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On coverings with convex domains

by

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In honour of Professor
L. J. Mordell

§ 1. Introduction. Let K be a plane closed bounded convex domain with the origin O as an inner point. In the usual vector notation, let $K+A$ denote the set of points $X+A$ with X in K.

Let Λ be a lattice in the plane. We say that (K, Λ) is a lattice covering of the plane, if the plane is covered by the sets $K+A$ with $A \in \Lambda$. We define the density $\theta(K, \Lambda)$ of this covering by

$$\theta(K, \Lambda) = a(K)/d(\Lambda), \tag{1}$$

where $a(K)$ is the area of K and $d(\Lambda)$ is the determinant of Λ. We define the density of the best lattice covering by K as

$$\theta(K) = \inf \theta(K, \Lambda), \tag{2}$$

the lower bound being taken over all lattice coverings (K, Λ).

If Σ is any set in the plane, we say that (K, Σ) is a covering by K, if the plane is covered by the sets $K+A$ with $A \in \Sigma$. If B is the square $|x| \leqslant t$, $|y| \leqslant t$, we define $\theta^*(K, \Sigma, B)$ as follows. Let $N(B)$ be the number of sets $K+A$, with $A \in \Sigma$, which have a point in common with B. Then we take $\theta^*(K, \Sigma, B) = a(K)N(B)/a(B)$, where $a(B)$ denotes the area of B. We define the density of the covering (K, Σ) by

$$\theta^*(K, \Sigma) = \liminf_{t\to\infty} \theta^*(K, \Sigma, B). \tag{3}$$

Note that, if Σ happens to coincide with a lattice Λ, then $\theta^*(K, \Lambda) = \theta(K, \Lambda)$. We define the density $\theta^*(K)$ of the best covering by K to be

$$\theta^*(K) = \inf \theta^*(K, \Sigma), \tag{4}$$

the lower bound being taken over all coverings (K, Σ).

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L. Fejes Tóth [1] has proved

THEOREM 1 (Fejes Tóth). *If $h(K)$ is the area of the largest hexagon inscribed in K, then*

$$\theta^*(K) \geqslant a(K)/h(K). \tag{5}$$

If K has O as centre

$$\theta^*(K) = a(K)/h(K) = \theta(K). \tag{6}$$

The proof depends on a construction replacing sets $K+A$ covering a hexagon H by polygonal subsets which cover H without overlapping, on Euler's formula $V-E+F=1$, and on an inequality of C. H. Dowker [2] about the areas of polygons inscribed in a convex domain. The construction of the non-overlapping polygonal covering is rather complicated and has been discussed in detail by R. P. Bambah and C. A. Rogers [3].

In Part I of this paper we prove the rather similar

THEOREM 2. *If $t(K)$ is the area of the largest triangle inscribed in K, then*

$$\theta^*(K) \geqslant a(K)/\big(2t(K)\big). \tag{7}$$

If K has O as centre

$$\theta^*(K) = a(K)/\big(2t(K)\big) = \theta(K). \tag{8}$$

Our proof is independent of Dowker's inequality and is based on a construction, which is similar to the one due to B. Delaunay [4] used by H. S. M. Coxeter, L. Few and C. A. Rogers [5] to discuss coverings of n-dimensional space by spheres, and which seems to us to have many advantages over that used by Fejes Tóth.

We remark that neither (5) nor (7) are best possible when K lacks a centre. Inequality (5) is the stronger in some cases, for example if K is a triangle, but (7) is stronger in other cases, for example if K is a regular pentagon. We also, in § 5 of Part I, make some remarks on the corresponding problem of packing convex domains.

In their paper [3], Bambah and Rogers also announced:

THEOREM 3. *Let K have O as centre. Let $A_0, A_1, \ldots, A_n = A_0, A_{n+1}, \ldots, A_{n+m}$ be points such that:*

(1) *the polygon $A_0A_1\ldots A_n$ is a Jordan polygon bounding a closed domain Π of area $a(\Pi)$;*

(2) *for each r, with $0 < r \leqslant n$, there is a point common to the sets $K+A_r$, $K+A_{r-1}$;*

(3) *the points $A_{n+1}, \ldots, A_{n+m}$ are in the interior of Π;*

(4) *for each point X of Π there is an integer r with $1 \leqslant r \leqslant n+m$, such that X is in $K+A_r$ and the segment XA_r is in Π.*

Then

$$a(\Pi) \leqslant (2m+n-2)t(K) \tag{9}$$

where $t(K)$ is the area of the largest triangle inscribed in K.

It is not difficult to deduce the result (8) of Theorem 2 from Theorem 3; but (9) is rather more precise and is best possible in many cases. Bambah and Rogers did not publish their very complicated proof of (9) as they felt that it had been superseded by Fejes Tóth's work. However, there has been interest in the inequality, especially since N. Oler's proof [6] of the analogous Zassenhaus-Oler inequality for packings, and it appears desirable to give a proof. In Part II of this paper we give a proof which is considerably simpler than the original one. Our new proof, like our proof of Theorem 2, makes use of regions connected with K, which Zassenhaus (7) has called 'domains of action'. These domains of action have been extensively used by the students of Zassenhaus in packing problems, and many of their properties are well known to the workers in the field. But we will prove all the properties that we require in § 3.

Part 1

§ 2. Some properties of strictly convex domains. In this section we suppose that K is strictly convex. We consider two sets

$$K_1 = \lambda_1 K + A_1 \quad\text{and}\quad K_2 = \lambda_2 K + A_2 \quad\text{with}\quad \lambda_1 > 0,\ \lambda_2 > 0 \quad\text{and}\quad A_1 \neq A_2.$$

We prove the existence of a line having separation properties similar to those of the radical axis of two circles.

