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# *SOME APPLICATIONS OF THE METHOD OF EXTREMAL POINTS**

BY

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**Introduction.** Let $E$ be a bounded closed set in the complex plane $C$. Let $b(z)$ be a real function defined and bounded on $E$. Put

$$V(\zeta^{(n)}, b) = \prod_{0\leqslant i<k\leqslant n} \{|\zeta_i - \zeta_k| \exp[-b(\zeta_i) - b(\zeta_k)]\},$$

where $\zeta^{(n)} = \{\zeta_0, \zeta_1, \ldots, \zeta_n\}$ is a system of $n+1$ points of $C$. Denote by $\boldsymbol{b}(z)$ the greatest lower semicontinuous minorant of $b(z)$. A system of points of $E$

$$\eta^{(n)} = \{\eta_0^{(n)}, \eta_1^{(n)}, \ldots, \eta_n^{(n)}\} \tag{1}$$

defined by

$$V(\eta^{(n)}, \boldsymbol{b}) = \max_{\zeta^{(n)}\subset E} V(\zeta^{(n)}, \boldsymbol{b}) = \sup_{\zeta^{(n)}\subset E} V(\zeta^{(n)}, b)$$

is called the *$n$-th extremal system of $E$ with respect to* $b(z)$.

Suppose the transfinite diameter $d(E)$ of $E$ is positive and define for every $n = 1, 2, \ldots$ the function

$$\Phi_n^{(1)}(z, E, b) = \max_{(j)} \left\{\left[\prod_{k=0(k\neq j)}^{n} \left|\frac{z-\eta_k^{(n)}}{\eta_j^{(n)}-\eta_k^{(n)}}\right|\right] \exp[nb(\eta_j^{(n)})]\right\}.$$

We prove that the sequence $\{\sqrt[n]{\Phi_n^{(1)}(z, E, b)}\}$ is convergent at every finite point $z \in C$ to $\Phi(z) = \Phi(z, E, b)$

$$\Phi(z) = \lim_{n\to\infty} \sqrt[n]{\Phi_n^{(1)}(z, E, b)}, \quad z \in C, \tag{2}$$

$\Phi(z)$ being the Leja's extremal function of $E$ with respect to $b(z)$ (comp. [9], [17]). Leja defines $\Phi$ as a limit of the sequence $\{\sqrt[n]{\Phi_n^{(3)}}\}$ given by

* This paper is a slight modification of the author's doctoral thesis (1960).

(2.5). The advantage of our definition of $\Phi$ relies on the fact that it admits a straightforward generalization to the case of the space $C^n$ of $n$ complex variables (see [22]).

In section 4 we give a generalization of Tchebycheff polynomials (with respect to $b(z)$) and prove some of their properties in terms of $\Phi$.

For $b(z) \equiv 0$ the extremal points (1) were introduced by Fekete [1]. In the case that $b(z)$ is continuous the extremal points (1) and the function $\Phi$ were introduced by Leja [9] and investigated later by him and his students in connection with the conformal mapping of simply or multiply connected domains on some canonical domains and with the Dirichlet problem (for bibliography see [17]).

The purpose of this paper is to prove new properties of the function $\Phi$ and to apply them to the effective construction of generalized (by Kellog-Wiener or Perron) solution of the Dirichlet problem.

One of the most important properties of $\Phi$ we prove in this paper is given by the following result:

*Let* $a_i \geqslant 0$, $i = 1, 2, \ldots, k$, *and let* $a = \Sigma a_i > 0$. *If*

$$b(z) = \frac{1}{a}\left[a_1 b_1(z) + \ldots + a_k b_k(z)\right],$$

*then*

$$\prod_{i=1}^{k} \Phi^{a_i}(z, E, b_i) \leqslant \Phi^{a}(z, E, b), \qquad z \in C.$$

As a simple corollary from this inequality we get:

*If real functions* $p(z)$ *and* $b(z)$ *are defined and bounded on* $E$ *and if* $0 < \lambda' \leqslant \lambda$, *then*

$$[\Phi(z, E, p+\lambda b)/\Phi(z, E, p)]^{1/\lambda} \leqslant [\Phi(z, E, p+\lambda' b)/\Phi(z, E, p)]^{1/\lambda'}, \qquad z \in C.$$

Moreover, the function

$$u(z) \equiv u(z, E, p, b) = \lim_{\lambda \downarrow 0} \operatorname{Log}[\Phi(z, E, p+\lambda b)/\Phi(z, E, p)] \tag{3}$$

is harmonic at every point outside of $E$.

This result and a theorem on approximation of continuous functions by harmonic functions, proved in section 8, enable us to obtain the following theorems:

I. *Let* $E$ *be a union of the boundaries of* $p+1$ $(p \geqslant 0)$ *domains* $D_0, D_1, \ldots, D_p$, *no two of which have common points. Let* $b(z)$ *be an arbitrary real function defined and lower semicontinuous on* $E$. *If*

$$p(z) = \operatorname{Log}\sqrt{1+|z|^2} \quad \text{or} \quad p(z) = \operatorname{Log}|(z-a_1)^{\sigma_1} \ldots (z-a_k)^{\sigma_k}|,$$

*where* $\sigma_i > 0$, $\sum_{i=1}^{k} \sigma_i < 1$, $a_i \in D_i$, $\infty \in D_0$, *then*

$$u(z, E, p, b) = b(z) \quad \text{for} \quad z \in E. \tag{4}$$

*If, moreover,* $b(z)$ *is continuous and all the domains are regular with respect to the Dirichlet problem, then* $u(z, E, p, b)$ *is a solution of the Dirichlet problem for every component of* $CE$ *with boundary values* $b(z)$.

II. *Let* $D$ *be a domain containing* $\infty$ *in its interior. Let* $D$ *be regular with respect to the Dirichlet problem and let* $b(z)$ *be an arbitrary real function defined and bounded on the boundary* $E$ *of* $D$. *Then*

$$u(z, E, b) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \operatorname{Log}[\Phi(z, E, \lambda b)/\Phi(z, E, 0)], \qquad z \in C, \tag{5}$$

*is the least Perron solution of the Dirichlet problem for* $D$ *with boundary values* $b(z)$.

III. *If* $b(z)$ *is continuous on* $E$ *and* $E$ *is a boundary of the unbounded component* $D = D(E)$ *of* $CE$ *and* $d(E) > 0$, *then the function* $u(z)$ *given by* (5) *is a Kellog-Wiener solution of the Dirichlet problem for* $D$ *with boundary values* $b(z)$.

Theorem I may be considered as a generalization of results by Leja [9], [12] and Inoue [5].

From the practical point of view it is useful to know what are the functions such that for some $\lambda > 0$ we have $\Phi(z, E, \lambda b) = \exp[\lambda b(z)]$ for $z \in E$. In section 5 we give a necessary and sufficient condition that this equality hold. In section 6 we characterize a family of functions for which the equation $\Phi = \exp[\lambda b(z)]$ for $z \in E$ holds for some $\lambda > 0$ and which is dense in the class of all continuous functions (under some general conditions on $E$).

The last section is devoted to remarks on the effectiveness of the method of the extremal points. We show that in principle the method is really effective; it is possible to compute the function $\Phi(z, E, b)$ (and other extremal functionals) with as small error as we wish performing only a finite number (may be very large) of practically realizable operations.

It is worth-while to mention that the main and almost the only tools used in this paper are the Lagrange interpolation formula and the maximum principle for subharmonic functions. These elementary tools enable us, however, to give the solution of the Dirichlet problem that plays the fundamental role in the theory of functions.

The author wishes to express here his gratitude to Professor Leja for the suggestion of the theme of this paper and for his valuable guidance while preparing it.

CONTENTS



**1. Extremal points of a set with respect to a function.** Let $C$ be the open complex plane, $E$ — a closed subset of $\bar{C} = C + \{\infty\}$ and $b(z)$ — a real function defined at every finite point of $E$ such that

(1.1)
$$0 < m = \inf_{z \in E (z \neq \infty)} \{e^{-b(z)} \max(1, |z|)\} \leqslant \sup_{z \in E (z \neq \infty)} \{e^{-b(z)} \max(1, |z|)\} = M < \infty.$$

If $E$ is bounded, inequalities (1.1) are equivalent to the assumption that the function $b(z)$ is bounded on $E$. Because of (1.1), the function $b(z)$ is bounded on every bounded subset of $E$.

If $E$ is unbounded, we assume $\infty$ is a limit point of $E$.

We define $\omega(z, \zeta)$ for $(z, \zeta) \in E \times E$ by

(1.2) $\omega(z, \zeta) =$
$$= \begin{cases} |z-\zeta| \exp[-b(z) - b(\zeta)], \text{ if } z \text{ and } \zeta \text{ are finite,} \\ \limsup\limits_{\zeta \to \infty, \zeta \in E - \{\infty\}} |z-\zeta| \exp[-b(z) - b(\zeta)] = m_0 e^{-b(z)}, \text{ if } z \text{ is finite and } \zeta = \infty, \\ \limsup\limits_{z \to \infty} m_0 e^{-b(z)} = 0, \quad \text{if} \quad z = \zeta = \infty. \end{cases}$$

By (1.1) we have $m \leqslant m_0 \leqslant M$, and

(1.1′) $$\omega(z, \zeta) \leqslant |z e^{-b(z)}| e^{-b(\zeta)} + |\zeta e^{-b(\zeta)}| e^{-b(z)} \leqslant 2M e^{-b_0} < \infty,$$

where $b_0 = \inf\limits_{z \in E} b(z)$.

Example 1. $E = \{z \mid |z| \geqslant 1\}$, $b(z) = \ln|z|$, $\omega(z, \zeta) = |z-\zeta| \dfrac{1}{|z\zeta|} = \left|\dfrac{1}{z} - \dfrac{1}{\zeta}\right|$.

Example 2. $E = \bar{C}$, $b(z) = \ln\sqrt{1+|z|^2}$, $\omega(z, \zeta) = \dfrac{|z-\zeta|}{\sqrt{1+|z|^2}\sqrt{1+|\zeta|^2}}$.



The function
$$\boldsymbol{b}(z) = \lim_{\delta \to 0} \{\inf_{|z'-z| < \delta, z, z' \in E} b(z')\} = \liminf_{z' \to z, z' \in E} b(z')$$
is lowersemicontinuous on $E$ and the function

(1.2′) $$\boldsymbol{\omega}(z, \zeta) = |z-\zeta| \exp[-\boldsymbol{b}(z) - \boldsymbol{b}(\zeta)], \quad z, \zeta \in E,$$

is uppersemicontinuous in $E \times E$.

