On a alors d'après (6) $Ae^{-i\beta_1}k(t) = Ae^{i\beta_2}\overline{k(t)}$, d'où

$$k^2(t) = e^{i(\beta_1+\beta_2)} = \frac{\Phi(z_1,t)\Phi(z_2,t)}{A^2}.$$

En admettant l'égalité

$$k^2(t) = \frac{z_1 z_2}{\varrho^2} \quad \text{pour} \quad 0 \leqslant t \leqslant t_0, \tag{8}$$

on a donc

$$\frac{\Phi(z_1,t)\Phi(z_2,t)}{A^2} = \frac{z_1 z_2}{\varrho^2}.$$

On peut déterminer à présent la fonction extrémale à l'aide de l'équation (1), dans laquelle la fonction $k(t)$ est celle donnée par (8). En intégrant l'équation (1), où $k(t)$ est la fonction en question, il vient

$$\int_0^t \frac{\Phi(z,t)-k(t)}{\Phi(z,t)+k(t)}\,\frac{dt}{\Phi(z,t)} = t,$$

d'où l'on tire par des calculs évidents l'équation fonctionnelle

$$\frac{[\Phi(z,t)+k(t)]^2}{\Phi(z,t)} = e^t\,\frac{[z+k(t)]^2}{z}.$$

En posant

$$k(t) = e^{i\gamma}, \quad \gamma = \frac{a_1+a_2}{2} \quad \text{et} \quad \varphi(z) = \frac{(1+z)^2}{z},$$

on vérifie aisément que la fonction extrémale $W^*(z) = m\Phi^*(z,t)$ est de la forme

$$W^*(z) = me^{i\gamma}\Phi^{-1}\left[\frac{1}{m}\varphi(ze^{-i\gamma})\right] = z+2e^{i\gamma}(1-m)+e^{2i\gamma}\frac{1-m^2}{z}+\ldots$$

TRAVAUX CITÉS

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TO THE MEMORY OF A. N. MILGRAM

ON THE INCLINATION OF A MINIMAL SURFACE $\varphi(x,y)$ *

BY

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It has been known since the pioneering work of J. Schauder that the derivatives of a solution of a linear elliptic partial differential equation

$$\text{(1)}\quad \begin{aligned} &\text{(a)} \quad a\varphi_{xx}+2b\varphi_{xy}+c\varphi_{yy}+d\varphi_x+e\varphi_y+f\varphi = 0,\\ &\text{(b)} \quad ac-b^2 > k^2 > 0, \end{aligned}$$

are bounded interior to the domain of definition if the solution $\varphi(x,y)$ is bounded in the whole domain. In fact, as has been shown by Bers and Nirenberg [2], the bound depends only on a bound for the coefficients, on k, and on distance to the boundary. In this form, the estimate can be used to study the (non-linear) case in which the coefficients depend not only on (x,y) but also on the solution and its derivatives.

On the other hand, I have shown by example in [3] that weaker assumptions will in general not suffice to obtain a bound even on the first derivatives of the solution.

Important non-linear equations of the form (1a), which arise in practice, satisfy (1b) only in the restricted sense that $ac-b^2 > 0$ for every solution. Since physical and heuristic considerations suggest that estimates of the sort we have mentioned will hold also for these equations, it seems desirable to study particular such equations with a view to developing an appropriate theory. An initial step in this direction was taken in my paper [3], in which, in particular, I derived bounds on the derivatives of the solutions of the *minimal surface equation*

$$\frac{1+v^2}{W}\varphi_{xx}-2\frac{uv}{W}\varphi_{xy}\frac{1+u^2}{W}\varphi_{yy} = 0, \tag{2}$$

$$u = \varphi_x, \quad v = \varphi_y, \quad W = \sqrt{1+u^2+v^2},$$

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depending only on a bound for the solution and distance to the boundary. The methods used to study (1a, b) cannot be applied to (2), and to obtain the result it was necessary to exploit the particular non-linearity of the equation and to use some general properties of conformal mappings.

The prototype for equations of the form (1a, b) is the Laplace equation

$$\varphi_{xx}+\varphi_{yy}=0. \tag{3}$$

Condition (1b) can be interpreted geometrically as the condition that the Riemannian metric

$$ds^2 \equiv c\,dx^2-2b\,dx\,dy+a\,dy^2 \tag{4}$$

induced over the (x, y)-plane by the coefficients is *quasi-conformally* related to the corresponding metric induced by (3), that is, to the Euclidean metric of the (x, y)-plane.

