}(c_1-c_2)$$

on subtracting $\frac{1}{3}(\varepsilon^2 c_1+\varepsilon c_2+c_3)=0$; and similarly

$$\begin{array}{lll} \sqrt{3}b_1=\mathrm{i}(c_3-c_4), & \sqrt{3}\varepsilon b_1=\mathrm{i}(c_2+c_4), & \sqrt{3}\varepsilon^2 b_1=-\mathrm{i}(c_2+c_3),\\ \sqrt{3}b_2=\mathrm{i}(c_3+c_4), & \sqrt{3}\varepsilon b_2=-\mathrm{i}(c_1+c_3), & \sqrt{3}\varepsilon^2 b_2=\mathrm{i}(c_1-c_4),\\ \sqrt{3}b_3=-\mathrm{i}(c_1+c_2), & \sqrt{3}\varepsilon b_3=\mathrm{i}(c_2-c_4), & \sqrt{3}\varepsilon^2 b_3=\mathrm{i}(c_1+c_4),\\ \sqrt{3}b_4=\mathrm{i}(c_1-c_2), & \sqrt{3}\varepsilon b_4=\mathrm{i}(c_3-c_1), & \sqrt{3}\varepsilon^2 b_4=\mathrm{i}(c_2-c_3). \end{array}\tag{15}$$

(14) means that if the points of the lattice $\frac{2}{3}\Omega$ are divided into rows parallel to ω_i, these are in cyclic order (from left to right, looking forwards along ω_i) a row of poles, a row of zeros, and a row of values b_i, εb_i, $\varepsilon^2 b_i$.

It may be remarked that from (9) it is possible to express the symmetric functions of the four quantities $(c_1\pm c_2)^2$, $(c_3\pm c_4)^2$ rationally in terms of g_2, g_3, and a cube root of $\Delta=g_2-27g_3^2$; and hence to show that the 24 quantities $\pm\varepsilon^j b_i$ $(i=1,2,3,4;\ j=0,1,2)$ are the roots of

$$t^6(t^6+8g_3)^3+8\Delta t^{12}+2\Delta t^6(t^6+8g_3)=\frac{\Delta^2}{27}.$$

8. If (ω_i,ω_j) denotes any one of the four ordered pairs (ω_1,ω_2), $(\omega_2,-\omega_1)$, $(\omega_3,-\omega_4)$, (ω_4,ω_3), we see from Liouville's theorem that $\mathfrak{g}_i(u)\cdot\mathfrak{g}_i(-u-\frac{2}{3}\omega_j)$ is a constant, the zeros of each factor coinciding with the poles of the other. As at $u=\frac{2}{3}\omega_j$ each factor has the value b_i, the constant value of the product is b_i^2. At the origin, as the pole of the one factor has residue 1, the derivative of the vanishing factor is b_i^2, i.e.

$$\mathfrak{g}_i'(-\tfrac{2}{3}\omega_j)=-b_i^2.$$

The product $\mathfrak{g}_i(u)\cdot\mathfrak{g}_i(u-\frac{2}{3}\omega_j)\cdot\mathfrak{g}_i(u+\frac{2}{3}\omega_j)$ is also a constant, the poles of each factor being the zeros of the next, in cyclic order. At the origin, the first factor has a pole with residue 1, the second a zero with derivative $-b_i^2$, and the third the value b_i; the constant value of the product is accordingly $-b_i^3$. We have thus

$$\mathfrak{g}_i(u+\tfrac{2}{3}\omega_j)=\frac{b_i^2}{\mathfrak{g}_i(-u)},\qquad \mathfrak{g}_i(u-\tfrac{2}{3}\omega_j)=-b_i\frac{\mathfrak{g}_i(-u)}{\mathfrak{g}_i(u)}.\tag{16}$$

9. Since the transformation $u\to u+\frac{2}{3}\omega_i$ multiplies $\mathfrak{g}_i(u)$, $\mathfrak{g}_i(-u)$ by factors ε, ε^2 respectively, it leaves $\mathfrak{g}_i^3(u)$, $\mathfrak{g}_i^3(-u)$, and $\mathfrak{g}_i(u)\cdot\mathfrak{g}_i(-u)$ unchanged; i.e. these are functions with the period lattice $\frac{2}{3}\Omega^{(i)}$ instead of 2Ω, and with poles only in points congruent to the origin $(\bmod\frac{2}{3}\Omega^{(i)})$, of orders 3 in the first two cases, and 2 in the last. Also as $\mathfrak{g}_i^3(u)+\mathfrak{g}_i^3(-u)$ is even, its poles are only of order 2. Thus this and $\mathfrak{g}_i(u)\cdot\mathfrak{g}_i(-u)$ are functions of order 2, with the same double pole; there is accordingly an identity, which we can write

$$\mathfrak{g}_i^3(u)+\mathfrak{g}_i^3(-u)=6m_i b_i\mathfrak{g}_i(u)\cdot\mathfrak{g}_i(-u)+b_i^3.\tag{17}$$

The value b_i^3 of the constant term is obvious, as the values of u that make one of $\mathfrak{g}_i(u)$, $\mathfrak{g}_i(-u)$ vanish make the other equal to a cube root of b_i^3. The constant m_i however remains to be found.