Denote by
$$p^{(n)} = \{p_0, p_1, \ldots, p_n\}, \quad n = 1, 2, \ldots,$$
a system of $n+1$ points (distinct or not) of the set $E$ and put
$$V(p^{(n)}) = \prod_{0 \leqslant i < k \leqslant n} |p_i - p_k|,$$

(1.3) $$V(p^{(n)}, b) = \prod_{0 \leqslant i < k \leqslant n} \omega(p_i, p_k) = V(p^{(n)}) \exp\Big[-n \sum_{k=0}^{n} b(p_k)\Big],$$

(1.4)
$$\Delta^{(i)}(p^{(n)}, b) = \prod_{\substack{k=0 \\ (k \neq i)}}^{n} \omega(p_i, p_k) = \Big(\prod_{\substack{k=0 \\ (k \neq i)}}^{n} |p_i - p_k|\Big) \exp\Big[-nb(p_i) - \sum_{\substack{k=0 \\ (k \neq i)}}^{n} b(p_k)\Big].$$

In the sequel we shall always assume that the points of the system $p^{(n)}$ are numbered in such a way that

(1.5) $$\Delta^{(0)}(p^{(n)}, b) \leqslant \Delta^{(i)}(p^{(n)}, b), \quad i = 1, 2, \ldots, n.$$

Let $\{l_n\}$ be a sequence of real numbers such that

(1.6) $$l_n > 1, \quad n = 1, 2, \ldots; \quad \lim_{n \to \infty} l_n = 1.$$

We shall denote by

(1.7) $$\xi^{(n)} = \{\xi_0^{(n)}, \xi_1^{(n)}, \ldots, \xi_n^{(n)}\}, \quad \text{or shortly} \quad \xi^{(n)} = \{\xi_0, \xi_1, \ldots, \xi_n\},$$

a system of $n+1$ finite points of $E$ such that

(1.8) $$V_n(E, b) = \sup_{p^{(n)} \subset E} V(p^{(n)}, b) \leqslant l_n V(\xi^{(n)}, b).$$

In view of (1.1′) and (1.6) such a system certainly exists.

The function $\boldsymbol{\omega}(z, \zeta)$ being uppersemicontinuous on $E \times E$, there is a system $\eta^{(n)} \subset E$

(1.9) $$\eta^{(n)} = \{\eta_0^{(n)}, \eta_1^{(n)}, \ldots, \eta_n^{(n)}\}, \quad \text{or shortly} \quad \eta^{(n)} = \{\eta_0, \eta_1, \ldots, \eta_n\},$$

such that

(1.10) $$V(\eta^{(n)}, \boldsymbol{b}) = \max_{p^{(n)} \subset E} V(p^{(n)}, \boldsymbol{b}).$$

One can easily check that

(1.11) $$V(\eta^{(n)}, \boldsymbol{b}) = V_n(E, b), \quad n = 1, 2, \ldots$$

Indeed, let

$$p^{(n,\nu)} = \{p_{\nu 0}, p_{\nu 1}, \ldots, p_{\nu n}\}, \quad \nu = 1, 2, \ldots,$$

be a sequence of systems of $n+1$ points of $E$ such that

$$\eta_i^{(n)} = \lim_{\nu\to\infty} p_{\nu i} \quad \text{and} \quad \lim_{\nu\to\infty} b(p_{\nu i}) = \boldsymbol{b}(\eta_i^{(n)}), \quad i = 0, 1, \ldots, n.$$

Then, because of (1.3), we have

$$\lim_{\nu\to\infty} V(p^{(n,\nu)}, b) = V(\eta^{(n)}, \boldsymbol{b}).$$

From (1.8) we get

$$V(p^{(n,\nu)}, b) \leqslant V_n(E, b), \quad \nu = 1, 2, \ldots,$$

whence

$$V(\eta^{(n)}, \boldsymbol{b}) \leqslant V_n(E, b).$$

Due to the obvious inequality $\boldsymbol{b}(z) \leqslant b(z)$, $z \in E$, we have $\boldsymbol{\omega}(z, \zeta) \leqslant \omega(z, \zeta)$, whence

$$V_n(E, b) \leqslant V_n(E, \boldsymbol{b}) = V(\eta^{(n)}, \boldsymbol{b}).$$

This completes the proof of (1.11).

A system $\eta^{(n)}$, given by (1.9) and (1.10), will be called an *n-th extremal system of $E$ with respect to $b$*. For a fixed $n$ there may of course exist more than one system (1.9).

A system $\xi^{(n)}$, given by (1.7) and (1.8), will be called an *n-th extremal system of $E$ with respect to $b$ and $\{l_n\}$*.

If $E$ is closed and bounded and $b(z) \equiv 0$, then points of the system (1.9) are well-known Fekete's points of $E$. If $b(z)$ is continuous the points (1.9) were first considered by F. Leja [9], [17]. It is known [17] that there exists the limit

(1.12) $$v(E, b) = \lim_{n\to\infty} [V(\eta^{(n)}, \boldsymbol{b})]^{2/n(n+1)}$$

which is called the *ecart* of the set $E$ with respect to $b$. Since $V(\xi^{(n)}, b) \leqslant V_n(E, b) = V(\eta^{(n)}, \boldsymbol{b})$, $n = 1, 2, \ldots$, we have

(1.13) $$v(E, b) = \lim_{n\to\infty} [V(\xi^{(n)}, b)]^{2/n(n+1)} = \lim [V_n(E, b)]^{2/n(n+1)}.$$

If $E$ is closed and bounded and $b(z) \equiv 0$, then $v(E, 0) = d(E)$ is called the *transfinite diameter of $E$*. It is obvious that if $E$ is bounded then a necessary and sufficient condition that $v(E, b) > 0$ is that $d(E) > 0$.



Let

(1.14) $$\Delta_n = \sup_{p^{(n)} \subset E} \{\min_i \Delta^{(i)}(p^{(n)}, b)\}, \quad n = 1, 2, \ldots,$$

where $\Delta^{(i)}(p^{(n)}, b)$ is defined by (1.4).

By the method used in the case that $E$ is bounded and $b$ is continuous (see [17], [9]) one can prove that in our case the sequence $\{\sqrt[n]{\Delta_n}\}$ is also convergent and

(1.15) $$v(E, b) = \lim_{n\to\infty} \sqrt[n]{\Delta_n}.$$

**2. The extremal function** $\Phi(z, E, b)$. Starting from this section we shall always assume that the set $E$ is not finite. Given $n$ and an arbitrary system $\zeta^{(n)} = \{\zeta_0, \zeta_1, \ldots, \zeta_n\}$ of $n+1$ distinct finite points of $E$, we define the polynomials

(2.1) $$L^{(i)}(z, \zeta^{(n)}) = \prod_{\substack{k=0\\k\neq i}}^{n} \frac{z-\zeta_k}{\zeta_i-\zeta_k}, \quad i = 0, 1, \ldots, n,$$

(2.2) $$\Phi^{(i)}(z, \zeta^{(n)}, b) = L^{(i)}(z, \zeta^{(n)}) e^{nb(\zeta_i)}, \quad i = 0, 1, \ldots, n.$$

Next we define for every $n = 1, 2, \ldots$ the functions

(2.3) $$\Phi_n^{(1)}(z, E, b) = \max_{(i)} |\Phi^{(i)}(z, \xi^{(n)}, b)|,$$

(2.4) $$\Phi_n^{(2)}(z, E, b) = \sum_{i=0}^{n} |\Phi^{(i)}(z, \xi^{(n)}, b)|,$$

(2.5) $$\Phi_n^{(3)}(z, E, b) = \inf_{\zeta^{(n)} \subset E} \{\max_{(i)} |\Phi^{(i)}(z, \zeta^{(n)}, b)|\},$$

(2.6) $$\Phi_n^{(4)}(z, E, b) = \inf_{\zeta^{(n)} \subset E} \sum_{i=0}^{n} |\Phi^{(i)}(z, \zeta^{(n)}, b)|,$$

(2.7) $$\Phi_n^{(5)}(z, E, b) = |\Phi^{(0)}(z, \xi^{(n)}, b)|,$$

where $\xi^{(n)}$ is given by (1.7).

Let $E^* = E^*(b)$ denote the set of all the limit points of the triangular sequence (1.7).

THEOREM 2.1. *If $v(E, b) > 0$, then:*

1° *The sequences $\{\sqrt[n]{\Phi_n^{(i)}(z, E, b)}\}$, $i = 1, 2, 3, 4$, are convergent to the same limit $\Phi(z, E, b)$ at any finite point $z \in C$,*

(2.8) $$\Phi(z, E, b) = \lim_{n\to\infty} \sqrt[n]{\Phi_n^{(i)}(z, E, b)}, \quad i = 1, 2, 3, 4.$$

2° *At any finite point of the complement of* $E^*$ *the sequence* $\{\sqrt[n]{\Phi_n^{(5)}(z,E,b)}\}$ *converges to* $\Phi(z,E,b)$

$$\Phi(z,E,b)=\lim_{n\to\infty}\sqrt[n]{\Phi_n^{(5)}(z,E,b)},\qquad z\in CE^*, \tag{2.9}$$

*the convergence being uniform in a neighborhood of any finite point* $z\in CE^*$. *The function* $\operatorname{Log}\Phi(z,E,b)$ *is harmonic in* $CE^*$.

Proof. The proof given by Leja [9] for the case that $E$ is bounded and $b(z)$ is continuous, is also valid for the more general situation we are dealing with. Leja shows at first in his proof that the sequence $\{\sqrt[n]{\Phi_n^{(3)}}\}$ is convergent. He does not consider at all the sequence $\{\sqrt[n]{\Phi_n^{(1)}}\}$. The proof we are giving here is different from that of Leja and its advantage relies on the fact that it can be repeated if one wants to prove the theorem 2.1 for extremal sequences defined suitably in the space $C^n$ of $n$ complex variables (see [22]).