In [3] I have used an analogous consideration to characterize a class of *equations of minimal surface type*. Equations

$$a(u,v)\varphi_{xx}+2b(u,v)\varphi_{xy}+c(u,v)\varphi_{yy}=0, \qquad ac-b^2>0, \tag{5}$$

are considered, which have the property that for any (u, v) the metric (4) induced over the (x, y)-plane by the coefficients is quasi-conformally related to the metric induced correspondingly by the coefficients of (2). Setting

$$\alpha=\frac{1+v^2}{W}, \quad \beta=-\frac{uv}{W}, \quad \gamma=\frac{1+u^2}{W}, \quad \Delta^2=ac-b^2,$$

the condition is expressed by requiring the existence of a fixed constant ε, such that uniformly in (u, v),

$$|a\gamma+c\alpha-2b\beta|\leqslant 2\varepsilon\Delta. \tag{6}$$

Alternatively, this condition can be interpreted as requiring the spherical image mapping defined by any solution to be a quasi-conformal mapping of the solution surface (as is known, this mapping is conformal on a minimal surface).

Under the hypothesis (6) and one additional assumption (1), I showed in [3] that the qualitative estimates obtained for solutions of (2) are equally valid for the solutions of (5), and as a consequence I was able to strengthen classical existence theorems for these equations. Thus, the theory of equations of minimal surface type parallels in an important respect the classical theory of uniformly elliptic equations.

(1) It seems likely that the additional assumption (stated in [3], §1, equation (8)) is a consequence of (6). This has been shown in cases of particular interest by Jenkins [5]. This assumption can also be interpreted geometrically with the aid of quasi-conformal mappings, cf. [4].

The estimates obtained in [3] exhibit the correct general structure and indicate that $|\nabla\varphi|$ can grow exponentially with the bound on $|\varphi|$, but they are not best possible, as some of the constants which appear are much too large. In an effort to sharpen these results I have now approached the question from a very different point of view. It turns out that a simple "comparison lemma", valid for a fairly general class of equations (5), leads to results which are both sharper and stronger than those I could give in [3] (2), and show, in particular, that it suffices to assume φ bounded on one side(3). These new results indicate some differences in behavior between the solutions of (5) and (6) and those of (1), notably in the manner by which the solution is controlled by its boundary values. In the special case of the minimal surface equation the new estimate, in an important sense, cannot be improved.

The following sections are devoted to a statement of results and a brief sketch of the method of proof. Detailed demonstrations will appear elsewhere.

1. The comparison lemma. Let $\varphi_1(x, y)$, $\varphi_2(x, y)$ be solutions of (5) in a region $\mathcal{G}$ bounded by a Jordan arc Γ, and suppose $ac-b^2>0$ in some convex region $\mathcal{R}$ of the (u, v)-plane which includes the values achieved by these solutions. It follows from a theorem of Bers [1] that (5) can be written as a divergence. That is, there exist functions $\theta(u, v)$ and $\Lambda(u, v)$ such that

$$\frac{\partial}{\partial x}\theta(u,v)+\frac{\partial}{\partial y}\Lambda(u,v)=0 \tag{7}$$

for any solution of (5) for which $(u, v)\in\mathcal{R}$. I shall assume that it is possible to choose these functions in such a way that

$$\theta^2(u,v)+\Lambda^2(u,v)\leqslant 1 \tag{8}$$

throughout $\mathcal{R}$ (4).

(2) Some of the results of [3], notably the bound on $|\nabla\varphi|$ in terms of the area of the solution surface, seem however not accessible to the methods of the present paper.

(3) Independently and somewhat previously, Jenkins and Serrin [6] have obtained results of this sort for those equations of minimal surface type which arise from variational principles. For the minimal surface equation, the estimates (9), (10) and (11) of the present paper are superior in the sense that the constant $\pi/2$ in the exponent is sharp. See, however, footnote 6.

(4) See footnote 1. In the special case of equation (2) we may write $\theta=u/W$, $\Lambda=v/W$, or, alternatively, $\theta=-uv/W$, $\Lambda=1+u^2/W$. Only the first choice satisfies (8).

LEMMA. *Let $\Phi(x,y) = \varphi_1(x,y) - \varphi_2(x,y)$ and suppose Γ can be divided into two parts by subarcs Γ^+ and Γ^-, such that*

$$\underline{\lim}_{(x,y)\to\Gamma^+} \Phi(x,y) > 0, \qquad \overline{\lim}_{(x,y)\to\Gamma^-} \Phi(x,y) < 0.$$

Then at any point $P \in \mathscr{G}$ at which $\Phi(P) = 0$, there holds $\nabla\Phi(P) \neq 0$.