10. To find explicitly the algebraic identity between any two elliptic functions, we have to eliminate $\wp u$, $\wp' u$ between the rational expressions for the given functions in terms of these, and the identity

$$\wp'^2 u=4\wp^3 u-g_2\wp u-g_3.\tag{18}$$

Now using the familiar identity

$$\frac{\wp' u}{\wp u - \wp v} = \zeta(u+v) + \zeta(u-v) - 2\zeta u$$

(which still holds if ζ is replaced by ζ_i, the linear terms on the right cancelling) and putting for each of the arguments u, v in turn the constant value $\frac{2}{3}\omega_i$, we have

$$\text{(19)}\qquad \mathrm{g}_i(u)+\mathrm{g}_i(-u) = \sqrt{3}\,\mathrm{i}\,[\zeta_i(u-\tfrac{2}{3}\omega_i) - \zeta_i(u+\tfrac{2}{3}\omega_i)]$$

$$= \sqrt{3}\,\mathrm{i}\left\{\frac{\wp'(\frac{2}{3}\omega_i)}{\wp u - \frac{1}{3}c_i^2} - \tfrac{2}{3}c_i\right\}$$

and

$$\text{(20)}\qquad \mathrm{g}_i(u)-\mathrm{g}_i(-u) = 2\zeta_i u - \zeta_i(u-\tfrac{2}{3}\omega_i) - \zeta_i(u+\tfrac{2}{3}\omega_i) = \frac{-\wp' u}{\wp u - \frac{1}{3}c_i^2};$$

and solving these for $\wp u$, $\wp' u$ in terms of $\mathrm{g}_i(u)$, $\mathrm{g}_i(-u)$, and substituting the result in (18), we obtain after quite straightforward simplification

$$\text{(21)}\qquad \mathrm{g}_i^3(u)+\mathrm{g}_i^3(-u) = 2\sqrt{3}\,\mathrm{i}\,\mathrm{g}_i(u)\cdot\mathrm{g}_i(-u) + 3\sqrt{3}\,\mathrm{i}\left(\frac{1}{4}\cdot\frac{g_2}{c_2} - \frac{1}{27}c_i^3\right).$$

11. Comparing (21) with (17) we obtain

$$\text{(22)}\qquad b_i^3 = 3\sqrt{3}\,\mathrm{i}\left(\frac{1}{4}\cdot\frac{g_2}{c_i} - \frac{1}{27}c_i^3\right),\qquad m_i = \frac{\mathrm{i}\,c_i}{\sqrt{3}\,b_i}.$$

These in turn give us further relations between the constants. On the one hand substituting from (22) in (6) we have

$$\text{(23)}\qquad \wp(\tfrac{2}{3}\omega_i) = -m_i^2 b_i^2,\qquad 3\sqrt{3}\,\wp'(\tfrac{2}{3}\omega_i) = \mathrm{i}\,b_i^3(8m_i^3+1),$$

and on the other hand substituting from (15) in the second of (22),

$$\text{(24)}\qquad \begin{array}{lll} m_1 = \dfrac{c_1}{c_3-c_4}, & \varepsilon m_1 = \dfrac{-c_1}{c_2+c_3}, & \varepsilon^2 m_1 = \dfrac{c_1}{c_2+c_4}, \\[2ex] m_2 = \dfrac{c_2}{c_3+c_4}, & \varepsilon m_2 = \dfrac{c_2}{c_1-c_4}, & \varepsilon^2 m_2 = \dfrac{-c_2}{c_1+c_3}, \\[2ex] m_3 = \dfrac{-c_3}{c_1+c_2}, & \varepsilon m_3 = \dfrac{c_3}{c_1+c_4}, & \varepsilon^2 m_3 = \dfrac{c_3}{c_2-c_4}, \\[2ex] m_4 = \dfrac{c_4}{c_1-c_2}, & \varepsilon m_4 = \dfrac{c_4}{c_2-c_3}, & \varepsilon^2 m_4 = \dfrac{c_4}{c_3-c_1}. \end{array}$$