1° At first we shall prove that the sequence $\{\sqrt[n]{\Phi_n^{(1)}}\}$ is convergent for every $z\in C$. For this purpose let us observe that

$$|\Phi^{(i)}(z,\xi^{(n)},b)|\leqslant l_n e^{nb(z)},\qquad z\in E,\ i=0,1,\dots,n\ (z\neq\infty). \tag{2.10}$$

Indeed, if this inequality were not true, there would exist a finite point $z'\in E$ and a subindex $i'$, $0\leqslant i'\leqslant n$, such that

$$\Big(\prod_{\substack{k=0\\(k\neq i)}}^{n}|z'-\xi_k|\Big)e^{-nb(z')}>l_n\Big(\prod_{\substack{k=0\\(k\neq i')}}^{n}|\xi_{i'}-\xi_k|\Big)e^{-nb(\xi_{i'})},$$

whence by (1.3) we would have

$$V(\{\xi_0,\dots,\xi_{i-1},z',\xi_{i+1},\dots,\xi_n\})>l_nV(\xi^{(n)},b)\geqslant V_n(E,b).$$

This, however, contradicts the definition of $V_n(E,b)$.

Let $z$ be an arbitrary fixed point of $C$, let $n$ be an arbitrary fixed positive integer and let $m$ be an arbitrary integer greater than or equal to $n$. There exist two unique integers $k$ and $r$ such that $m=kn+r$ and $0\leqslant r<n$. By the Lagrange interpolation formula and because of (2.10) we have

$$|\Phi^{(i)}(z,\xi^{(n)},b)|^k\leqslant l_n^k\sum_{j=0}^{m}|\Phi^{(j)}(z,\xi^{(m)},b)|e^{-rb_0},\qquad i=0,1,\dots,n, \tag{2.11}$$

whence

$$(\sqrt[n]{\Phi_n^{(1)}})^{nk/m}\leqslant l_n^{k/m}(m+1)^{1/m}\sqrt[m]{\Phi_m^{(1)}}\cdot e^{-b_0r/m},$$

where $b_0=\inf_{z\in E}b(z)$. Our assumptions on $b(z)$ imply that $b_0$ is finite. Since



$nk/m\to 1$, $m/k\to n$, $b_0r/m\to 0$, as $k\to\infty$, we have

$$\sqrt[n]{\Phi_n^{(1)}}\leqslant\sqrt[n]{l_n}\liminf_{m\to\infty}\sqrt[m]{\Phi_m^{(1)}},\qquad n=1,2,\dots \tag{2.12}$$

So

$$\limsup\sqrt[n]{\Phi_n^{(1)}}\leqslant\liminf\sqrt[n]{\Phi_n^{(1)}}.$$

Therefore the sequence $\{\sqrt[n]{\Phi_n^{(1)}}\}$ is convergent to a limit $\Phi(z,E,b)$ (finite or not). Observe that as far we did not use the assumption that $v(E,b)>0$. To prove that the sequences $\{\Phi_n^{(i)}\}$, $i=2,3,4$, are convergent to $\Phi(z)=\Phi(z,E,b)$ it is enough to show that

$$\Phi_n^{(1)}\leqslant l_n\Phi_n^{(4)}\leqslant l_n\Phi_n^{(2)}\leqslant(n+1)^2l_n^2\Phi_n^{(3)}\leqslant(n+1)^2l_n^2\Phi_n^{(1)},\qquad n=1,2,\dots \tag{2.13}$$

Let $\zeta^{(n)}=\{\zeta_0,\zeta_1,\dots,\zeta_n\}$ be an arbitrary system of $n+1$ finite and distinct points of $E$. Then by the interpolation formula of Lagrange and owing to (2.10) we have

$$|\Phi^{(i)}(z,\xi^{(n)},b)|\leqslant l_n\sum_{k=0}^{n}|\Phi^{(k)}(z,\zeta^{(n)},b)|,\qquad i=0,1,\dots,n,$$

whence the inequalities $\Phi_n^{(1)}\leqslant l_n\Phi_n^{(4)}$ and $\Phi_n^{(2)}\leqslant(n+1)^2l_n\Phi_n^{(3)}$ follow. The inequalities $\Phi_n^{(4)}\leqslant\Phi_n^{(2)}$ and $\Phi_n^{(3)}\leqslant\Phi_n^{(1)}$ are a direct consequence of the definitions (2.3)-(2.6).

In order to prove that the function $\Phi(z)$ is finite at every point $z\in C$, let us observe that

$$|\Phi^{(i)}(z,\zeta^{(n)},b)|=\Big(\prod_{\substack{k=0\\(k\neq i)}}^{n}\frac{|z-\zeta_k|}{|e^{b(\zeta_k)}|}\Big)\frac{1}{\Delta^{(i)}(\zeta^{(n)},b)},\qquad i=0,1,\dots,n,$$

whence

$$\Phi_n^{(3)}(z,E,b)\leqslant R^n(z)/\Delta_n, \tag{2.14}$$

where

$$R(z)=\sup_{\zeta\in E}|ze^{-b(\zeta)}-\zeta e^{-b(\zeta)}|\leqslant|z|e^{-b_0}+M. \tag{2.15}$$

Therefore

$$\Phi(z,E,b)\leqslant R(z)/v(E,b)\leqslant\infty,\qquad z\in C. \tag{2.16}$$

2° To prove (2.9) we shall first show that for sufficiently large $n$

$$m(z)\Phi_n^{(1)}(z,E,b)\leqslant|\Phi_n^{(0)}(z,\xi^{(n)},b)|\leqslant\Phi_n^{(2)}(z,E,b),\qquad z\in CE^*, \tag{2.17}$$

where $m(z)>0$ depends only on $z$. In view of (2.4) and (2.7) the second inequality is obvious. In order to prove the first one, let us observe that

$$|\Phi^{(0)}(z,\xi^{(n)},b)|$$
$$=|\Phi^{(i)}(z,\xi^{(n)},b)|\frac{\Delta^{(i)}(\xi^{(n)},b)}{\Delta^{(0)}(\xi^{(n)},b)}\,\frac{|z-\xi_i|}{e^{b(\xi_i)}}\,\frac{e^{b(\xi_0)}}{|z-\xi_0|},\qquad i=0,1,\dots,n.$$

By (1.5), $\Delta^{(i)}(\xi^{(n)}, b) \geqslant \Delta^{(0)}(\xi^{(n)}, b)$, $i = 0, 1, \ldots, n$, and in virtue of (1.1) we have

$$|z-\xi_i|e^{-b(\xi_i)}/|z-\xi_0|e^{-b(\xi_0)} \geqslant \inf_{\zeta\in E^*}(|z-\zeta|e^{-b(\zeta)})/\sup_{\zeta\in E^*}(|z-\zeta|e^{-b(\zeta)}) = m(z) > 0$$

for $i = 0, 1, \ldots, n$ and $n$ sufficiently large. Hence and from (2.3) also the first inequality of (2.17) follows. Thus (2.9) is proved. To prove that the convergence in (2.9) is loco uniform and that the function $\operatorname{Log}\Phi(z)$ is harmonic at any finite point of $CE^*$ it is enough to observe that for every finite point $z \in CE^*$ there is a neighborhood $U$ and an integer $n_0$ such that for $n \geqslant n_0$ the functions $\operatorname{Log}\sqrt[n]{|\Phi^{(0)}(x, \xi^{(n)}, b)|}$ are harmonic and uniformly bounded in $U$.

LEMMA 2.1. *If $P(z)$ is a polynomial of degree less than or equal to $n$ and $|P(z)| \leqslant M \exp[nb(z)]$ for $z \in E$, then*

$$|P(z)| \leqslant M\Phi^n(z, E, b), \quad z \in C. \tag{2.18}$$

Proof. By the interpolation formula of Lagrange

$$|P(z)|^k \leqslant M^k \sum_{i=0}^{kn} |\Phi^{(i)}(z, \xi^{(kn)}, b)| = M^k \Phi_{kn}^{(2)}(z, E, b), \quad k = 1, 2, \ldots,$$

whence (2.18) follows by (2.8).

Let $E$ be a bounded closed set with $d(E) > 0$. Denote by $D(E)$ the unbounded component of $CE$. Then $\operatorname{Log}\Phi(z, E, 0)$ is a generalized Green's function of $D(E)$ with its logarithmic pole at $\infty$ (see [8], [2]). Therefore inequality (21.8) is a generalization of well known Bernstein-Walsh inequality (see [23]).

From (2.10) and lemma 2.1 we get the following

THEOREM 2.2. *The function $\Phi(z, E, b)$ is the least upper bound of all the functions $\sqrt[n]{|P_n(z)|}$, $n = 1, 2, \ldots$, where $P_n(z)$ denotes an arbitrary polynomial of degree $n$ such that $|P_n(z)| \leqslant \exp[nb(z)]$ for $z \in E$.*

COROLLARY 2.1. *$\Phi(z, E, \boldsymbol{b}) = \Phi(z, E, b)$, $z \in C$, where $\boldsymbol{b}(z) = \lim_{\delta\to 0}\{\inf_{|z-z'|<\delta} b(z')\}$ is the greatest lower semicontinuous minorant of $b(z)$.*

This implies that for constructing $\Phi(z, E, b)$ we may take extremal points of $E$ with respect to $\boldsymbol{b}$.

**3. Some fundamental properties of $\Phi(z, E, b)$.** From this section on we shall always assume that $v(E, b) > 0$ (except the contrary is clearly stated),

$$\Phi(z, E, b) \leqslant e^{b(z)}, \quad z \in E. \tag{3.1}$$

This inequality follows from (2.10) and theorem 2.2.

$$\Phi(z, E, b) \geqslant e^{b_0}, \quad z \in C; \quad b_0 = \inf_{z\in E} b(z). \tag{3.2}$$



Indeed,

$$|\Phi^{(i)}(z, \xi^{(n)}, b)| \geqslant |L^{(i)}(z, \xi^{(n)})|e^{nb_0}, \quad i = 0, 1, \ldots, n; \quad z \in C,$$

whence

$$\Phi_n^{(2)}(z, E, b) = \sum_{i=0}^{n} |\Phi^{(i)}(z, \xi^{(n)}, b)| \geqslant \sum_{i=0}^{n} |L^{(i)}(z, \xi^{(n)})|e^{nb_0} \geqslant e^{nb_0}$$

since

$$1 \equiv \sum_{i=0}^{n} L^{(i)}(z, \xi^{(n)}) \leqslant \sum_{i=0}^{n} |L^{(i)}(z, \xi^{(n)})|.$$

Therefore (3.2) follows from (2.8).

(3.3) *If $b$ and $b_1$ satisfy* (1.1) *and $b(z) \leqslant b_1(z)$ for $z \in E$, then $\Phi(z, E, b) \leqslant \Phi(z, E, b_1)$ for $z \in C$.*

(3.4) *If $b_1(z) = b(z) + c$, $c = \text{const}$, then $\Phi(z, E, b_1) = e^c\Phi(z, E, b)$.*

(3.5) *If $E_1 \subset E$, then $\Phi(z, E, b) \leqslant \Phi(z, E_1, b)$ for $z \in C$.*

The last three properties of $\Phi$ follow directly from (2.6) and (2.8).