LEMMA 1. *Let K_1 and K_2 have inner points in common. If one is contained in the other their boundaries intersect in at most one point, otherwise their boundaries intersect in exactly two points.*

Proof. Let F, F_1 and F_2 be the boundaries of K, K_1 and K_2. First suppose that one, say K_1, is contained in the other. Suppose that P is a point common to F_1 and F_2. Let γ be a tac-line to K_2 at P. Then γ is also a tac-line to K_1 at P. Further K_1 and K_2 lie on the same side of γ and meet γ only at the point P. Hence P is a self-corresponding point in the direct similitude taking K_1 into K_2. As $A_1 \neq A_2$ this similitude is not the identity. Since K is strictly convex, it follows that P is the only point common to F_1 and F_2.

We can now suppose that neither of K_1, K_2 is contained in the other. Then we can choose a point H on F_1 not in K_2, a point C in the interiors of K_1 and K_2, and a point J on F_2 not in K_1. Then the segment CJ will contain a point, H' say, of F_1 in the interior of K_2. So both the arcs fo

F_1 joining H and H' have points in common with F_2. Hence $F_1 \cap F_2$ contains at least two points. It remains to show that $F_1 \cap F_2$ cannot have more than two points.

Suppose that $F_1 \cap F_2$ contains three points X, Y, Z. As K_1 is strictly convex XYZ is a proper triangle. The points

$$X_i = \lambda_i^{-1}(X - A_i), \quad Y_i = \lambda_i^{-1}(Y - A_i), \quad Z_i = \lambda_i^{-1}(Z - A_i),$$

$i = 1, 2$, lie on F, and the triangles $X_1 Y_1 Z_1$, $X_2 Y_2 Z_2$ are similar and similarly situated. Suppose $\lambda_1 \neq \lambda_2$. Without loss of generality we can suppose that $\lambda_1 < \lambda_2$. Then $X_1 Y_1 Z_1$ and $X_2 Y_2 Z_2$ are in direct similitude with a centre, C say, and C, X_1, X_2 lie in this order on the line CX_1X_2. Similarly for CY_1Y_2 and CZ_1Z_2. As F is strictly convex, C cannot lie on a side of the triangle $X_2Y_2Z_2$, for if C is on Y_2Z_2 then Y_1, Z_1, Y_2, Z_2 are at least three distinct collinear points on F. If C was interior to the triangle $X_2Y_2Z_2$, X_1, Y_1, Z_1 would be also interior to this triangle and so interior to K. Hence C is exterior to the triangle $X_2Y_2Z_2$. We may thus suppose, without loss of generality, that CZ_2 lies in the angle less than π between CX_2 and CY_2. For convenience of description we will suppose that X_2Y_2 is horizontal and above C. Then X_1Y_1 is horizontal and between X_2Y_2 and C. Let X_1Y_1 and X_2Y_2 meet the line CZ_1Z_2 in points U_1 and U_2. Then U_1 and U_2 are inner points of K and so the segment U_1U_2 is a proper sub-segment of the segment Z_1Z_2 joining the two distinct points Z_1Z_2 of F. Hence one of Z_1, Z_2 is above X_2Y_2 and the other is below X_1Y_1. This is impossible as $X_1Y_1Z_1$ and $X_2Y_2Z_2$ are in direct similitude.

We obtain a similar contradiction when $\lambda_1 = \lambda_2$ and one triangle is a translation of the other.

Lemma 2. *Suppose that K_1, K_2 have common inner points but neither is contained in the other. Let X, Y be the two points of $F_1 \cap F_2$. Then the part of F_1 on one side of XY lies in K_2 and the part of F_2 on the other side of XY lies in K_1.*

Proof. Let H be a point of F_1 not in K_2. Then the part of F_1 on the side of XY remote from H lies in K_2. Consequently the part of F_2 on the same side of XY as H lies in K_1.

Lemma 3. *Suppose that neither of K_1, K_2 is contained in the other. Let E_i be the set of points not in the interior of K_i, for $i = 1, 2$. Then the sets*

$$K_1 \cap E_2 \quad \textit{and} \quad K_2 \cap E_1$$

can be separated by a straight line.

Proof. If K_1 and K_2 have no inner points in common, the result is classical. If K_1 and K_2 have common inner points the result follows from Lemma 2.

§ 3. Some properties of the "domains of action". In this section we continue to assume that K is strictly convex with O as an inner point. Thus there will be a unique function $f(X)$ such that:

(1) K is the set of points X with

$$f(X) \leqslant 1;$$

(2) $f(tX) = tf(X)$ for all X and all $t \geqslant 0$.

Further this $f(X)$ will be continuous for all X and will satisfy

$$f(X+Y) \leqslant f(X) + f(Y),$$

for all points X, Y, with strict inequality unless $X = tY$ for some $t \geqslant 0$ or $Y = O$.

Let Σ be a discrete set and suppose that (K, Σ) is a covering. We define the domain of action $D(K, A, \Sigma)$, or simply $D(A)$, of K at A with respect to Σ by

$$D(A) = D(K, A, \Sigma) = \{X \mid f(X-A) \leqslant f(X-B) \text{ for all } B \in \Sigma\}.$$

Lemma 4. *If $A \in \Sigma$, $D(A)$ contains A as an inner point and is a closed subset of $K+A$ bounded by a continuous curve. Each ray through A meets the boundary of $D(A)$ in a single point. If $B \in \Sigma$ and $B \neq A$ then $D(A)$ and $D(B)$ have no inner points in common. Further*

$$\bigcup_{A \in \Sigma} D(A)$$

is the whole plane.

Proof. Since (K, Σ) is a covering, given any point X of the plane, there will be a point B of Σ with $f(X-B) \leqslant 1$. As Σ is discrete there will be only finitely many such points B of Σ. Hence there will be a point A of Σ for which $f(X-A)$ takes its least value. Then $X \in D(A)$. This proves the last assertion of the lemma; it also shows that, if $X \in D(A)$ then $f(X-A) \leqslant 1$, so that $X \in K+A$. Hence $D(A) \subset K+A$.