12. The function $\mathrm{g}_1(u)+\mathrm{g}_1(-u)$ is even, and of order 2, having no pole at the origin, but simple poles with residues $\pm\sqrt{3}\,\mathrm{i}$ at $u = \pm\frac{2}{3}\omega_1$ $(\mathrm{mod}\, 2\Omega)$. Similarly, $\varepsilon^2\mathrm{g}_1(u)+\varepsilon\mathrm{g}_1(-u)$ has no pole at $u = \frac{2}{3}\omega_1 (\mathrm{mod}\, 2\Omega)$ but simple poles at $u = 0$, $-\frac{2}{3}\omega_1(\mathrm{mod}\, 2\Omega)$. Thus the quotient

$$\frac{\varepsilon^2\mathrm{g}_1(u)+\varepsilon\mathrm{g}_1(-u)-b_1}{\mathrm{g}_1(u)+\mathrm{g}_1(-u)-b_1}$$

has simple poles at $u = 0(\mathrm{mod}\, 2\Omega)$, simple zeros at $u = \frac{2}{3}\omega_1(\mathrm{mod}\, 2\Omega)$, and the finite value -1 at $u = -\frac{2}{3}\omega_1(\mathrm{mod}\, 2\Omega)$. Its values at the remaining residue classes $\pm\frac{2}{3}\omega_j(\mathrm{mod}\, 2\Omega)$ can be written down directly from (14), and are found to be those of $-\mathrm{g}_2(u)/b_2$; and as these values include all the poles and zeros of both functions, and some finite values as well, it follows from Liouville's theorem that the two functions are identically equal. In the same way, any of the eight functions $\mathrm{g}_i(\pm u)$ $(i = 1, 2, 3, 4)$ can be expressed homographically in terms of any of the pairs $\mathrm{g}_j(\pm u)$ $(j \neq i)$. In particular,

$$\text{(25)}\qquad \begin{aligned} -\frac{\mathrm{g}_2(u)}{b_2} &= \frac{\varepsilon^2\mathrm{g}_1(u)+\varepsilon\mathrm{g}_1(-u)-b_1}{\mathrm{g}_1(u)+\mathrm{g}_1(-u)-b_1}, \\ -\frac{\mathrm{g}_3(u)}{b_3} &= \frac{\varepsilon\mathrm{g}_1(u)+\varepsilon^2\mathrm{g}_1(-u)-\varepsilon b_1}{\varepsilon^2\mathrm{g}_1(u)+\varepsilon^2\mathrm{g}_1(-u)-b_1}, \\ -\frac{\mathrm{g}_4(u)}{b_4} &= \frac{\varepsilon\mathrm{g}_1(u)+\varepsilon^2\mathrm{g}_1(-u)-\varepsilon^2 b_1}{\varepsilon\mathrm{g}_1(u)+\varepsilon\mathrm{g}_1(-u)-b_1}, \end{aligned}$$

and the other homographies all follow from these by replacing u by $-u$, together with ordinary linear transformation theory.

13. If we substitute

$$\mathrm{g}_2(u) : \mathrm{g}_2(-u) : -b_2$$

$$= \varepsilon^2\mathrm{g}_1(u)+\varepsilon\mathrm{g}_1(-u)-b_1 : \varepsilon\mathrm{g}_1(u)+\varepsilon^2\mathrm{g}_1(-u)-b_1 : \mathrm{g}_1(u)+\mathrm{g}_1(-u)-b_1$$

in the homogeneous equation (17) for $i = 2$, it becomes

$$(3+6m_2)\big(\mathrm{g}_1^3(u)+\mathrm{g}_1^3(-u)-b_1^3\big) = 18(1-m_2)\,b_1\cdot\mathrm{g}_1(u)\cdot\mathrm{g}_1(-u),$$

which must be the same as (17) for $i = 1$. Thus we see that

$$m_1 = \frac{1-m_2}{1+2m_2},$$

and similarly

$$m_1 = \frac{\varepsilon^2-\varepsilon m_3}{\varepsilon+2m_3} = \frac{\varepsilon-\varepsilon^2 m_4}{\varepsilon^2+2m_4}.$$

In fact the twelve quantities $\varepsilon^j m_i$ $(i = 1, 2, 3, 4;\ j = 0, 1, 2)$ are the transforms of any one of them under the twelve homographies

$$(26)\qquad \begin{cases} m \to \quad m, & \dfrac{1-m}{1+2m}, & \dfrac{\varepsilon^2-\varepsilon m}{\varepsilon+2m}, & \dfrac{\varepsilon-\varepsilon^2 m}{\varepsilon^2+2m}, \\ \qquad \varepsilon m, & \dfrac{1-\varepsilon^2 m}{\varepsilon+2m}, & \dfrac{\varepsilon^2-m}{\varepsilon^2+2m}, & \dfrac{\varepsilon(1-m)}{1+2m}, \\ \qquad \varepsilon^2 m, & \dfrac{1-\varepsilon m}{\varepsilon^2+2m}, & \dfrac{\varepsilon^2(1-m)}{1+2m}, & \dfrac{\varepsilon-m}{\varepsilon+2m}. \end{cases}$$