(3.6) *The function $\Phi(z, E, b)$ is lower semicontinuous in $C$.*

Indeed, by (2.8) and (2.12)

$$\Phi(z, E, b) = \sup_{n=1,2,\ldots} \sqrt[n]{l_n^{-1}\Phi_n^{(1)}(z, E, b)} \quad \text{for} \quad z \in C.$$

Thus $\Phi$ is an upper bound of continuous functions $\sqrt[n]{\Phi_n^{(1)}}$, whence the result.

(3.7) *Let $a_i \geqslant 0$, $i = 1, \ldots, k$, and let $a = \sum_{i=1}^{k} a_i > 0$. If*

$$b(z) = \frac{1}{a}[a_1 b_1(z) + \ldots + a_k b_k(z)],$$

*then*

$$\prod_{i=1}^{k} \Phi^{a_i}(z, E, b_i) \leqslant \Phi^a(z, E, b), \quad z \in C.$$

Without loss of generality it is sufficient to prove the inequality for $k = 2$ and for rational $a_i$, $i = 1, 2$. Let $a_1 = p_1/q_1$, $a_2 = p_2/q_2$, where $p_1, q_1, p_2, q_2$ are positive integers. We may assume $q_1 = q_2 = q$. Put $p = p_1 + p_2$ and let $\xi^{(n,i)} = \{\xi_0^i, \xi_1^i, \ldots, \xi_n^i\}$, $i = 1, 2$, be $n$-th extremal systems of $E$ with respect to $b_1$ and $b_2$, respectively. Let $\xi^{(pn)} = \{\xi_0, \xi_1, \ldots, \xi_{pn}\}$ be a $pn$-th extremal system of $E$ with respect to

$$b(z) = \frac{1}{a}[a_1 b_1(z) + a_2 b_2(z)].$$

Given $z_0 \in C$, let $i_1$ and $i_2$ be such that

$$|\Phi^{(i_k)}(z_0, \xi^{(n,k)}, b_k)| = \Phi_n^{(1)}(z_0, E, b_k), \quad k = 1, 2.$$

By the Lagrange interpolation formula and owing to (2.10) we have

$$|[\Phi^{(i_1)}(z,\xi^{(n,1)},b_1)]^{p_1}[\Phi^{(i_2)}(z,\xi^{(n,2)},b_2)]^{p_2}|$$
$$\leqslant\sum_{j=0}^{pn} l_n^{pn}\exp\left(pn\,\frac{\frac{p_1}{q}b_1+\frac{p_2}{q}b_2}{p/q}\right)|L^{(j)}(z,\xi^{(pn)})|,$$

whence

$$[\Phi_n^{(1)}(z_0,E,b_1)]^{p_1}[\Phi_n^{(1)}(z_0,E,b_2)]^{p_2}\leqslant l_n^{pn}\Phi_{pn}^{(2)}(z_0,E,b).$$

Thus, by (2.8) and (1.6), $\Phi^{p_1/q}(z_0,E,b_1)\Phi^{p_2/q}(z_0,E,b_2)\leqslant\Phi^{p/q}(z_0,E,b)$. The proof is completed.

(3.8) *Let $e(z)$ satisfy* (1.1) *and let $v(E,e)>0$. Let $b(z)$ be a real function defined and bounded on $E$ and let $0<\lambda'\leqslant\lambda$. Then*

$$\left[\frac{\Phi(z,E,e+\lambda b)}{\Phi(z,E,e)}\right]^{1/\lambda}\leqslant\left[\frac{\Phi(z,E,e+\lambda' b)}{\Phi(z,E,e)}\right]^{1/\lambda'},\qquad z\in C.$$

Suppose at first $\lambda$ and $\lambda'$ are rational: $\lambda=p/q$, $\lambda'=p'/q'$. We have

$$e+\frac{p'}{q'}b=\frac{1}{q'p}\left[p'q\left(e+\frac{p}{q}b\right)+(q'p-p'q)e\right].$$

Therefore by (3.7)

$$\Phi^{p'q}\left(z,E,e+\frac{p}{q}b\right)\Phi^{q'p-p'q}(z,E,e)\leqslant\Phi^{q'p}\left(z,E,e+\frac{p'}{q'}b\right),$$

whence the desired inequality follows, if $\lambda$ and $\lambda'$ are rational. If $\lambda$ and $\lambda'$ are arbitrary, there are sequences of rational numbers $\{\lambda_n\}$ and $\{\lambda_n'\}$ such that $\lambda_n\searrow\lambda$ and $\lambda_n'\nearrow\lambda'$. Let $b_1(z)=b(z)+b_0$, where $b_0=\inf_{z\in E}b(z)$. By (3.3)

$$\Phi(z,E,e+\lambda_n'b_1)\leqslant\Phi(z,E,e+\lambda'b_1)\leqslant\Phi(z,E,e+\lambda b_1)\leqslant\Phi(z,E,e+\lambda_n b_1),$$

whence owing to the first part of our proof

$$\left[\frac{\Phi(z,E,e+\lambda b_1)}{\Phi(z,E,e)}\right]^{1/\lambda_n}\leqslant\left[\frac{\Phi(z,E,e+\lambda_n b_1)}{\Phi(z,E,e)}\right]^{1/\lambda_n}$$
$$\leqslant\left[\frac{\Phi(z,E,e+\lambda_n' b_1)}{\Phi(z,E,e)}\right]^{1/\lambda_n'}\leqslant\left[\frac{\Phi(z,E,e+\lambda' b_1)}{\Phi(z,E,e)}\right]^{1/\lambda_n'}.$$

These inequalities and (3.4) imply

$$\exp(\lambda/\lambda_n)\left[\frac{\Phi(z,E,e+\lambda b)}{\Phi(z,E,e)}\right]^{1/\lambda_n}\leqslant\exp(\lambda'/\lambda_n)\left[\frac{\Phi(z,E,e+\lambda b)}{\Phi(z,E,e)}\right]^{1/\lambda_n'},$$

whence the result follows.



(3.9) *Let $E$ be bounded and let $b_0=\inf_{z\in E}b(z)$, $B_0=\sup_{z\in E}b(z)$, $-\infty<b_0\leqslant B_0<\infty$. Then*

$$b_0\leqslant u(z,E,e,b)=\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\operatorname{Log}\frac{\Phi(z,E,e+\lambda b)}{\Phi(z,E,e)}\leqslant B_0,\qquad z\in C,$$

*and the function $u$ is harmonic in $CE$.*

The limit $u(z,E,e,b)$ exists because of proposition (3.8). The inequalities $b_0\leqslant u\leqslant B_0$, $z\in C$, follow from the inequalities $e^{\lambda b_0}\Phi(z,E,e)\leqslant\Phi(z,E,e+\lambda b)\leqslant\Phi(z,E,e)e^{\lambda B_0}$, which are consequences of (3.3) and (3.4). The function $u$ is harmonic in $CE$ because of 2° of theorem 2.1, of (3.8) and by the Harnack principle.

(3.10) *If $E$ is closed and bounded and $d(E)>0$, then $\operatorname{Log}\Phi(z,E,b)$ is harmonic outside of $E^*(b)$ and it has a pole of order one at $\infty$, i. e.*

$$\lim_{z\to\infty}\frac{\Phi(z,E,b)}{|z|}=\frac{1}{\varrho},\qquad\varrho=\varrho(E,b)>0.$$

*Moreover,*

$$\varrho=\lim_{n\to\infty}\left[\sqrt[n]{|(\xi_0-\xi_1)\dots(\xi_0-\xi_n)|}\,e^{-b(\xi_0)}\right].$$

Indeed, the functions $\operatorname{Log}[|\Phi^{(0)}(z,\xi^{(n)},b)|^{1/n}/|z|]$, $n=1,2,\dots$, are harmonic and uniformly bounded in a neighborhood of $\infty$; the result therefore follows from (2.9).

Property $L$. Let $E_r(z_0)=\{z\,|\,z\in E,|z-z_0|<r\}$, $r>0$. We say *$E$ has the property $L$ at $z_0\in E$*, if for arbitrary two real numbers $\varepsilon>0$ and $r>0$ there exist two numbers $\delta>0$ and $N>0$ such that every polynomial $P_n(z)$ of degree less than or equal to $n$ satisfying the inequality

$$|P_n(z)|\leqslant M,\qquad z\in E_r(z_0),$$

satisfies also the inequality

$$|P_n(z)|\leqslant M(1+\varepsilon)^n,\quad\text{if}\quad|z-z_0|<\delta\ \text{and}\ n\geqslant N.$$

It is known [7] that every continuum (not reduced to a single point) has the property $L$ at every its point. It is also known [11] that if $E$ is a boundary of a domain $D(E)$ containing point $\infty$ in its interior, then $E$ has the property $L$ at $z_0\in E$ if and only if $z_0$ is regular with respect to the Dirichlet problem for $D(E)$. We shall write $E\subset L$, if and only if $E$ has the property $L$ at every its point.