If $A \in \Sigma$ and $X \in K+A$, then

$$f(X-B) > 1 \geqslant f(X-A)$$

for all but a finite number of points of Σ, say the points $A, B_1, \ldots, B_r$. Thus

$$D(A) = \{X \mid f(X-A) \leqslant f(X-B_i),\ i = 1, 2, \ldots, r\}.$$

Writing

$$\varphi_A(X) = \max_{1 \leqslant i \leqslant r}\{f(X-A) - f(X-B_i)\},$$

we see that $D(A)$ is the set of points X with $\varphi_A(X) \leqslant 0$. As $f(X)$ is continuous it follows that $\varphi_A(X)$ is continuous. Hence $D(A)$ is closed, and any point X satisfying $\varphi_A(X) < 0$ is an inner point of $D(A)$; in particular, A is an inner point of $D(A)$.

Let γ be any half-ray with A as end-point. As $D(A)$ is closed and bounded, there will be a point P of γ in $D(A)$ furthest from A. Let Q be an inner point of the segment AP. For $i = 1, 2, \ldots, r$ we have

$$f(P-A) \leqslant f(P-B_i).$$

If B_i is not on the segment AP nor on PA produced beyond A,

$$f(Q-A)+f(P-Q) = f(P-A) \leqslant f(B_i-A) = f(B_i-Q+Q-A)$$
$$< f(B_i-Q)+f(Q-A).$$

If B_i were on AP we would have

$$f(P-B_i) < f(P-A),$$

and P would not be in $D(A)$. If B_i is on AP produced beyond A,

$$f(Q-A)+f(P-Q) = f(P-A) < f(B_i-A) = f(B_i-Q+Q-A)$$
$$< f(B_i-Q)+f(Q-A).$$

So, in the possible cases,

$$f(Q-A) < f(B_i-Q).$$

As this holds for $i = 1, 2, \ldots, r$, we have $\varphi_A(Q) < 0$, and Q is an inner point of $D(A)$. Thus each ray through A meets the boundary of $D(A)$ in a single point. It follows that $D(A)$ is bounded by a continuous curve.

If A and B belong to Σ, and $D(A)$ and $D(B)$ have a common inner point Q, then Q lies on segments AP and BP' lying in $D(A)$ and $D(B)$, and, as above,

$$\varphi_A(Q) < 0, \quad \varphi_B(Q) < 0.$$

Hence

$$f(Q-A) < f(Q-B), \quad f(Q-B) < f(Q-A),$$

which is impossible. This completes the proof.

We now define a point to be a *vertex*, if it is common to three or more domains of action.

LEMMA 5. *Let V be common to domains $D(A)$, $D(B)$, $D(C)$. Then $f(V-A) = f(V-B) = f(V-C) \leqslant 1$ and ABC is a proper triangle inscribed in*

$$-f(V-A)K+V.$$

Proof. As V is common to $D(A)$, $D(B)$ and $D(C)$, we have

$$f(V-A) = f(V-B) = f(V-C) \leqslant 1.$$

Hence A, B, C lie on the boundary of $-f(V-A)K+V$. As K is strictly convex, it follows that the triangle ABC is proper.

Remark. Note that V does not necessarily lie in the triangle ABC.

LEMMA 6. *If $A \neq B$ the set D of points X with $f(X-A) = f(X-B)$ has points on both sides of the line AB arbitrarily far from this line.*

Proof. Without loss of generality take $A = O$, $B = (1, 0)$. Then it is enough to show that, for each y, there exists a point $(x(y), y)$ in D.

Let y_0 be fixed. Choose $\lambda > 0$ so large that $y = y_0$ meets the boundary of λK in two points P, Q with x coordinates $x(P), x(Q)$ with $x(P) < x(Q)$. Then $R = P+B$ and $S = Q+B$ lie on the boundary of $\lambda K+B$. Then P does not lie in $\lambda K+B$ and S does not lie in λK. Thus

$$f(P)-f(P-B) < 0, \quad f(S)-f(S-B) > 0.$$

Since $f(X)$ is continuous, $f(W) = f(W-B)$ for some point W on the segment PS. So this segment meets D and the result follows.

LEMMA 7. *$D(A)$ has at least one vertex. If $D(A)$ and $D(B)$ meet there are points C, C' of Σ on opposite sides of the line AB, such that $D(A)$, $D(B)$, $D(C)$ and $D(A)$, $D(B)$, $D(C')$ have common vertices.*

Proof. Given $A \epsilon \Sigma$, we can choose P on the boundary of $D(A)$ and then choose a second point B of Σ with $P \epsilon D(B)$. Thus the first assertion follows from the second.

Let P be a point common to $D(A)$ and $D(B)$. We take $A = O$ and $B = (1, 0)$, as we may without loss of generality. Let γ be the ray AP; we shall measure angles from this ray in the anticlockwise direction. Since the set D of points X with $f(X-A) = f(X-B)$ is unbounded and has points with y arbitrarily large and positive, while $D(A)$ is bounded, we can choose a point Z of D, not in $D(A)$, which has positive y coordinate and lies above the broken line APB. Then the ray AZ makes a positive angle θ_1 with γ and θ_1 is less than the angle between γ and the negative x-axis. Now the ray AZ meets the boundary of $D(A)$ at a point U short of Z. Just as in the proof of Lemma 4, it follows that $f(U-A) < f(U-B)$, so that U is not in D. Let θ_0 be the lower bound of the angles θ, such that $0 \leqslant \theta \leqslant \theta_1$ and the ray AT which makes an angle θ with γ meets the boundary of $D(A)$ at a point T not on D. Let AV, the ray that makes the angle θ_0 with γ, meet the boundary of $D(A)$ at V.