These form a tetrahedral group. Its invariant vierer subgroup, whose elements interchange m_1, m_2, m_3, m_4 by pairs, consists of the top row of (26), the other rows being its cosets. We shall call a set of twelve values of m, which are in this way the transforms of any one of them by the twelve homographies (26), an *m-set*.

14. The three m-sets in which not all the twelve values are distinct are the fixed points of the involutory elements:

$$m = -\tfrac{1}{2}(1\pm\sqrt{3}), \qquad -\tfrac{1}{2}\varepsilon(1\pm\sqrt{3}), \qquad -\tfrac{1}{2}\varepsilon^2(1\pm\sqrt{3}),$$

counted twice; and two equianharmonic sets, each counted three times, and each consisting of one fixed point of each of the four pairs of inverse elements of order 3 in the group, namely

$$m = 0, 1, \varepsilon, \varepsilon^2; \quad \text{and} \quad m = \infty, -\tfrac{1}{2}, -\tfrac{1}{2}\varepsilon, -\tfrac{1}{2}\varepsilon^2.$$

The equations

$$(27)\quad (8m^6+20m^3-1)^2 = 0, \qquad m^3(m^3-1)^3 = 0, \qquad (8m^3+1)^3 = 0$$

of these three singular m-sets satisfy the linear identity

$$(8m^6+20m^3-1)^2 - 64m^3(m^3-1)^3 - (8m^3+1)^3 = 0,$$

and every m-set has an equation which can be expressed as a linear combination of any two of (27).

15. The four stationary values of any elliptic function of order 2 are the roots of a quartic equation whose absolute invariant J is equal to the absolute invariant $J = g_2^3/27g_3^2$ of (1). Now $g_i(u)+g_i(-u)$ is of order 2, and being even, its stationary points are the origin and the three half period points. But from (19), as the origin is the pole of $\wp u$, the value of $g_i(u)+g_i(-u)$ there is $-\frac{2}{3}\sqrt{3}ic_i = -2m_ib_i$ by (22); and by (20) the half period points, being the zeros of $\wp' u$, are the zeros of $g_i(u)-g_i(-u)$. Thus, as (17) can be written in the form

$$[g_i(u)+g_i(-u)]^3 - 6m_ib_i[g_i(u)+g_i(-u)]^2 - 4b_i^3$$
$$= -3[g_i(u)-g_i(-u)]^2[g_i(u)+g_i(-u)+2m_ib_i],$$

the four stationary values of $g_i(u)+g_i(-u)$ are the roots of

$$(t+2m_ib_i)(t^3-6m_ib_it^2-4b_i^3) = 0.$$

The invariants of this quartic are

$$G_2 = 12b_i^4m_i(m_i^3-1), \qquad G_3 = b_i^6(8m_i^6+20m_i^3-1),$$
$$\Delta = G_2^3-27G_3^2 = -27b_i^{12}(8m_i^3-1)^3$$

and hence

$$(28)\qquad J = G_2^3/27G_3^2 = \frac{64m_i^3(m_i^3-1)^3}{(8m_i^6+20m_i^3-1)^3}.$$

Thus the twelve quantities $\varepsilon^j m_i$ $(i = 1, 2, 3, 4;\ j = 0, 1, 2)$ are the roots of an equation which can be written

$$(29)\qquad \begin{aligned} 64m^3(m^3-1)^3 - J(8m^6+20m^3-1)^2 &= 0, \\ (8m^3+1)^3 + (J-1)(8m^6+20m^3-1)^2 &= 0, \\ J(8m^3+1)^3 + 64(J-1)m^3(m^3-1)^3 &= 0. \end{aligned}$$