Let $\zeta^{(n)}=\{\zeta_0^{(n)},\zeta_1^{(n)},\dots,\zeta_n^{(n)}\}$, $n=1,2,\dots$, be an arbitrary triangular sequence of finite points of $E$ such that $\zeta_i^{(n)}\neq\zeta_j^{(n)}$ for $i\neq j$, $n=1,2,\dots$ Let

$$L^{(j)}(z,\zeta^{(n)})=\prod_{\substack{k=0\\(k\neq j)}}^{n}(z-\zeta_k^{(n)})/(\zeta_j^{(n)}-\zeta_k^{(n)}),\qquad j=0,1,\dots,n,$$

and let

$$M(z_0, r, \zeta^{(n)}) = \max_{(j')} |L^{(j')}(z, \zeta^{(n)})|,$$

where $j'$ denotes an arbitrary integer such that $|\zeta_{j'}^{(n)} - z_0| < r$. If no point of $\zeta^{(n)}$ lies in $|z - z_0| < r$, we put $M(z_0, r, \zeta^{(n)}) = 0$. The following lemma has been proved in [18]:

LEMMA 3.1. *If $z_0$ is a finite limit point of the triangular sequence $\{\zeta_j^{(n)}\}$, $j = 0, 1, \ldots, n$, $n = 1, 2, \ldots$, then for every $r > 0$ we have*

$$\limsup_{n\to\infty} \sqrt[n]{M(z_0, r, \zeta^{(n)})} > 1. \tag{3.11}$$

We shall now prove the following property of $\Phi(z, E, b)$:

(3.12) *If $b(z)$ is lower semicontinuous at $z_0 \in E^*(b)$, then*

$$\Phi(z_0, E, b) = \exp b(z_0).$$

In fact, by the lower semicontinuity of $b(z)$ for every $\varepsilon > 0$ there is $r > 0$ such that $b(z) > b(z_0) - \varepsilon$ for $z \in E_r(z_0)$. If a point $\xi_j^{(n)}$ of the extremal system (1.7) belongs to $E_r(z_0)$, we have by (2.2)

$$|\Phi^{(j)}(z, \xi^{(n)}, b)| \geqslant |L^{(j)}(z, \xi^{(n)})| \exp\{n[b(z_0) - \varepsilon]\}, \quad z \in C,$$

whence

$$\Phi_n^{(2)}(z_0, E, b) \geqslant M(z_0, r, \xi^{(n)}) \exp\{n[b(z_0) - \varepsilon]\}.$$

Therefore, by (3.11) and (2.8), $\Phi(z_0, E, b) \geqslant \exp[b(z_0) - \varepsilon]$, whence because of the arbitrariness of $\varepsilon > 0$, we have $\Phi(z_0, E, b) \geqslant \exp b(z_0)$. The result follows now from (3.1).

(3.13) *If $E$ has the property $L$ at $z_0 \in E$, $b(z)$ is upper semicontinuous at $z_0$ and $\Phi(z_0, E, b) = \exp b(z_0)$, then $\Phi(z)$ is continuous at $z_0$.*

Since, by (3.6), $\Phi$ is lower semicontinuous, it is sufficient to prove that $\Phi$ is upper semicontinuous at $z_0$. Let $\varepsilon > 0$; by the upper semicontinuity of $b(z)$ at $z_0$ there is $r$ such that $b(z) < b(z_0) + \varepsilon$, $z \in E_r(z_0)$, whence and from (2.10)

$$\left| \frac{1}{l_n} \Phi^{(i)}(z, \xi^{(n)}, b) e^{-n[b(z_0)+\varepsilon]} \right| \leqslant 1, \quad z \in E_r(z_0).$$

By the property $L$ of $E$ at $z_0$ there are $\delta > 0$ and $N > 0$ such that

$$\left| \frac{1}{l_n} \Phi^{(i)}(z, \xi^{(n)}, b) e^{-n[b(z_0)+\varepsilon]} \right| \leqslant (1+\varepsilon)^n,$$

if $|z - z_0| < \delta$ and $n \geqslant N$. Thus

$$\Phi_n^{(1)}(z, E, b) \leqslant l_n (1+\varepsilon)^n \exp\{n[b(z_0) + \varepsilon]\}, \quad |z - z_0| < \delta, \ n \geqslant N,$$



whence

$$\Phi(z, E, b) \leqslant (1+\varepsilon) e^{\varepsilon} \Phi(z_0, E, b), \quad |z - z_0| < \delta,$$

which gives the result in view of the arbitrariness of $\varepsilon > 0$.

Since $\operatorname{Log} \Phi(z)$ is harmonic outside of $E^*(b)$, (3.12) and (3.13) imply

(3.14) *If $E \subset L$ and $b(z)$ is continuous in $E$, then $\Phi(z, E, b)$ is continuous in $C$ and $\Phi(z) = \exp b(z)$ for $z \in E^*(b)$.*

COROLLARY 3.1. *If $E \subset L$ is bounded and $b(z) = 0$, then* $\operatorname{Log} \Phi(z, E, b)$ *is the Green function of $D(E)$ with its logarithmic pole at $\infty$*, [8].

All the properties of $\Phi$, except 3.7, 3.8 and 3.9, proved in this section, were found earlier by Leja (in the case of bounded $E$ and continuous $b(z)$; see the quoted papers).

**4. A generalization of Tchebycheff polynomials.** Throughout this section we shall assume that $E$ is closed and bounded and that $b(z)$ is a real function defined and lower semicontinuous in $E$. Let $c^{(n)} = \{c_1, \ldots, c_n\}$ be an arbitrary system of $n$ finite points. The absolute value of

$$W(z, c^{(n)}, b) = (z - c_1) \ldots (z - c^{(n)}) \exp[-nb(z)], \quad z \in E, \tag{4.1}$$

is an upper semicontinuous function in $E$ and therefore it takes its maximum at a point of $E$. Similarly as in the case of $b(z) \equiv 0$ one can prove that for every $n = 1, 2, \ldots$ there is a system $\dot{c}^{(n)} = \{\dot{c}_1, \ldots, \dot{c}_n\} \subset C$ such that

$$\dot{\mu}_n = \min_{c^{(n)} \subset C} \{\max_{z\in C} |W(z, c^{(n)}, b)|\} = \max_{z\in E} |W(z, \dot{c}^{(n)}, b)|, \tag{4.2}$$

and, moreover, the points of $\dot{c}^{(n)}$ lie in the convex envelope of $E$ (see [10], [23]). One can also show that $|W(z, \dot{c}^{(n)}, b)|$ takes its maximum on $E$ at least at $n+1$ points of $E$, whence it follows that the function $W(z, \dot{c}^{(n)}, b)$ (satisfying (4.2)) is unique.

The polynomial $\tilde{T}_n(z) \equiv \tilde{T}_n(z, E, b) = \exp[nb(z)] W(z, \dot{c}^{(n)}, b)$, $n = 1, 2, \ldots$, will be called the *n-th Tchebycheff polynomial of $E$ with respect to $b(z)$*. The polynomial $\tilde{T}_n(z, E, 0)$ is, of course, the $n$-th Tchebycheff polynomial of $E$ in the ordinary sense.

We shall also consider the polynomial $T_n(z)$ defined by

$$\mu_n = \min_{c^{(n)} \subset E} \{\max_{z\in E} |W(z, c^{(n)}, b)|\} = \max_{z\in E} |T_n(z) \exp[-nb(z)]|. \tag{4.3}$$

In (4.3) the minimum is taken with respect to $c^{(n)} \subset E$, while in (4.2) it is taken with respect to all systems $c^{(n)} \subset C$. The polynomial $T_n(z)$ need not be unique.

It follows directly from the definition of $\dot{\mu}_n$ and $\mu_n$ that $\dot{\mu}_{m+n} \leqslant \dot{\mu}_m\dot{\mu}_n$ and $\mu_{m+n} \leqslant \mu_m\mu_n$, $m, n = 1, 2, \ldots$ This implies the existence of the limits

$$\dot{\mu}(E, b) = \lim_{n\to\infty} \sqrt[n]{\dot{\mu}_n}, \tag{4.4}$$

$$\mu(E, b) = \lim_{n\to\infty} \sqrt[n]{\mu_n}. \tag{4.5}$$

It is obvious that

$$\dot{\mu}(E, b) \leqslant \mu(E, b). \tag{4.6}$$

THEOREM 4.1. *If $E$ is closed and bounded and $d(E) > 0$, then the sequence $\{\sqrt[n]{|T_n(z)|/\mu_n}\}$ is convergent at any point $z \in D(E)$ ($D(E)$ is an unbounded component of $CE$) and*

$$\lim_{n\to\infty} \sqrt[n]{|T_n(z)|/\mu_n} = \Phi(z, E, b), \quad z \in D(E), \tag{4.7}$$

$$\mu(E, b) = \varrho(E, b), \quad \varrho^{-1}(E, b) = \lim_{z\to\infty} [\Phi(z, E, b)/|z|]. \tag{4.8}$$

Proof. By (1.10) we have

$$\max_{z\in E} |(z-\eta_1)\ldots(z-\eta_n)\exp[-nb(z)]| = |(\eta_0-\eta_1)\ldots(\eta_0-\eta_n)\exp[-nb(\eta_0)]|, \tag{4.9}$$

whence and from (4.3)

$$\mu_n \leqslant |(\eta_0-\eta_1)\ldots(\eta_0-\eta_n)|\exp[-nb(\eta_0)], \quad n = 1, 2, \ldots,$$

and, by (3.10),

$$\mu(E, b) \leqslant \varrho(E, b). \tag{4.10}$$

From (4.3) we have $|T_n(z)/\mu_n| \leqslant \exp[nb(z)]$, $z \in E$, and therefore by (2.18)

$$|T_n(z)/\mu_n|^{1/n} \leqslant \Phi(z, E, b), \quad n = 1, 2, \ldots, \tag{4.11}$$

whence

$$(\sqrt[n]{\mu_n})^{-1} = \lim_{z\to\infty} \frac{1}{|z|} |T_n(z)/\mu_n|^{1/n} \leqslant \lim_{z\to\infty} [\Phi(z, E, b)/|z|] = \varrho^{-1}(E, b).$$

Thus $\varrho(E, b) \leqslant \mu(E, b)$ and this together with (4.10) implies equation (4.8).

To prove (4.7) consider functions

$$R_n(z) = \operatorname{Log}\Phi(z, E, b) - \frac{1}{n}\operatorname{Log}|T_n(z)/\mu_n|, \quad n = 1, 2, \ldots,$$



that are harmonic in $D(E)$ and uniformly bounded in every compact subset of $D(E)$. Owing to (4.11) and (4.8) we have $R_n(z) \geqslant 0$, $z \in D(E)$, and $\lim R_n(\infty) = \lim[\varrho^{-1} - \sqrt[n]{\mu_n}] = 0$, respectively. Therefore, by the principle of Harnack, we have $R_n(z) \to 0$ uniformly in every compact subset of $D(E)$. Thus (4.7) follows.

By the same argument one can show that $\dot{\mu}(E, b) = \varrho(E, b)$ and that in the complement of the convex hull of $E$ the sequence $\{\sqrt[n]{|T_n(z)|/\dot{\mu}_n}\}$ is convergent to $\Phi(z, E, b)$ (see [10], [23]).