Since $D(A) \cap D$ is compact, it follows that V is in $D(A) \cap D$ and so is in $D(A) \cap D(B)$. Further, for each $\alpha > 0$, there is a θ in $\theta_0 < \theta \leqslant \theta_0+\alpha$, such that the ray making angle θ with γ meets the boundary of $D(A)$ at a point not on D and so in some $D(C)$ with $C \neq A$, $C \neq B$. As Σ is discrete, there are only a finite number of choices for the point C of Σ. So for a suitable choice of C in Σ, for each $\alpha > 0$, there will be a θ with $\theta_0 < \theta \leqslant \theta_0 + \alpha$ for which the ray making an angle θ with γ has a point $U(\theta)$ in both $D(A)$ and $D(C)$. Since $D(C)$ is compact and V is the limit of such points $U(\theta)$ of $D(C)$, it follows that $V \epsilon D(C)$. Thus C is a vertex common to $D(A)$, $D(B)$ and $D(C)$.

Now the contour Γ, consisting of the negative x-axis, the broken line AVB, and the positive x-axis beyond B, divides the plane into two domains, which we may call the upper and lower domains. Further for suitable values of θ, just larger than θ_0, the point $U(\theta)$ is a point of $D(C)$ in the upper domain. Now the segment $CU(\theta)$ lies in the interior of $D(C)$, except for the point $U(\theta)$. So this segment meets neither the segment AV in $D(A)$ nor the segment VB in $D(B)$. Further $CU(\theta)$ lies in the convex set $-f(V-A)K+V$ with A, B and C on its boundary. Hence $CU(\theta)$ lies in the part of $-f(V-A)K+V$ in the upper domain. As C is on the boundary of $-f(V-A)K+V$ it must have positive y-coordinate.

Repetition of the argument shows that there must be a vertex V', not necessarily different from V, common to $D(A)$, $D(B)$ and $D(C')$, where C' is a point of Σ with a negative y-coordinate. This proves the lemma.

§ 4. The triangulation of the plane and the proof of Theorem 2.

We shall obtain Theorem 2 as a simple consequence of

THEOREM 4. *Let K be strictly convex and let (K, Σ) be a covering. Then there exists a triangulation of the plane with vertices at points of Σ, such that the area of each triangle does not exceed $t(K)$.*

Proof. Consider the set of vertices of the domains of action $D(A)$. By Lemma 7, this set is non-empty. For each V in this set, we construct a set $T(V)$ as follows:

Let $A_1, \ldots, A_k$ be the points of Σ for which the sets $D(A_i)$ meet at V, and let $T(V)$ be the convex hull of these points $A_1, \ldots, A_k$.

By Lemma 5, the set $T(V)$ is a proper polygon for each V. Further $A_1, \ldots, A_k$ lie on the boundary of the set $-\lambda K+V$, where $0 < \lambda = f(V-A_1) \leqslant 1$, and $f(V-A) > \lambda$ for all A of Σ with $A \neq A_1, \ldots,$ or A_k.

If we had $T(U) = T(V)$ and $U \neq V$, then there would be at least three distinct points A_1, A_2, A_3 common to the sets

$$-\lambda K+U \quad \text{and} \quad -\mu K+V$$

for some positive λ, μ. This is impossible by Lemma 1. Thus different sets $T(V)$ correspond to different vertices V. The number of subsets of Σ, which can be chosen in any bounded region, is finite; and hence the set of vertices V is discrete.

Let $T(U)$, $T(V)$ be two sets corresponding to distinct vertices U and V. Let $A_1, \ldots, A_r$ be the vertices of $T(U)$ and $B_1, \ldots, B_s$ those of $T(V)$. Then $A_1, \ldots, A_r$ lie on the boundary of a set $-\lambda K+U$, with $\lambda > 0$, and no points of Σ lie in the interior of this set. In particular, $B_1, \ldots, B_s$ do not lie in the interior of the set. Similarly, $B_1, \ldots, B_s$ lie on the boundary of a set $-\mu K+V$, and $A_1, \ldots, A_r$ do not lie in the interior of this set. By Lemma 3, the points $A_1, \ldots, A_r$ can be separated from the points $B_1, \ldots, B_s$ by a straight line. This shows that $T(U)$ and $T(V)$ do not overlap, i.e. they have no common inner points.

The set $\bigcup T(V)$ is non-empty. Our aim is to prove that it is the whole plane by proving that it has no boundary. This boundary is clearly a subset of the union of the boundaries of the sets $T(V)$. Let AB be an edge of one of the polygons $T(V)$. Then $D(A)$ and $D(B)$ meet at V. By Lemma 7, there will be a point C of Σ on the side of AB not containing $T(V)$, such that $D(A)$, $D(B)$, $D(C)$ have a common point, V' say. Then the triangle ABC is a proper triangle and is a subset of $T(V')$. Thus the inner points of the segment AB are inner points of $T(V) \cup T(V')$ and do not belong to the boundary of $\bigcup T(V)$. Further, if A is any vertex of one of the polygons $T(V)$, then we can find a succession of polygons with A as common vertex, each meeting the previous one along an edge through A. As at most finitely many polygons meet A, this succession must lead back to $T(V)$. Hence A is an inner point of $\bigcup T(V)$. Thus $\bigcup T(V)$ has no boundary, and so is the whole plane.

Since $T(V)$ lies in $-K+V$, on triangulating the polygons $T(V)$ in the natural way, we obtain Theorem 4.