16. Each of the twelve quantities $\varepsilon^j m_i$ $(i = 1, 2, 3, 4;\ j = 0, 1, 2)$ is by (24) a function of $\tau = \omega_2/\omega_1$, analytic throughout the open upper half of the τ plane, and having the group Γ_{12} as group of automorphisms. If $m = m(\tau)$ is any one of these, $\tau \to m(\tau)$ is a mapping of the fundamental region of Γ_{12}, as shown for instance in Klein's Fig. 81 (loc. cit.) onto the m plane, which can be taken to be his Fig. 80 (but turned through a right angle, and with the shaded and unshaded regions correspondingly interchanged); the lines and circles in the latter figure are the loci

$$I(m) = 0, \qquad I(\varepsilon m) = 0, \qquad I(\varepsilon^2 m) = 0,$$
$$|2m+1|^2 = 3, \qquad |2m+\varepsilon|^2 = 3, \qquad |2m+\varepsilon^2|^2 = 3,$$

whose intersections are the fourteen points forming the three singular m-sets (27); and a fundamental region of the group (26) consists of any unshaded region of the figure together with any shaded region adjacent to it (with suitable inclusion of only half the boundary of the region.) Which of the regions in Fig. 80 corresponds to which of those in Fig. 81 depends of course on which of the quantities $\varepsilon^j m_i$ we have taken to be $m(\tau)$.

17. We now define $\varphi_i(u)$ to be an elliptic function of order 3, with period lattice 2Ω, having a triple pole at the origin, with leading term $1/u^3$, and a triple zero at $u = \frac{2}{3}\omega_i$. It is evidently of the form $-\frac{1}{2}\wp' u + A\wp u + B$, where A, B are constants, determined by the conditions that

$\varphi_i(\frac{2}{3}\omega_i) = \varphi_i'(\frac{2}{3}\omega_i) = 0$. It is easily found that

$$\varphi_i(u) = -\tfrac{1}{2}\wp' u - c_i \wp u + \frac{1}{8}\cdot\frac{g_2}{c_i} + \frac{1}{3}c_i^3. \tag{30}$$

It is clear that each function $\mathfrak{g}_i^3(u)$ is one of these functions $\varphi_j(u)$, but constructed for the period lattice $\frac{2}{3}\Omega^{(i)}$ instead of 2Ω. In fact, if (i, j) is either of the ordered pairs $(1, 2)$, $(4, 3)$, we have

$$\mathfrak{g}_j^3(u \mid 2\Omega) = \varphi_i(u \mid \tfrac{2}{3}\Omega^{(j)}), \qquad \mathfrak{g}_i^3(u \mid 2\Omega) = -\varphi_j(u \mid \tfrac{2}{3}\Omega^{(i)}).$$

Replacing Ω by $\Omega^{(i)}$, $\Omega^{(j)}$, these give in virtue of (4)

$$\varphi_i(u \mid 2\Omega) = \mathfrak{g}_j^3(u \mid 2\Omega^{(i)}), \qquad \varphi_j(u \mid 2\Omega) = -\mathfrak{g}_i^3(u \mid 2\Omega^{(j)}).$$

Thus, noting that as the poles and zeros of $\varphi_i(u)$ are all triple its cube roots are three distinct functions, and defining $\mathfrak{f}_i(u)$ to be that one of these whose leading term at the origin is $1/u$, we see that

$$\mathfrak{f}_i(u \mid 2\Omega) = \mathfrak{g}_j(u \mid 2\Omega^{(i)}), \qquad \mathfrak{f}_j(u \mid 2\Omega) = -\mathfrak{g}_i(u \mid 2\Omega^{(j)}), \tag{31}$$

(i, j) still being either of the ordered pairs $(1, 2)$, $(4, 3)$.

18. Applying the property (13) of $\mathfrak{g}_i(u)$ to $\mathfrak{f}_i(u)$, we have the results

$$\mathfrak{f}_i(u + 2\omega_j) = \varepsilon \mathfrak{f}_i(u), \qquad \mathfrak{f}_j(u + 2\omega_i) = \varepsilon^2 \mathfrak{f}_j(u), \tag{32}$$

where now (i, j) is any one of the ordered pairs $(2, 3)$, $(3, 1)$, $(1, 2)$, $(4, 1)$, $(4, 2)$, $(4, 3)$; and also

$$\mathfrak{f}_i(u + 2\omega_i) = \mathfrak{f}_i(u) \qquad (i = 1, 2, 3, 4), \tag{33}$$

since $2\omega_i$ is an element of $2\Omega^{(i)}$.