To end this section we shall prove that the sequence $\{\sqrt[n]{\gamma_n}\}$, given by

$$\gamma_n = \sup_{\zeta^{(n)}\subset E} \{\min_{(i)} |(\zeta_i-\zeta_0)\ldots(\zeta_i-\zeta_{i-1})(\zeta_i-\zeta_{i+1})\ldots(\zeta_i-\zeta_n)|\, e^{-nb(\zeta_i)}\} \tag{4.12}$$

is convergent to $\varrho(E, b)$:

$$\lim_{n\to\infty} \sqrt[n]{\gamma_n} = \varrho(E, b). \tag{4.13}$$

Indeed, if

$$r(z) = \inf_{\zeta\in E} |z-\zeta|, \quad R(z) = \max_{\zeta\in E} |z-\zeta|,$$

then

$$r(z)^n/\gamma_n \leqslant \inf_{\zeta^{(n)}\subset E} \{\max_{(i)} |\Phi^{(i)}(z, \zeta^{(n)}, b)|\} \leqslant R(z)^n/\gamma_n,$$

whence

$$r(z)/[|z|\sqrt[n]{\gamma_n}] \leqslant \sqrt[n]{\Phi_n^{(3)}(z, E, b)}/|z| \leqslant R(z)/[|z|\sqrt[n]{\gamma_n}], \quad n = 1, 2, \ldots$$

Since $\lim_{z\to\infty} r(z)/|z| = \lim R(z)/|z| = 1$, this implies that

$$1/\liminf_{n\to\infty} \sqrt[n]{\gamma_n} \leqslant \varrho^{-1}(E, b) \leqslant 1/\limsup_{n\to\infty} \sqrt[n]{\gamma_n},$$

whence (4.13) follows.

**5. A necessary and sufficient condition that $\Phi(z, E, b) = \exp b(z)$ for $z \in E$.** Having in view some applications of $\Phi(Z, E, b)$ to the construction of solution of the Dirichlet boundary value problem, it is important to know what are the functions $b(z)$ such that $\Phi(z, E, b) = \exp b(z)$ for $z \in E$. At first we shall state the following sufficient condition:

THEOREM 5.1. *Let $E$ be closed and bounded and let $d(E) > 0$. Let $b(z)$ be a real function defined and continuous in $E$. A sufficient condition that $\Phi(z, E, b) = \exp b(z)$, $z \in E$, is that there exist function $V(z) = V(z, E, b)$ such that:*

(i) $V(z) = b(z)$, $z \in E$;

(ii) *$V(z)$ is continuous and subharmonic in $C$;*

(iii) *$\lim_{z\to\infty} [V(z) - \operatorname{Log}|z|]$ exists and is finite.*

Proof. By (3.10) and (3.12) $\operatorname{Log}\Phi(z)$, where $\Phi(z) = \Phi(z, E, b)$, is harmonic outside of $E^*(b)$, $\lim_{z\to\infty}\operatorname{Log}[\Phi(z)/|z|]$ is finite and $\Phi(z) = e^{b(z)}$ for $z \in E^*(b)$. Therefore the function $R(z) = V(z) - \operatorname{Log}\Phi(z)$ is subharmonic in $CE^*$ and $R(z) = 0$ for $z \in E^*$, whence by the maximum principle $R(z) \leqslant 0$ for $z \in CE^*$. Thus $R(z) \leqslant 0$ for $z \in E$, i. e. $V(z) = b(z) \leqslant \operatorname{Log}\Phi(z)$, $z \in E$, whence by (3.1) the result follows.

We shall now show that the condition the sufficiency of which we have just proved is also necessary, if $E$ satisfies the property $L$ at each its point.

THEOREM 5.2. *If $E \subset L$ and $\Phi(z, E, b) = \exp b(z)$, $z \in E$, then the function $V(z) = \operatorname{Log}\Phi(z, E, b)$ satisfies conditions* (i), (ii), (iii) *of theorem* 5.1.

Proof. Condition (i) is satisfied by the assumption. In view of (3.13) and (3.10), $V(z)$ is continuous in $C$ and satisfies (iii). It therefore remains to show that $V(z) = \operatorname{Log}\Phi(z, E, b)$ is subharmonic in $C$. Let $K = \{z \mid |z-z_0| < r\}$ and $K' = \{z \mid |z-z_0| = r\}$. Let $v(z)$ be a harmonic function in $K$, continuous in $K+K'$ such that

$$\operatorname{Log}\Phi(z, E, b) = v(z), \quad z \in K'.$$

We shall show that

$$v(z) \geqslant \operatorname{Log}\Phi(z, E, b), \quad z \in K.$$

The function

$$\omega(z, \zeta) = |z-\zeta|/[\Phi(z, E, b)\Phi(\zeta, E, b)], \quad z, \zeta \in \bar{C},$$

is continuous in $\bar{C}\times\bar{C}$ and it is an absolute value of a holomorphic function (of one variable when the other is fixed) in a neighborhood of an arbitrary point of $CE^*$. This implies by the maximum principle that all extremal points of $\bar{C}$ with respect to $\varphi(z) = \operatorname{Log}\Phi(z, E, b)$ lie in $E^*(b)$, whence $E^*(b) = \bar{C}^*(\varphi)$, and consequently $\operatorname{Log}\Phi(z, \bar{C}, \varphi) = \operatorname{Log}\Phi(z, E, b)$, $z \in C$. By (3.5) we have

$$\operatorname{Log}\Phi(z, K', \varphi) \geqslant \log\Phi(z, \bar{C}, \varphi).$$

The function $\operatorname{Log}\Phi(z, K', \varphi)$ is by (3.14) and (3.10) continuous on $\bar{K}$, harmonic in $K$ and (because of (3.1))

$$\operatorname{Log}\Phi(z, K', \varphi) \leqslant \varphi(z) = \operatorname{Log}\Phi(z, E, b) = v(z), \quad z \in K',$$

whence $\operatorname{Log}\Phi(z, K', \varphi) \leqslant v(z)$, $z \in K$. Therefore $\operatorname{Log}\Phi(z, E, b) \leqslant v(z)$, $z \in K$. The proof is completed.

COROLLARY 5.1. *When proving that $\operatorname{Log}\Phi(z, E, b)$ is subharmonic, we did not use the assumption that $\Phi(z, E, b) = \exp(b(z))$, $z \in E$. Therefore: If $E \subset L$ and $b(z)$ is continuous, then $\operatorname{Log}\Phi(z, E, b)$ is a continuous subharmonic function in $C$.*



COROLLARY 5.2. *If the function $\Phi(z, E, b)$ is continuous in $C$, then $\operatorname{Log}\Phi(z, E, b)$ is an upper envelope of all functions $U(z)$ that satisfy* (ii) *and* (iii) *of Theorem 5.2 and the inequality $U(z) \leqslant b(z)$ for $z \in E$.*

Indeed, $R(z) = \operatorname{Log}\Phi(z, E, b) - U(z)$ is superharmonic in $CE^*(b)$ and $R(z) \geqslant 0$ for $z \in E^*(b)$, therefore $R(z) \geqslant 0$ everywhere, whence $U(z) \leqslant \operatorname{Log}\Phi(z, E, b)$ for $z \in C$. This completes the proof of the corollary.

Remark 5.1. By more sophisticated tools one can prove that the function

$$V(z) = \limsup_{z'\to z}\operatorname{Log}\Phi(z', E, b)$$

is subharmonic in $C$ for an arbitrary closed and bounded set $E$ with $d(E) > 0$ and for arbitrary real and bounded $b(z)$. This is a consequence of Theorem 2.2 and of the general theorem which states that an upper envelope of a family of loco uniformly bounded subharmonic functions is a subharmonic function (see [19]).

**6. Some families of functions $b(z)$ such that $\Phi(z, E, b) = \exp b(z)$, $z \in E$.**

(a) We shall write $b(z) \in R(E)$, if $\Phi(z, E, b) = \exp b(z)$, $z \in E$.

(6.1) *Let $T$ be a subfamily of $R(E)$. Then*

$$\tilde{b}(z) = \sup_{b\in T} b(z)$$

*belongs to $R(E)$.*

Indeed, since $\tilde{b}(z) \geqslant b(z)$, $z \in E$, $b \in T$, we have $\Phi(z, E, \tilde{b}) \geqslant \Phi(z, E, b)$. Hence, owing to (3.1) we have $\exp b(z) = \Phi(z, E, b) \leqslant \Phi(z, E, \tilde{b}) \leqslant \exp\tilde{b}(z)$, $z \in E$, $b \in T$, and thus (6.1) follows.

(6.2) *If $e(z)$, $e(z)+b(z) \in R(E)$ and $0 \leqslant \lambda \leqslant 1$, then $e+\lambda b \in R(E)$.*

Indeed, by proposition (3.8) we have $\Phi(z, E, e+b)/\Phi(z, E, e) \leqslant [\Phi(z, E, e+\lambda b)/\Phi(z, E, e)]^{1/\lambda}$, $z \in E$, whence by (3.1)

$$\Phi(z, E, e+\lambda b) = \Phi(z, E, e)\exp\lambda b(z) = \exp[e(z)+\lambda b(z)], \quad z \in E,$$

q. e. d.

(6.3) *If $b_1, \ldots, b_k \in R(E)$ and $a_1, a_2, \ldots, a_k$ are non-negative real numbers such that $a = \sum_{i=1}^{k} a_i > 0$, then*

$$b = \frac{1}{a}\sum_{i=1}^{k} a_i b_i \in R(E),$$

*and, moreover,*

$$\prod_{i=1}^{k}\Phi^{a_i}(z, E, b_i) = \Phi^{a}(z, E, b), \quad z \in C.$$

Let

$$\varphi(z) = \frac{1}{a}\operatorname{Log}\prod_{i=1}^{k}\Phi^{a_i}(z, E, b_i), \quad z\in C.$$

By assumption we have $\varphi(z) = b(z)$ for $z\in E$. The function $|z-\zeta|\times\times\exp[-\varphi(z)-\varphi(\zeta)]$ is by (3.10) the absolute value of an analytic function at any point $z\notin E$. Therefore all extremal points of the closed plane $\overline{C}$ with respect to $\varphi(z)$ are contained in $E$, whence $\Phi(z, E, b) = \Phi z, \overline{C}, \varphi)$. Since, by (3.1), $\Phi(z, \overline{C}, \varphi)\leqslant\exp\varphi(z)$, $z\in\overline{C}$, we have, in virtue of (3.7),

$$\Phi(z, E, b)\equiv\exp\varphi(z) = \Big[\prod_{i=1}^{k}\Phi^{a_i}(z, E, b_i)\Big]^{1/a}.$$

(6.4) *If $F\subset E$ and $b\in R(E)$, then $b\in R(F)$.*

This follows from (3.1) and (3.5).