The application of this lemma lies in the following consideration. Suppose one is given a solution $\varphi(x,y)$ of (5) in a unit disc Σ, such that $|\varphi| < M$ in Σ, $\varphi(0,0) = m$. Suppose $\mathscr{R}$ is the entire u, v plane and that (6) and (8) hold throughout $\mathscr{R}$. One shows first that for any division of the boundary Γ of Σ into subarcs Γ^+, Γ^-, and any $\varepsilon > 0$, there will be a solution φ_M of (5) in Σ such that

$$\varphi_M = \begin{cases} M+\varepsilon & \text{on} \quad \Gamma^+, \\ -M-\varepsilon & \text{on} \quad \Gamma^-. \end{cases}$$

The construction of such a solution is not difficult for the minimal surface equation, but I have had to invoke a considerable apparatus to prove the existence in the more general case. In general, no such solution will exist if the hypotheses (6) and (8) are not satisfied.

The next step is to show that Γ^+, Γ^- can be so chosen that $\varphi_M(0,0) = m$, and that the direction of $\nabla\varphi_M(0,0)$ coincides with that of $\nabla\varphi$ at this point.

If now $|\nabla\varphi_M(0,0)| < |\nabla\varphi(0,0)|$, we let $\varepsilon\to\infty$, keeping $\varphi_M(0,0) = m$ and the direction of $\nabla\varphi_M(0,0)$ fixed. One shows that then $|\nabla\varphi_M(0,0)| \to \infty$. Thus, at some point in this process there would hold $\nabla\varphi_M(0,0) = \nabla\varphi(0,0)$, contradicting the lemma.

Thus, the inclination of the (relatively simple) comparison solution $\varphi_M(x,y)$ dominates that of any given solution whose magnitude does not exceed M. (Since ε is arbitrary, we may assume $\varepsilon = 0$.) This is the central idea of the method.

2. Estimation of the comparison solution. For reasons which will become apparent, it is convenient to replace the comparison surface defined above by a solution defined only in an inscribed quadrilateral Q, and achieving constant values $\pm M$ on the sides. The lemma, together with the maximum principle for solutions of (5), shows that such a surface cannot be less steep at the origin than the one originally chosen.

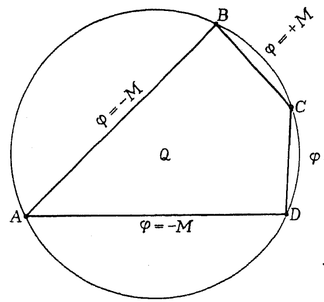
Equation (7) is an integrability condition assuring the existence of a function $\psi(x,y)$ such that $\psi_y = \Theta(u,y)$, $-\psi_x = \Lambda(u,v)$. Since, by (8), $|\Delta\psi| < 1$ for any solution, it follows that $\psi_M(x,y)$ tends to finite limits at the two points B and D of discontinuity on the boundary of Q. We may assume these limits have values $\pm\delta$, and one may show $\delta\to 1$ as $M\to\infty$. Thus, the boundary of Q is mapped by $\varphi_M + i\psi_M$ onto the

boundary of the rectangle R: $\{\varphi_M = \pm M,\ -\delta \leqslant \psi_M \leqslant \delta;\ \psi_M = \pm\delta,\ -M \leqslant \varphi_M \leqslant M\}$. In fact, *$\varphi_M + i\psi_M$ maps the surface $\mathscr{S}_M$ defined by $\varphi_M(x,y)$ one-to-one onto R.* Because of (8), *the mapping is quasi-conformal of $\mathscr{S}_M$ onto R. In the special case of* (2), *$\varphi_M + i\psi_M$ defines a conformal map of the surface.*

Consider now the spherical image of $\mathscr{S}_M$, and map this stereographically onto the equatorial plane. There results the function

$$\chi(x,y) = \frac{u-iv}{1+W},$$

which is analytic on a minimal surface, quasi-conformal in the more general case (5) and (6). Form the function $-\log\chi(x,y) = \tau + i\vartheta$, and observe that ϑ is known on the boundary segments of Q, $\tau\to 0$ at B and D. Considered as a function of $\varphi_M + i\psi_M$, $\tau = 0$ on the segments $\psi_M = \pm\delta$, while ϑ is known on the other segments as soon as the points of discontinuity of this function can be determined. These points can, however, be estimated (for equations with symmetry properties — in particular (2) — they are known exactly). This information suffices to yield an estimate for τ at the image of the origin in the mapping, and hence the desired estimate for $|\nabla\varphi_M|$ at the origin. For the minimal surface equation, this final estimate can be obtained from the series development of the analytic function $\tau + i\vartheta$; for the general case I have found it necessary to apply relatively sophisticated results from the theory of quasi-conformal mappings.