19. From (30) we have

$$\begin{aligned} \varphi_i(u) + \varphi_i(-u) &= -2c_i(\wp u - \tfrac{1}{3}c_i^2) + (\tfrac{1}{4}g_2/c_i - \tfrac{1}{3}c_i^3), \\ \varphi_i(u)\cdot\varphi_i(-u) &= (-\wp^3 u + \tfrac{1}{4}g_2\wp u + \tfrac{1}{4}g_3) + (c_i\wp u - \tfrac{1}{8}g_2/c_i - \tfrac{1}{6}c_i^3)^2 \\ &= -(\wp u - \tfrac{1}{3}c_i^2)^3, \end{aligned} \tag{34}$$

after some simplification, and making use of the fact that $\frac{1}{3}c_i^2$ is a root of (10). Identifying cube roots on both sides of (35) that have the same leading term $-1/u^2$ at the origin,

$$\mathfrak{f}_i(u)\cdot\mathfrak{f}_i(-u) = -(\wp u - \tfrac{1}{3}c_i^2),$$

so that

$$\mathfrak{f}_i^3(u) + \mathfrak{f}_i^3(-u) = 2c_i\mathfrak{f}_i(u)\cdot\mathfrak{f}_i(-u) + (\tfrac{1}{4}g_2/c_i - \tfrac{1}{3}c_i^3).$$

This is of course the cubic identity satisfied by $\mathfrak{f}_i(u)$ corresponding to (17) for $\mathfrak{g}_i(u)$. Writing it analogously

$$\mathfrak{f}_i^3(u) + \mathfrak{f}_i^3(-u) = 6m_i' b_i' \mathfrak{f}_i(u)\cdot\mathfrak{f}_i(-u) + b_i'^3,$$

we see that

$$\begin{aligned} b_i'^3 &= \tfrac{1}{4}g_2 c_i - \tfrac{1}{3}c_i^3 = \wp'(\tfrac{2}{3}\omega_i), \\ 3m_i' b_i' &= c_i. \end{aligned} \tag{35}$$

20. Differentiating $\mathfrak{f}_i^3(u) = \varphi_i(u)$ as given in (30), and recalling that $\wp'' u = 6\wp^2 u - \frac{1}{2}g_2$, we obtain

$$\begin{aligned} 3\mathfrak{f}_i^2(u)\cdot\mathfrak{f}_i'(u) &= -3(\wp u - \tfrac{1}{3}c_i^2)^2 - c_i\wp' u - 2c_i^2\wp u + \tfrac{1}{4}g_2 + \tfrac{1}{3}c_i^4 \\ &= -3\mathfrak{f}_i^2(u)\cdot\mathfrak{f}_i^2(-u) + 2c_i\mathfrak{f}_i^3(u), \end{aligned}$$

whence

$$\mathfrak{f}_i'(u) = 2m_i' b_i' \mathfrak{f}_i(u) - \mathfrak{f}_i^2(-u).$$

Further, since $\mathfrak{g}_i(u)$ is the same function as $\mathfrak{f}_i(u)$, with a different period lattice, and with the constants m_i, b_i in place of m_i', b_i',

$$\mathfrak{g}_i'(u) = 2m_i b_i \mathfrak{g}_i(u) - \mathfrak{g}_i^2(-u). \tag{36}$$

Eliminating $\mathfrak{g}_i(-u)$ between this and (17), we see that $x = \mathfrak{g}_i(u)$ is a solution of

$$\left(\frac{dx}{du} - 2m_i b_i x\right)\left(\frac{dx}{du} + 4m_i b_i x\right)^2 = (x^3 - b_i^3)^2.$$

21. The addition and duplication formulae for these functions can be found from the fact that for any constants A, B, C the function $A\mathfrak{g}_i(u) + B\mathfrak{g}_i(-u) + Cb_i$ has zeros whose sum is $\equiv 0 \pmod{2\Omega}$, say α, β, $-(\alpha+\beta)$. Putting for brevity

$$\begin{aligned} b_i x_1 &= \mathfrak{g}_i(\alpha), & b_i x_2 &= \mathfrak{g}_i(\beta), & b_i x_3 &= \mathfrak{g}_i(-\alpha-\beta), \\ b_i y_1 &= \mathfrak{g}_i(-\alpha), & b_i y_2 &= \mathfrak{g}_i(-\beta), & b_i y_3 &= \mathfrak{g}_i(\alpha+\beta); \end{aligned}$$

we see that (x_1, y_1), (x_2, y_2), (x_3, y_3) are the common solutions of

$$x^3 + y^3 - 2m_i xy - 1 = 0, \qquad Ax + By + C = 0,$$

whence, eliminating x, y_1, y_2, y_3 are the roots of

$$(A^3 - B^3)y^3 + 3B(2m_i A^2 - BC)y^2 + 3C(2m_i A^2 - BC)y - (A^3 + C^3) = 0;$$

and from the coefficients of the two middle terms

$$y_3 = -\frac{By_1y_2 + C(y_1 + y_2)}{B(y_1 + y_2) + C}.$$

But since (x_1, y_1), (x_2, y_2) satisfy $Ax+By+C=0$,

$$A:B:C = (y_1-y_2):(x_2-x_1):(x_1y_2-x_2y_1),$$

whence

$$y_3 = \frac{x_1y_2^2-x_2y_1^2}{x_1y_1-x_2y_2}$$

or

$$\mathrm{g}_i(a+\beta) = \frac{\mathrm{g}_i(a)\mathrm{g}_i^2(-\beta)-\mathrm{g}_i(\beta)\mathrm{g}_i^2(-a)}{\mathrm{g}_i(a)\mathrm{g}_i(-a)-\mathrm{g}_i(\beta)\mathrm{g}_i(-\beta)}.$$