Let $c(r)$ be a convex function defined for $r\geqslant 0$ such that $c'(r)$ is positive and continuous. Let $w(z)$ be a function defined and continuous in $C$ such that $\operatorname{Log}w(z)$ is subharmonic in $C$ and the limit

$$\lim_{z\to\infty}[w(z)/|z|^k] = \varkappa$$

exists and is positive (as an example of $w(z)$ we can take the absolute value of an arbitrary polynomial of positive degree, and for $c(r)$ we may take $\exp r$ or $r^a$, $a\geqslant 1$).

THEOREM 6.1. *Under the above assumptions, if $b(z) = c[w(z)]$, then there is $\lambda_0 > 0$ such that $\lambda b\in R(E)$ for $0\leqslant\lambda\leqslant\lambda_0$.*

**Proof.** Let $r_0$ be so large that $E$ is contained in $F = \{z\mid w(z)\leqslant r_0\}$. We shall now prove that $\lambda b\in R(F)$, if $0\leqslant\lambda<\lambda_0 = 1/[r_0kc'(r_0)]$ and the result will follow from (6.4). Let

$$V(z) = \begin{cases}\lambda c(r_0)+\dfrac{1}{k}\operatorname{Log}\dfrac{w(z)}{r_0}, & \text{if } z\in CF,\\ \lambda c[w(z)], & \text{if } z\in F.\end{cases}$$

In virtue of theorem 5.1 it is enough to show that $V(z)$ is a continuous subharmonic function in $C$ such that $\operatorname{Lim}[V(z)-\operatorname{Log}|z|]$ exists and is finite. The limit exists and is finte by the definition of $V(z)$. It is also obvious that $V(z)$ is continuous in $C$ and subharmonic at every finite point which does not belong to the boundary $F'$ of $F$. We shall now prove that it is also subharmonic at every point $z_0\in F'$. Let $C_\varrho = \{z|\,|z-z_0| = \varrho\}$, $\varrho>0$. The theorem will be proved, if we show that for sufficiently small $\varrho$

(i) $$V(z_0)\leqslant\frac{1}{2\pi}\int_0^{2\pi}V(z_0+\varrho e^{it})\,dt.$$



The function

$$g(r) = \lambda c(r_0)+\frac{1}{k}\operatorname{Log}\frac{r}{r_0}-\lambda c(r)$$

is defined and continuously differentiable for $r>0$. Since $g(r_0) = 0$ and $g'(r_0) = (\lambda_0-\lambda)c'(r_0)>0$, we have $g(r)>0$ for $r_0<r<r_0+\delta$, provided $\delta>0$ is sufficiently small. Therefore

$$g[w(z)] = \lambda c(r_0)+\frac{1}{k}\operatorname{Log}\frac{w(z)}{r_0}-\lambda c[w(z)]>0$$

in $G = \{z\mid r_0<w(z)<r_0+\delta\}$, whence $V(z)>\lambda c[w(z)]$ in $G$. If $\varrho>0$ is so small that $C_\varrho\subset\{z\mid w(z)<r_0+\delta\}$, then

(ii) $$\frac{1}{2\pi}\int_0^{2\pi}\lambda c[w(z_0+\varrho e^{it})]\,dt\leqslant\frac{1}{2\pi}\int_0^{2\pi}V(z_0+\varrho e^{it})\,dt.$$

The function $c[w(z)]$ is subharmonic in $C$ (see [19]), therefore

$$V(z_0) = \lambda c[w(z_0)]\leqslant\frac{1}{2\pi}\int_0^{2\pi}\lambda c[w(z_0+\varrho e^{it})]\,dt,$$

whence (i) follows because of (ii). The proof is completed.

COROLLARY 6.1. *Let $c(r) = r$, and let $w(z)$ be an absolute value of a polynomial of degree $k\geqslant 1$. Let $E$ be closed and bounded with $d(E)>0$. In virtue of the theorem, $\operatorname{Log}\Phi(z, E, \lambda w) = \lambda w(z)$ for $z\in E$ and $0<\lambda\leqslant\lambda_0$, $\lambda_0$ being sufficiently small. The function $w(z)$ is strictly subharmonic. Therefore every interior point of $E$ belongs to $E^*(b)$, $b(z) = \lambda w(z)$, because otherwise (in view of (3.10)), $\operatorname{Log}\Phi(z, E, b) = \lambda w(z)$ would be harmonic at $z_0$.*

COROLLARY 6.2. *If $E\in L$, $b(z)$ is continuous in $E$ and $b\in R(E)$, then $[\exp b(z)]\in R(E)$.*

This follows by putting $w(z) = \operatorname{Log}\Phi(z, E, b)$ and applying Theorem 6.1 with $c(r) = \exp r$.

(b) Let $E$ be a bounded closed set such that $d(E)>0$. Let $F$ be a continuum such that $F\subset E$ and $E-F$ is closed. Let $p(z)$ be a real function defined in a neighborhood of $E$ such that $\Phi(z, E, p) = \exp p(z)$, $z\in F$. Assume there is a neighborhood $U$ of $F$ such that $\exp[-p(z)]\Phi(z, E, p)>1$ for $z\in U-F$ and $p(z)$ is continuous and subharmonic in $U$. Then the following lemma holds:

LEMMA 6.1. *If $q(z)$ is an arbitrary real function defined and bounded on $E$ such that $q(z) = c = \text{const}$ for $z\in F$, then there is a real number $\lambda_0>0$ such that*

(6.6) $$\Phi(z, E, p+\lambda q) = \exp[p(z)+\lambda q(z)] \quad \textit{for} \quad z\in F,\ |\lambda|\leqslant\lambda_0.$$

Proof. At first we shall prove that (6.6) holds for $0 \leqslant \lambda \leqslant \lambda_0$, $\lambda_0$ being sufficiently small. Let $q_0 = \min_{z \in E} q(z)$. Then, by (3.3) and (3.4),

$$\Phi(z, E, p+\lambda q)\exp[-\lambda q_0] \geqslant \Phi(z, E, p), \qquad z \in C.$$

If $\mu > 0$ is sufficiently small, then the set $\{z \mid \Phi(z, E, p)e^{-p(z)} < e^{\mu}\}$ contains a component $\Delta$ such that $F \subset \Delta \subset U$. The function

$$H(z) = \operatorname{Log}\Phi(z, E, p+\lambda q) - p(z) - \lambda q_0$$

is superharmonic in $\Delta - F^*$, $F^* = E^*(p+\lambda q) \cap F$. On the boundary of $\Delta$ we have $H(z) \geqslant \mu$ and on $F^*$ we have $H(z) = \lambda(c-q_0)$. Choose $\lambda_0 > 0$ such that $\mu > \lambda_0(c-q_0)$. We claim that $H(z) = p(z)+\lambda q(z)-\lambda q_0 = p(z)+ +\lambda(c-q_0)$ for $0 \leqslant \lambda \leqslant \lambda_0$ and $z \in F$. Indeed, the function $H(z)$ is superharmonic in $\Delta - F^*$ and $H(z) \geqslant \mu$, if $z$ belongs to the boundary of $\Delta$, while $H(z) = \lambda(c-q_0)$, if $z \in F^*$. Therefore $H(z) \geqslant \min[\mu, \lambda(c-q_0)] = \lambda(c-q_0)$ on the boundary of $\Delta - F^*$. By the minimum principle for superharmonic functions we have $H(z) \geqslant \lambda(c-q_0)$ in $\Delta - F^*$, whence $\Phi(z, E, p+\lambda q) \geqslant \exp[\lambda c + p(z)]$ for $z \in F$ and $0 \leqslant \lambda \leqslant \lambda_0$. Since, by (3.1), $\Phi(z, E, p+\lambda q) \leqslant \exp[p(z)+\lambda q(z)]$, we infer that (6.6) holds for $0 \leqslant \lambda \leqslant \lambda_0$. To prove (6.6) for $-\lambda_0 \leqslant \lambda \leqslant 0$ it is enough to consider $-q(z)$ instead of $q(z)$.

COROLLARY 6.3 (see [4], [15]). *Let $E = E_1 \cup E_2 \cup \ldots \cup E_n$, where $E_i$, $i = 1, 2, \ldots, n$, are continua such that $E_i \cap E_j = \emptyset$ $(i \neq j)$. Let for any $i = 1, 2, \ldots, n$ the sum*

$$F_i = \bigcup_{\substack{k=1 \\ (k \neq i)}}^{n} E_k$$

*be contained in $D(E_i)$ $\big(D(E_i)$ denotes the unbounded component of $CE_i\big)$ and let $q(z) = c_i = \mathrm{const}$ for $z \in E_i$, $i = 1, 2, \ldots, n$. Then there is $\lambda_0 > 0$ such that $\Phi(z, E, \lambda q) = \exp[\lambda q(z)]$ for $z \in E$ and $|\lambda| \leqslant \lambda_0$.*

Remark 6.1. The assumption that $E_i$ $(i = 1, 2, \ldots, n)$ are continua is not essential, since if $E_i$ are arbitrary closed bounded sets, one can replace $E_i$ by continua $F_i$ containing $E_i$, respectively, such that $F = \bigcup_{i=1}^{n} F_i$ satisfies all assumptions of Corollary 6.3 and the equation $\Phi(z, E, \lambda b) = \exp[\lambda b(z)]$ follows then from (3.1) and (3.5).

Remark 6.2. If $E_1$ is an arbitrary closed set such that $d(E_1) > 0$, then there is a Jordan curve $E_2$ containing $E_1$ in its interior such that the function $\lambda b(z) = \lambda[(-1)^i+1]$, $z \in E_i$, $i = 1, 2$, does not belong to $R(E_1+E_2)$ for any $\lambda > 0$ (see [21]).

LEMMA 6.2. *Let $E$ be a bounded closed set such that $d(E) > 0$. If $p(z) = \operatorname{Log}\sqrt{1+|z|^2}$ and $q(z) = \operatorname{Log}|z-a|$, $a \notin E$, then there is a real number $\lambda_0 > 0$ such that*

$$(6.7) \qquad \Phi(z, E, p+\lambda q) = \exp[p(z)+\lambda q(z)] \quad \text{for } z \in E \text{ and } |\lambda| \leqslant \lambda_0.$$



Proof. Let $r > 0$ and $R > 0$ be two real numbers such that $E$ is contained in the annulus $A_{rR} = \{z \mid r < |z-a| < R\}$. Put $K_\varrho = \{z \mid |z-a| = \varrho\}$, $\varrho > 0$, and $\tilde{E} = E+K_r+K_R$. In virtue of Lemma 6.1 there is $\lambda_0 > 0$ such that

$$\operatorname{Log}\Phi(z, \tilde{E}, p+\lambda q) = \exp[p(z)+\lambda q(z)] \quad \text{for} \quad z \in K_r+K_R \text{ and } |\lambda| \leqslant \lambda_0$$

The function

$$H(z) = \operatorname{Log}\Phi(z, \tilde{E}, p+\lambda q) - p(z) - \lambda q(z)$$

$$= \operatorname{Log}\Phi(z, \tilde{E}, p+\lambda q) - \operatorname{Log}\frac{\sqrt{1+|z|^2}}{|z-a|^{\lambda}}$$

is superharmonic in the open set $A_{rR} - E^*$, $E^* = E^*(p+\lambda q)$, and equals zero on the boundary of it. Therefore by the minimum principle for superharmonic functions $H(z) \geqslant 0$ for $z \in A_{rR}$. Since $H(z) \leqslant 0$ for $z \in \tilde{E}$, it follows that $H(z) \equiv 0$ for $z \in \tilde{E}$ and $|\lambda| \leqslant \lambda_0$. As $E \subset \tilde{E}$, (6.7) follows now from (3.1) and (3.5).