Proof of Theorem 2. Suppose that K is strictly convex. Choose k so that K is contained in $|x| \leqslant k$, $|y| \leqslant k$. Let t be large and consider the square B of points with $|x| \leqslant t$, $|y| \leqslant t$. The vertices of those triangles of the triangulation provided by Theorem 4 which meet B, lie in the square B': $|x| \leqslant t+2k$, $|y| \leqslant t+2k$. The number of these vertices is thus at most $N(B')$. Let V, E, F denote the numbers of vertices edges and faces of this configuration covering B. As the configuration is connected, by Euler's theorem

$$V-E+F = 1.$$

When we add the edges of each triangle, each edge occurs at most twice. Hence

$$3F \leqslant 2E.$$

So

$$V = 1+E-F \geqslant 1+\tfrac{1}{2}F > \tfrac{1}{2}F.$$

Thus $F < 2V \leqslant 2N(B')$, and

$$a(B) \leqslant Ft(K) \leqslant 2N(B')t(K).$$

Hence

$$N(B')/a(B') \geqslant \big(t/(t+4k)\big)^2/2t(K).$$

Taking limits as $t \to \infty$ we obtain

$$\theta^*(K, \Sigma) = \liminf_{t\to\infty} \theta^*(K, \Sigma, B') = \liminf_{t\to\infty} N(B')a(K)/a(B') \geqslant a(K)/\big(2t(K)\big),$$

as required.

The extension to the case when K is not strictly convex is immediate as a general convex set K can be covered by arbitrarily close strictly convex sets.

§ 5. Packings of convex domains. Fejes Tóth [1] proved that there is no packing of an open convex domain K that is closer than the closest lattice packing of K. In this section we show how this result can be obtained by use of the proof of Theorem 4 above and Lemma 4 of the paper [8] of C. A. Rogers. In this way we obtain a considerable simplification of Rogers' proof of Fejes Tóth's result.

By replacing a general convex set K by half its difference set, we may reduce the general case to the case when K has O as centre. By replacing K by a sequence of strictly convex sets converging to K from within, we may reduce this case to the case when K is strictly convex and has O as centre. In proving that the density of a packing $K+A$ with $A \in \Sigma$ is not too large, we may suppose that it is not possible to adjoin any point to Σ without two of the resultant sets overlapping. This ensures that the sets $2K+A$ with $A \in \Sigma$ form a covering of the plane. By introducing the domains of action $D(A)$ corresponding to the points of Σ and the closure of the set $2K$, we obtain a system of vertices V and sets $T(V)$. This leads to a complete triangulation of the plane. If ABC is one of the triangles of this triangulation, the corresponding sets $K+A$, $K+B$, $K+C$ are disjoint and there is a vertex V such that A, B, C lie on the boundary of a set $\lambda K+V$ with $0<\lambda \leqslant 2$. Indeed, as $K+A$, $K+B$ have no common point, we have $1 \leqslant \lambda$; and, as V is in the interior of some set $2K+A_i$ with $A_i \in \Sigma$, we have $\lambda < 2$. It now follows from Lemma 4 of [8] that $a(ABC) \geqslant \frac{1}{2}d(K)$, where $d(K)$ is the lower bound of the determinants of the lattices giving rise to lattice packings of K. The proof can now be completed by a slightly modified form of the proof of Theorem 2.

Part II

§ 6. Modified domains of action. For the proof of Theorem 3, we define a new type of domain of action which take into account the finite nature of the situation. Let Π, $A_1, \ldots, A_{n+m}$ be as in the statement of Theorem 3. Let Σ denote the set $\{A_i\}$ and let Γ denote the bounding polygon $A_0A_1\ldots A_n$. For $A \in \Sigma$ we define $D^*(A) = D^*(K, \Pi, \Sigma, A)$ to be the set of all points X such that (i) the segment AX lies in Π and (ii) $f(X-A) \leqslant f(X-A_i)$ for all A_i in Σ for which the segment XA_i lies in Π. It is convenient to say that a point X is covered within Π by the set $K+A$ if the segment XA lies in $\Pi \cap (K+A)$. By condition (4) of Theorem 3, the points of Π are covered within Π by the sets $K+A$ with $A \in \Sigma$. It follows that $D^*(A) \subset K+A$ for each A and that $\Pi = \bigcup_{A \in \Sigma} D^*(A)$.

LEMMA 8. *If K is strictly convex:*

(a) *If P lies in $D^*(A)$, the whole segment PA lies in $D^*(A)$;*

(b) *If P lies in $D^*(A)$, the segment AP cannot contain a point Q, other than P, of any $D^*(B)$ with $B \neq A$;*

(c) *$D^*(A)$ is closed.*

Proof. Let P be a point of $D^*(A)$. Then the segment AP lies in Π and $f(P-A) \leqslant f(P-A_i)$ for all A_i of Σ with PA_i in Π.

Let $Q \neq P$ lie on AP. Suppose $Q \in D^*(B)$ with $B \neq A$. Then $Q \neq A$. Also QB lies in Π and $f(Q-B) \leqslant f(Q-A)$. Hence B does not lie on BA produced beyond A. If B were on the segment AP then we would have PB in Π and $f(P-B) < f(P-A)$ and P could not lie in $D^*(A)$. So B does not lie on the ray with end-point P obtained by producing PA beyond A. Hence

$$f(P-A) = f(Q-A)+f(P-Q) \geqslant f(Q-B)+f(P-Q) > f(\{Q-B\}+\{P-Q\}) = f(P-B),$$

using the strict convexity. As $P \in D^*(A)$, this implies that PB does not lie completely in Π. But the broken line PQ, QB does lie in Π. Hence PQB is a proper triangle which must contain a vertex of the boundary Γ of Π. Since Γ has only a finite number of vertices, and P is not a vertex, by choosing a vertex C in QPB such that CP makes the smallest angle with QP, we can ensure that Γ has a vertex C in the triangle QPB such that CP lies in Π. Then, since $f(P-B) < f(P-A)$, it follows from the convexity that $f(P-C) < f(P-A)$. Hence P cannot lie in $D^*(A)$. This contradiction shows that Q cannot belong to $D^*(B)$ with $B \neq A$. Thus (b) is proved and (a) follows.