Similarly, for the case $\beta = a$, using (36) we find that the condition for $A\mathrm{g}_i(u)+B\mathrm{g}_i(-u)+Cb_i$ to have a double zero at $u=a$, so that the remaining zero is at $u=-2a$, is

$$A:B:C = (2m_ix_1-y_1^2):(2m_iy_1-x_1^2):(2m_ix_1y_1+1),$$

whence

$$y_3 = \frac{y_1(x_1^3+1)}{x_1^3-y_1^3},$$

or

$$\mathrm{g}_i(2a) = \frac{\mathrm{g}_i(-a)[\mathrm{g}_i^3(a)+b_i^3]}{\mathrm{g}_i^3(a)-\mathrm{g}_i^3(-a)}.$$

22. The rational expression of $\mathrm{g}_i(u)$ in terms of $\mathrm{f}_i(u)$ is an elliptic function transformation of order 3, expressing a function with period lattice 2Ω in terms of the same function with period lattice $2\Omega^{(i)}$. One form of this is

$$\mathrm{g}_i(u) = \frac{\mathrm{f}_j(-u)\cdot\mathrm{f}_k(u)\cdot\mathrm{f}_4(u)}{\mathrm{f}_i(u)\cdot\mathrm{f}_i(-u)}$$

where (i,j,k) is an even permutation of $(1,2,3)$, and

$$\mathrm{g}_4(u) = \frac{\mathrm{f}_1(-u)\cdot\mathrm{f}_2(-u)\cdot\mathrm{f}_3(-u)}{\mathrm{f}_4(u)\cdot\mathrm{f}_4(-u)}.$$

These are easily proved by remarking that the product of the multipliers on the right, for any of the translations $u\to u+2\omega_i$ $(i=1,2,3,4)$, is unity, by (32), so that the right hand member has period lattice 2Ω; that the zeros and poles are the same for both members; and that both members have residue 1 at the origin.

Another form of the transformation follows from the fact that the even function

$$\mathrm{g}_i(u)+\mathrm{g}_i(-u)+2m_ib_i \tag{37}$$

has a double zero at the origin, and simple poles with residue $\pm\sqrt{3}i$ at $u=\pm\frac{2}{3}\omega_i$; the same function of $(u-\frac{2}{3}\omega_i)$ is

$$\varepsilon^2\mathrm{g}_i(u)+\varepsilon\mathrm{g}_i(-u)+2m_ib_i; \tag{38}$$

thus the quotient of (38) over (37) has a triple pole at the origin, a triple zero at $u=\frac{2}{3}\omega_i$, and the value -1 at $u=-\frac{2}{3}\omega_i$; and as $\varphi_i(u)=\mathrm{f}_i^3(u)$ has the same poles and zeros, and the value $b_{i'}^3$ at $u=-\frac{2}{3}\omega_i$, it follows that

$$\mathrm{f}_i^3(u) = -b_{i'}^3\frac{\varepsilon^2\mathrm{g}_i(u)+\varepsilon\mathrm{g}_i(-u)+2m_ib_i}{\mathrm{g}_i(u)+\mathrm{g}_i(-u)+2m_ib_i}.$$

Replacing u by $-u$ in this, and eliminating $\mathrm{g}_i(-u)$ between the two, we have

$$\mathrm{g}_i(u) = 2m_ib_i\frac{\varepsilon\mathrm{f}_i^3(u)+\varepsilon^2\mathrm{f}_i^3(-u)-b_i'^3}{\mathrm{f}_i^3(u)+\mathrm{f}_i^3(-u)-b_i'^3}. \tag{39}$$

23. We can regard as the corresponding modular relation the equation connecting m_i, m_i' obtained by eliminating b_i, b_i', c_i from (22), (35); since m_i plays for these functions the same part, of a constant determining the shape of the lattice, as the modulus k does for the functions $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$. This relation can be written in either of the forms

$$\begin{cases} (8m_i^3+1)(8m_i'^3+1)=1, \\ \dfrac{1}{m_i^3}+\dfrac{1}{m_i'^3}+8=0, \\ -\dfrac{m_i^3}{m_i'^3} = 8m_i^3+1 = \dfrac{1}{8m_i'^3+1}. \end{cases} \tag{40}$$