3. The principal results. We study first a solution $\varphi(x,y)$ of the minimal surface equation, defined in the unit disc $x^2+y^2<1$, and bounded in magnitude by M. Let $\varphi(0,0) = m$, set $\mu = M-m$, define $\gamma_m = e^{\sigma m}\operatorname{sech}\sigma m$, $\sigma = \frac{1}{2}\pi$.

THEOREM 1. *Under the above condition, there is an absolute constant C such that*

$$|\nabla\varphi(0,0)| < \tfrac{1}{2}\gamma_m e^{(\pi/2)\mu} + C\mu. \tag{9}$$

The constant C can be estimated explicitly.

COROLLARY. *Suppose $m = 0$. Then*

$$|\nabla\varphi(0,0)| < \tfrac{1}{2}e^{(\pi/2)M} + CM. \tag{10}$$

In these results, the factor $\pi/2$ in the exponent cannot be improved ([5]), as can be shown by example.

In Theorem 1, note that $\gamma_m < 2$ for all m, M, so that the estimate can be made independent of these quantities. Since the addition of an arbitrary constant does not change the property of $\varphi(x, y)$ to be a solution, we are led to an estimate requiring only a one-sided bound on $\varphi(x, y)$.

THEOREM 3. *Let $\varphi(x, y)$ be a positive solution in the unit disc of the minimal surface equation* (2), *and suppose $\varphi(0, 0) = m$. Then*

$$|\nabla\varphi(0, 0)| < e^{(\pi/2)m} + Cm, \tag{11}$$

where C is an explicitly known absolute constant ([6]).

In extending the above results to the general equations (5), (6) and (8), it is convenient to introduce the dilation ratio K of the mapping defined by $\varphi + i\psi$ as a function of $\tau + i\vartheta$. Because of the assumptions (6) and (8), K is bounded, depending only on the equation and not on the particular solution considered.

THEOREM 4. *Suppose* (6) *and* (8) *satisfied by the coefficients of* (5), *uniformly in all* (u, v). *Let $\varphi(x, y)$ be a positive solution of* (5) *in the unit disc $x^2 + y^2 < 1$, and suppose $\varphi(0, 0) = m$. Then, for any $\gamma > 1$, there holds*

$$|\nabla\varphi(0, 0)| < Ce^{(\pi/2)K^2\gamma m}, \tag{12}$$

where C is an absolute constant, independent of the solution considered.

This estimate leads to a form of Harnack inequality, which holds for all equations of minimal surface type. Let

$$\sigma = \frac{\pi}{2} K^2 \gamma .$$

Then there is an absolute constant C with the following property:

THEOREM 5. *Suppose* (6) *and* (8) *satisfied by the coefficients of* (5), *uniformly in all* (u, v). *Let $\varphi(x, y)$ be a positive solution of* (5) *in the unit disc $x^2 + y^2 < 1$, with $\varphi(0, 0) = m$. Then at distance r from the origin there holds*

$$\exp\left(\sigma\frac{\varphi}{1-r}\right) \leqslant \frac{e^{\sigma m}}{1 + Ce^{\sigma m}\log(1-r)} \tag{13}$$

for all r sufficiently small that the denominator on the right is positive.

([5]) I am indebted to Professor J. B. Keller for an interesting heuristic reasoning which suggests that $(\pi/2)M$ is the best exponent. This discussion will appear in conjunction with the full exposition of my results.

([6]) Professor J. B. Serrin has observed that a different choice of comparison surface leads, with more elementary proof, to a similar result.

Thus, for all positive solutions bounded by m at the origin, there is a uniform subcircle interior to which these solutions remain bounded. Jenkins and Serrin have shown by example that, unlike the case of harmonic functions, this subcircle cannot be extended to the entire disc $x^2 + y^2 < 1$.

In [3] I have given various existence theorems, which are consequences of the estimates of that paper. New and stronger existence theorems, which follow from the improved estimates in the present work, will appear in a forthcoming paper.

Added in proof. After this material was sent to press, I obtained sharper and more general results than those indicated here. Details are included in the full exposition, which appears in Archive for Rational Mechanics and Analysis 14 (1963), p. 337-375.

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