To prove (c), let $\{X_n\}$ be a sequence in $D^*(A)$, converging to a point X. Then, for fixed λ, with $0 \leqslant \lambda \leqslant 1$, the point $A+\lambda\{X_n-A\}$ of AX_n converges to the point $A+\lambda\{X-A\}$ of AX. As AX_n lies in Π and Π is closed it follows that AX lies in Π. If X is a vertex A^i of Γ, let δ be the distance from X to the broken line $A_{i+1}A_{i+2}\ldots A_nA_1\ldots A_{i-1}$. If X is not a vertex of Γ, but lies on $A_{i-1}A_i$, let δ be the distance from X to the broken line $A_iA_{i+1}\ldots A_nA_1\ldots A_{i-1}$. If X is not on Γ let δ be the distance from X of Γ. Then, in each case $\delta > 0$, and, if $Y \epsilon \Pi$ and Y is within the distance δ of X, the whole segment XY lies in Π. Hence the segment XX_n lies in Π for all sufficiently large values of n. Now suppose that X is not in $D^*(A)$. Since AX lies in Π, there is a point $B \neq A$ in Σ such that $X \epsilon D^*(B)$, BX lies in Π and $f(X-B) < f(X-A)$. By the continuity of $f(X)$, we have $f(X_n-B) < f(X_n-A)$, for all sufficiently large n. Since $X_n \epsilon D^*(A)$, this implies that BX_n does not lie completely in Π. Since the broken line BXX_n lies in Π for n sufficiently large, it follows that the closed triangle BXX_n contains a vertex of Γ not coinciding with B. Since Γ has only a finite number of vertices and X_n converges to X, it follows that there is a vertex, C say, of Γ on BX not coinciding with B. This is impossible as $X \epsilon D^*(B)$. Consequently X must be in $D^*(A)$; it follows that $D^*(A)$ is closed.

Remark. We refer to the property (b) as the non-overlapping property of the sets $D^*(A)$.

§ 7. The triangulation of Π and the proof of Theorem 3. In this section we prove:

THEOREM 5. *Let K be strictly convex and suppose that the conditions of Theorem 2 are satisfied. Then Π can be triangulated by a system $n+2m-2$ triangles each lying in a set $K+X$ for some X.*

Proof. We prove the theorem by induction on $k = n+2m-2$. First suppose that $k = 1$. Then $n = 3$, $m = 0$. For convenience we write $A_0 = A$, $A_1 = B$, $A_2 = C$. Now Π is the triangle ABC and Γ is its boundary.

Suppose that a side, BC say, of ABC contains a point D with

$$f(D-A) \leqslant \min\{f(D-B), f(D-C)\}.$$

Let M be the midpoint of BC. As K is symmetric and $K+B$, $K+C$ have a common point, we have $f(B-M) = f(C-M) \leqslant 1$. We may suppose, without loss of generality, that D lies on BM. Then

$$f(A-M) \leqslant f(A-D)+f(D-M) \leqslant f(B-D)+f(D-M) = f(B-M) \leqslant 1.$$

Thus A, B, C lie in $K+M$ and the required result holds in this case.

We can, therefore, suppose that none of the sets $D^*(A)$, $D^*(B)$, $D^*(C)$ meets corresponding side BC, CA, AB of ABC. Now the three closed sets $D^*(A)$, $D^*(B)$, $D^*(C)$ cover the triangle ABC and

$$A \notin D^*(B) \cup D^*(C), \quad B \notin D^*(C) \cup D^*(A), \quad C \notin D^*(A) \cup D^*(B),$$

$$BC \cap D^*(A) = \emptyset, \quad CA \cap D^*(B) = \emptyset, \quad AB \cap D^*(C) = \emptyset.$$

It follows, by a well-known topological result[1] (see, for example the lemma on page 54 of [9]) that the three sets have a common point, M say. Then

$$ABC \subset -K+M,$$

and the result follows.

Now suppose that $k = 2m+n-2 > 1$ and that the theorem has been proved for all m, n with $2m+n-2 < k$. We consider two cases:

I: some edge A_iA_{i+1} of Γ has a point D which lies in a domain of action $D^*(A)$ with $A \neq A_i$ and $A \neq A_{i+1}$;

II: no edge of Γ contains such a point.

Case I. Suppose that BC is an edge of Γ that contains a point D of a set $D^*(A)$ with $A \neq B$ and $A \neq C$. Then D belongs to the set Δ of points X of BC with $AX \subset \Pi$ and

$$f(X-A) \leqslant \min\{f(X-B), f(X-C)\}.$$

As $f(X)$ is continuous the set Δ is closed. So we can replace D, if necessary, by a point D of Δ at which $f(X-A)$ assumes its minimum value for X in Δ. As Σ has only a finite number of points, we may suppose that A is chosen from the possible points to ensure that the corresponding minimum value $f(D-A)$ has its least possible value. Then we have:

(1) DA lies in Π;

(2) $f(D-A) \leqslant \min\{f(D-B), f(D-C)\}$;

(3) if D' is on BC, $A' \epsilon \Sigma$, $A' \neq B$, $A' \neq C$, $D'A'$ lies in Π, and $f(D'-A') \leqslant \min\{f(D'-B), f(D'-C)\}$, then $f(D-A) \leqslant f(D'-A')$.

It is clear from (2) that $D \neq B$ and $D \neq C$. We now show that the triangle ABC lies in Π. We shall prove that ADB lies in Π; the proof that ADC lies in Π is identical.

Suppose that ADB is not contained in Π. Since the broken line AD, DB is contained in Π, it follows that the triangle ADB contains a vertex P of Γ not on AB. Let the line through P parallel to AB meet DA at A' and DB at B'. If the triangle $A'DB'$ were not contained in Π, the argument we have just used would yield a vertex P' of Γ in $A'DB'$ not on $A'B'$, contrary to the choice or P. Hence $A'DB'$ lies in Π. Let the line through P parallel to AD meet $B'D$ at the point D'. Then PD' lies in $A'DB'$ and so lies in Π. By similar triangles, $PD'/AD = D'B'/DB$. Since $f(D-A) \leqslant f(D-B)$, this implies that $f(D'-P) \leqslant f(D'-B') \leqslant f(D'-B)$. Also $f(D'-P) < f(D-A) \leqslant f(D-C) \leqslant f(D'-C)$. Thus $D'P$ lies in Π and $f(D'-P) \leqslant \min\{f(D'-B), f(D'-C)\}$. It follows from property (3) that $f(D-A) \leqslant f(D'-P)$. This is impossible as PD' is pa-

[1] It is in fact quite easy to avoid an appeal to this result by giving a proof, using the methods of this paper, and making use of special properties of our sets.

rallel to and shorter than AD. Hence ADB lies in Π as required.