In terms of any one of the twelve modular functions which we denote by $m(\tau)$ this relation has the expression

$$\tau\tau' = -\tfrac{1}{3} \Rightarrow \big(8m^3(\tau)+1\big)\big(8m^3(\tau')+1\big)=1.$$

We also have from (22), (35) the relation

$$m_ib_i = \sqrt{3}im_i'b_i'.$$

24. The application of these results to the cubic curve is obvious. The parametric equations

$$x:y:z = \mathrm{g}_i(u):\mathrm{g}_i(-u):-b_i$$

lead to the equation of the curve parametrised, in the form

$$x^3+y^3+z^3+6m_ixyz = 0$$

by (17). The lines $x = 0, y = 0, z = 0$, cut the curve in its nine inflexions $u = 0, \pm\frac{2}{3}\omega_j$ $(j = 1, 2, 3, 4)$. The 18 projective transformations of the curve into itself

(41) $(x, y, z) \to$ all permutations of (x, y, z), $(\varepsilon x, \varepsilon^2 y, z)$ and $(\varepsilon^2 x, \varepsilon y, z)$

correspond to the 18 transformations

$$u \to \pm u, \quad \pm u \pm \tfrac{2}{3}\omega_j \qquad (j = 1, 2, 3, 4)$$

with the following correspondences between the generators

$$\begin{aligned} &u \to -u, && (x, y, z) \to (y, x, z), \\ (42)\qquad &u \to u + \tfrac{2}{3}\omega_i, && (x, y, z) \to (\varepsilon x, \varepsilon^2 y, z), \\ &u \to u + \tfrac{2}{3}\omega_j, && (x, y, z) \to (z, x, y), \end{aligned}$$

the last of these following from (16), with (i, j) related as there.

25. The 216 projective self-transformations of the inflexion configuration transform the cubic into the twelve curves

$$x^3 + y^3 + z^3 + 6\varepsilon^j m_i xyz = 0 \qquad (i = 1, 2, 3, 4;\ j = 0, 1, 2),$$

which are all the members of the hessian pencil that have the same absolute invariant J. In this group, (41) is an invariant subgroup whose cosets correspond to the individual elements of (26); in particular, $(x, y, z) \to (x, y, \varepsilon z)$ corresponds to $m \to \varepsilon m$, and

$$(x, y, z) \to (\varepsilon^2 x + \varepsilon y + z,\ \varepsilon x + \varepsilon^2 y + z,\ x + y + z)$$

to $m \to (1 - m)/(1 + 2m)$; and these, with (41), generate the whole group of order 216.

26. From (36) we see that the stationary points of the rational function x/z on the curve are the zeros on it of the polar of $(0, 1, 0)$, $y^2 + 2m_i zx$, as we expect. Calculating the derivative

$$\frac{d}{du}\left(\frac{A g_i(u) + B g_i(-u) + C b_i}{A' g_i(u) + B' g_i(-u) + C' b_i}\right)$$

in the obvious way from (36), we can verify also that the stationary points of the rational function $\dfrac{Ax + By + Cz}{A'x + B'y + C'z}$ are the zeros on the curve of the polar of $(BC' - CB',\ CA' - AC',\ AB' - BA')$.

27. The results of Sections 22, 23 give us the following theorem, which I have not seen stated elsewhere (though it is rash to claim as new any result in so thoroughly explored a field):

If m, m' are related as in (40), then

$$X : Y : -2m'Z = \varepsilon x^3 + \varepsilon^2 y^3 + z^3 : \varepsilon^2 x^3 + \varepsilon y^3 + z^3 : x^3 + y^3 + z^3$$

is a unirational transformation of the curve

$$(43)\qquad x^3 + y^3 + z^3 + 6mxyz = 0$$

into

$$(44)\qquad X^3 + Y^3 + Z^3 + 6m'XYZ = 0,$$

in which each point of the latter curve is the image of three points of the former; the linear system

$$(45)\qquad \lambda(\varepsilon x^3 + \varepsilon^2 y^3 + z^3) + \mu(\varepsilon^2 x^3 + \varepsilon y^2 + z^3) + \nu(x^3 + y^3 + z^3) = 5$$

trace on (43) a linear series compunded with an involution of order 3, whose projective model is (44). The system (45) is in fact (being a net) compounded with an involution of order 9, each of whose sets lies by threes on three curves, (43) and those obtained by replacing m by εm, $\varepsilon^2 m$, all three of which are equally transformed into (44).

Reçu par la Rédaction le 14. 8. 1963