We now know that ABC lies in Π. If ABC contain a point A' of Σ other than A, B or C, a simplified version of the above argument would lead to a contradiction to the choice of A and D. Hence no member of Σ other than A, B, C lie in the triangle ABC.

Now there are three possibilities to consider:

(i) A, B, C are consecutive vertices of Γ in some order;

(ii) B and C are vertices of Γ and A is in the interior of Π;

(iii) B and C are consecutive vertices of Γ while A is a vertex of Γ adjacent to neither B nor C.

In case (i), we may suppose that A, B, C are consecutive vertices of Γ. If we remove the points of ABC, not on AC, from Π, we get a closed domain Π^* bounded by the Jordan polygon Γ^* obtained from Γ by replacing the edges AB, BC by the edge AC. Let $A_0^*, \ldots, A_{n-1}^* = A_0^*$ be the vertices of Γ^* and let Σ^* be the set of points $A_0^*, \ldots, A_{n-1}^* = A_0^*$, $A_{n+1}, \ldots, A_{n+m}$. Then $A_{n+1}, \ldots, A_{n+m}$ lie in the interior of Π^*. Since DA is in $D^*(A)$, while $D^*(B)$ lies in the angle bounded by the lines BA, BC, it follows from the non-overlapping property of the sets D^* that $D^*(B)$ is contained in the triangle ABD. So Π^* is covered by the original sets $D^*(A) = D^*(K, \Pi, \Sigma, A)$ with $A \in \Sigma^*$. But if two points X, A lie in Π^* and XA lies in Π, then clearly XA lies in Π^*. Hence Π^* is covered by the sets $D^*(A) = D^*(K, \Pi^*, \Sigma^*, A)$ with $A \in \Sigma^*$.

Let n^*, m^* be the numbers for Π^*, Σ^* corresponding to n, m for Π, Σ. Then

$$n^*+2m^*-2 = (n-1)+2m-2 = k-1,$$

so that, by the inductive hypothesis, Π^* can be covered by $k-1$ non-overlapping triangles each lying in a set of the form $K+X$. Since $f(D-A) \leqslant \min\{f(D-B),\ f(D-C)\}$, and $f(A-B) \leqslant 2$, $f(B-C) \leqslant 2$, it follows as in the case $n+2m-2 = 1$, that ABC lies in $K+M$, where M is the mid-point of BC. Adjoining the triangle ABC to the triangulation of Π^* we obtain the required triangulation of Π. So the induction is complete in this case.

In case (ii) we form a closed domain Π^* by removing from Π the points of the triangle ABC not on the sides AB and AC. Then Π^* is bounded by the Jordan polygon Γ^* obtained from Γ by replacing the edge BC by the two edges BA, AC. We take $\Sigma^* = \Sigma$. Then Π^* is covered by the domains $D^*(A) = D^*(K, \Pi, \Sigma, A)$ with $A \in \Sigma^*$. Suppose that a point X of Π^* does not belong to any of the domains $D^*(K, \Pi^*, \Sigma^*, A)$ with $A \in \Sigma^*$. Then for some $A^* \in \Sigma^*$ we have $X \in D^*(K, \Pi, \Sigma, A^*)$ but $X \notin D^*(K, \Pi^*, \Sigma^*, A^*)$. Hence the segment A^*X of $D^*(K, \Pi, \Sigma, A^*)$ lies in Π but not in Π^*. So X lies outside the triangle ABC or on one of

the sides AB, AC, while some point of A^*X lies in the interior of ABC. Since A, B, C are the only points of Σ in ABC, it follows that A^*X must meet AD at some point other than A. This is contrary to the non-overlapping property of the domains $D^*(K, \Pi, \Sigma, A)$. Consequently, the points of Π^* are covered within Π^* by the sets $K+A^*$ with $A^* \in \Sigma^*$. Defining the integers n^*, m^* for Π^*, Σ^* in the natural way, we have

$$n^*+2m^*-2 = (n+1)+2(m-1)-2 = n+2m-3 = k-1.$$

Now the induction can be completed as in case (i).

In case (iii) the removal of ABC from Π splits it into two polygons Π^*, Π^{**} say. Now Σ splits into two sets $\Sigma^* \subset \Pi^*$ and $\Sigma^{**} \subseteq \Pi^{**}$. As before the non-overlapping property of the domains $D^*(K, \Pi, \Sigma, A)$ ensures that Π^* is covered from within by the sets $K+A^*$ with $A^* \in \Sigma^*$, and similarly for Π^{**} and Σ^{**}. Let n^*, m^*, n^{**}, m^{**} be the integers associated with these pairs of sets. Then $n^*+n^{**} = n+1$ and $m^*+m^{**} = m$. As $n^* \geqslant 3$, $n^{**} \geqslant 3$ this implies $n^*+2m^*-2 < k$ and $n^{**}+2m^{**}-2 < k$. By the inductive hypothesis, the polygons Π^* and Π^{**} can be covered by n^*+2m^*-2 and $n^{**}+2m^{**}-2$ non-overlapping triangles, respectively, each contained in a set $K+X$ for some X. Adjoining the triangle ABC to the union of the triangulations of Π^* and Π^{**} we obtain the required triangulation, as

$$\begin{aligned} 1+(n^*+2m^*-2)+(n^{**}+2m^{**}-2) \\ = n^*+n^{**}-1+2(m^*+m^{**})-2 = k. \end{aligned}$$

This completes the proof for case I.

Case II. We now suppose that no edge of Γ contains a point in a domain of action that is associated with a vertex of Γ other than the end-points of the edge. Let BC be any edge of Γ. The mid-point D of BC lies in the closed set $D^*(B) \cap D^*(C)$. As B is not in $D^*(B) \cap D^*(C)$ the angle $\theta(X)$ that the ray BX makes with BC is a continuous function of X on $D^*(B) \cap D^*(C)$. So we can choose a point V of $D^*(B) \cap D^*(C)$, so that $\theta(X)$ attains its maximum value on $D^*(B) \cap D^*(C)$, say the value θ_0, at V. As we are in Case II, the point V is not on Γ, unless it coincides with D. In either case, if α is sufficiently small and if X is sufficiently close to B on the ray BX making the angle $\theta_0+\alpha$ with BC, the point X will be in the interior of Π and in $D^*(B)$. So, if α is sufficiently small the ray starting from B at an angle $\theta_0+\alpha$ with BC meets the boundary of $D^*(B)$ at a point $U(\alpha)$ other than B. As we are in Case II, $U(\alpha)$ is not on Γ and so belongs to some $D^*(A_\alpha)$ with $A_\alpha \in \Sigma$, $A_\alpha \neq B$. For $\alpha > 0$, the ray $BU(\alpha)$ does not meet $D^*(B) \cap D^*(C)$. Hence $A_\alpha \neq C$. Letting α tend to zero through a suitable sequence we easily obtain a point A of

Σ other than B, C, with V in $D^*(A)$. Now VA, VB, VC all lie in Π, and $f(V-A) = f(V-B) = f(V-C) = \lambda \leqslant 1$ for some λ. Further A, B, C lie on the boundary of $-\lambda K+V$.

Since VB, VC and BC lie in Π, while $V = D$ or V is an inner point of Π, the whole triangle VBC lies in Π. Now consider VBA. Suppose that VBA does not lie in Π. Since AV, VB lie in Π, this implies that there is a vertex of Γ in the triangle VBA not on AB. Let P be a vertex of Γ in VBA at the greatest possible distance from AB. As V is D or is an inner point of Π, it follows that PV is in Π. But, by strict convexity, $f(V-P) < f(V-A) = f(V-B)$, which is impossible as $V \epsilon D^*(B)$. Thus VBA lies in Π. Similarly, VAC lies in Π.

As $V \epsilon D^*(A) \cap D^*(B) \cap D^*(C)$, it follows by this last argument, that no point of Σ other than A, B, C lies in VAB, VBC or VCA.

We consider two cases: (a) when V lies in ABC; (b) when V does not lie in ABC. In case (a) we can complete the proof just as in Case I using the non-overlapping property of the domains $D^*(A_i)$ with $A_i \epsilon \Sigma$ and the fact that the segments VA, VB and VC lie in $D^*(A)$, $D^*(B)$ and $D^*(C)$, in place of the result in Case I that DA is in $D^*(A)$.

In case (b), as VA lies in Π but V is not in the triangle ABC, it follows that V lies in the angle at B or in the angle at C. Without loss of generality, we can take V in the angle at B. Then

$$f(V-C) \leqslant 1, \quad f(V-A) \leqslant 1, \quad f(C-A) \leqslant 2.$$

It follows that the triangle AVC, lying in Π outside the triangle ABC, is covered within itself by $K+A$ and $K+C$. The triangulation can again be completed, just as in Case I, using this fact and the fact that VA, VB and VC lie in $D^*(A)$, $D^*(B)$ and $D^*(C)$, in place of the result that DA is in $D^*(A)$.

This completes the proof of Theorem 5.

Proof of Theorem 3. When K is strictly convex the result follows immediately from Theorem 5. When K is not necessarily strictly convex the result follows on applying the strictly convex case to strictly convex domains approximating K from without.

§ 8. A result on packing. In [8] C. A. Rogers has given an outline of his proof of a result (Theorem 2 of [8]) on the packing of convex domains that is very similar to our Theorem 3. In this section, we indicate how his result may be obtained from our Theorem 5 and some of his simpler lemmas. In the first place the general case can be reduced in the usual way to the case when K has O as centre. So we may suppose that K is open and strictly convex with O as centre and that $A_0, A_1, \ldots, A_n = A_0$, $A_{n+1}, \ldots, A_{n+m}$ satisfy the conditions of Rogers' theorem. We may also suppose that there is no point which can be adjoined to the points $A_0, \ldots$, A_{n+m} to form a system $A_0, \ldots, A_{n+m}, A_{n+m+1}$ also satisfying the conditions of the theorem. If the sets $2K+A_i$ with $0 \leqslant i \leqslant n+m$ do not cover the polygon bounded by the Jordan polygon $A_0A_1\ldots A_n$ from within, we can find a point, A^* say, in Π which is not covered within Π by the sets $2K+A_i$, with $0 \leqslant i \leqslant n+m$. Using the special case $\sigma = 0$ of Lemma 3 of (8), it follows that A^* could be taken as an extra point A_{n+m+1}. This shows that Π is covered from within by the sets $2K+A_i$, with $0 \leqslant i \leqslant n+m$. Now the conditions of Theorem 5 are satisfied and we obtain a triangulation of Π into $n+2m-2$ triangles each within a set $2K+X$ for some X.

Now consider a triangle, say ABC, of the triangulation. Let K' be the closure of K. Then we can choose λ with $0 \leqslant \lambda < 2$ and a point V so that $ABC \subset \lambda K'+V$ and λ has the smallest possible value. Then A, B, C lie on the boundary of $\lambda K+V$. As in §5 above, it follows that $a(ABC) \geqslant \frac{1}{2}d(K)$. As this holds for all ABC of the triangulation, the required result follows.

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