LEMMA 6.3. *Let $E$ be a bounded closed set with $d(E) > 0$. Let*

$$p(z) = \operatorname{Log}|(z-a_1)^{\sigma_1}\ldots(z-a_k)^{\sigma_k}|, \quad \text{where} \quad a_i \notin E,\ \sigma_i \geqslant 0,\ \sum_{i=1}^{k}\sigma_i = \sigma < 1.$$

*(If all $\sigma_i = 0$, we put $p(z) \equiv 0$). Let $q(z) = \operatorname{Log}|z-a|$, where $a$ belongs to the component of $CE$ which contains at least one of the points $a_1, a_2, \ldots, a_k$ and $\infty$. Then there is $\lambda_0 > 0$ such that*

$$(6.8) \qquad \Phi(z, E, p+\lambda q) = \exp[p(z)+\lambda q(z)] \quad \text{for} \quad z \in E \text{ and } |\lambda| \leqslant \lambda_0$$

(see [3] and [14]).

Proof. Let $A_{rR}$, $K_r$, $K_R$ and $\tilde{E}$ have the same meaning as in the proof of Lemma 6.2. We may assume that $r$ and $R$ are chosen in such a way that $E \subset A_{rR}$ and $a_i \in A_{rR}$, $i = 1, 2, \ldots, k$. By Lemma 6.1 there is $\lambda_0 > 0$ such that

$$\operatorname{Log}\Phi(z, \tilde{E}, p+\lambda q) = \exp[p(z)+\lambda q(z)]$$

for $z \in K_r+K_R$ and $|\lambda| \leqslant \lambda_0$ (here the set $K_r+K_R$ plays the role of $F$ in Lemma 6.1). Then the function $H(z) = \operatorname{Log}\Phi(z, E, p+\lambda q) - p(z) - \lambda q(z)$ is superharmonic in the open set $A_{rR} - \tilde{E}^* - \{a_1, a_2, \ldots, a_k\}$, is equal to zero on the boundary of $A_{rR} - \tilde{E}^*$ and $\lim_{z \to a_i} H(z) = \infty$, $i = 1, \ldots, k$. Therefore by the minimum principle we have $H(z) \geqslant 0$ for $z \in A_{rR}$, whence we deduce equality (6.8) in the same way as in the proof of Lemma 6.2.

As a direct consequence of Lemma 6.2. and of (6.3) we obtain the following

THEOREM 6.2. *If $E$ and $p(z)$ are the same as in Lemma 6.2 and*

$$q(z) = \mathrm{Log}\left[|(z-b_1)^{\beta_1}\dots(z-b_r)^{\beta_r}|/|(z-c_1)^{\gamma_1}\dots(z-c_s)^{\gamma_s}|\right],$$

*where $b_1, \dots, b_r, c_1, \dots, c_s \notin E$, $\beta_i \geqslant 0$, $\gamma_j \geqslant 0$, then there is a $\lambda_0 > 0$ such that*

$$\Phi(z, E, p+\lambda q) = \exp[p(z)+\lambda q(z)], \quad z \in E, \ |\lambda| \leqslant \lambda_0.$$

Lemma 6.3 and (6.3) imply the following

THEOREM 6.3. *Let $E$ and $p(z)$ be the same as in Lemma 6.3 and let all points $b_1, \dots, b_r, c_1, \dots, c_s$ lie in the sum of the components of $CE$ such that each of them contains at least one of the points $a_1, \dots, a_k, \infty$. Then there is $\lambda_0 > 0$ such that*

$$\Phi(z, E, p+\lambda q) = \exp[p(z)+\lambda q(z)], \quad z \in E, \ |\lambda| \leqslant \lambda_0.$$

**7. Solution of the Dirichlet problem for a class of domains and for a class of boundary values.** Let $E$ be a bounded and closed set such that $d(E) > 0$ and let $p(z)$ and $b(z)$ be real functions defined and bounded on $E$. If $\lambda > 0$, then by (3.9) the function $u(z) = u(z, E, b)$, given by

$$(7.1) \qquad u(z) = \lim_{\lambda \searrow 0} \frac{1}{\lambda} \mathrm{Log}[\Phi(z, E, p+\lambda b)/\Phi(z, E, p)], \quad z \in \bar{C},$$

is harmonic in $\bar{C}-E$ and

$$\inf_{\zeta \in E} b(\zeta) \leqslant b_0 \leqslant u(z) \leqslant B_0 = \sup_{\zeta \in E} b(\zeta), \quad z \in \bar{C}.$$

The following lemma is a generalization of a result contained in [9], [5], [12], [13]:

LEMMA 7.1. *If $E \subset L$, if $p(z)$ and $b(z)$ are continuous on $E$ and if*

$$(7.2) \qquad \mathrm{Lim}_{\lambda \downarrow 0} \frac{1}{\lambda} \mathrm{Log}[\Phi(z, E, p+\lambda q)/\Phi(z, E, p)] = b(z), \quad z \in E,$$

*then the function $u(z)$ given by (7.1) is a solution of the Dirichlet problem for every component of $CE$ with boundary values $b(z)$.*

Proof. Since $E$ is contained in $L$ and $p(z)$ and $b(z)$ are continuous on $E$, the function

$$u_\lambda(z) = \frac{1}{\lambda} \mathrm{Log}[\Phi(z, E, p+\lambda q)/\Phi(z, E, p)]$$

is by (3.14) continuous in $\bar{C}$ and harmonic in $CE$. Moreover, $b_0 \leqslant u_\lambda(z) \leqslant B_0$, $z \in \bar{C}$. If $\lambda \downarrow 0$, then in virtue of (3.8) and of (7.2) we have $u_\lambda(z) \nearrow b(z)$ for $z \in E$. Therefore by well known Dini's theorem $u_\lambda(z)$ is convergent to $b(z)$ uniformly on $E$ and hence — by the theorem of Harnack — uniformly in $\bar{C}$. Therefore $u(z)$ is continuous in $\bar{C}$, harmonic in $CE$ and $u(z) = b(z)$ for $z \in E$, i. e. $u(z)$ is a solution of the Dirichlet problem for $CE$ with the boundary values $b(z)$. The proof is completed.



If $E \subset L$, $p(z)$ and $b(z)$ are continuous on $E$ and there is a number $\lambda_0 > 0$ such that $p+\lambda_0 q \in R(E)$, then of course $u_\lambda(z) \equiv u(z)$ for $0 < \lambda \leqslant \lambda_0$ and the procedure of finding the solution of the Dirichlet problem is simplified. We shall give now some examples which show that this favourable situation does not always hold.

Example 1. Let $E = \{z \mid |z| \leqslant 1\}$, $p(z) \equiv 0$, $b(z) = -|z|$. The function $\lambda b(z)$ is strictly superharmonic for every $\lambda > 0$, so the equation $\mathrm{Log}\,\Phi(z, E, \lambda b) \equiv \lambda b(z)$, $z \in E$, cannot hold, because $\mathrm{Log}\,\Phi(z, E, \lambda b)$ is subharmonic. Moreover, it follows from (3.1) and (3.2) that $\mathrm{Log}\,\Phi(z, E, \lambda b) = -\lambda$ for $|z| = 1$. Therefore $\mathrm{Log}\,\Phi(z, E, \lambda b) \equiv -\lambda$ for $|z| \leqslant 1$.

Example 2. Let $E_1 = \{z \mid |z| = 1\}$, $E_2 = \{z \mid 1 \leqslant z = x \leqslant 2\}$, $E = E_1 + E_2$, $b(z) = 0$ for $z \in E_1$ and $b(z) = \sqrt{\frac{1}{4} - (z-\frac{3}{2})^2}$ for $z \in E_2$. It is easy to check that $\mathrm{Log}\,\Phi(z, E_1, \lambda b) = \max(0, \mathrm{Log}|z|)$. According to (3.5) we have $\Phi(z, E_1, \lambda b) \geqslant \mathrm{Log}\,\Phi(z, E, \lambda b)$. But

$$\mathrm{Log}\,\Phi(z, E_1, \lambda b) = \mathrm{Log}|z| < \lambda\sqrt{\tfrac{1}{4} - (z-\tfrac{3}{2})^2},$$

if $1 < z < 1+\delta$, $\delta > 0$ depending on $\lambda$. Hence $\mathrm{Log}\,\Phi(z, E, \lambda b) < \lambda b(z)$, $z \in (1, 1+\delta)$. In view of (3.12) this implies also that there are no extremal points of $E$ with respect to $\lambda b$ in the unterval $(1, 1+\delta)$.

It may, however, happen that though $p+\lambda b \notin R(E)$ for any $\lambda > 0$ equation (7.2) holds. We shall write $b \in R^*(E, p)$, if and only if (7.2) holds.

LEMMA 7.2. *Let $E$ be a bounded closed set with $d(E) > 0$ and let $p(z) = \mathrm{Log}\sqrt{1+|z|^2}$. Suppose for every $z_0 \in E$ there is a circle $K = \{z \mid |z-a| < r\}$ such that $K \subset CE$ and $\bar{K} \cap E = \{z_0\}$. Then every function $b(z)$ bounded and lower semicontinuous on $E$ belongs to $R^*(E, p)$.*

Proof. Let $z_0$ be a fixed point of $E$ and let $\varepsilon > 0$. There is $\delta > 0$ *such that*

$$b(z) > b(z_0) - \varepsilon, \quad |z-z_0| \leqslant \delta, \quad z \in E.$$

Since $(r/|z-a|) < 1$ for $|z-a| > r$, there is $n_0$ such that $(r/|z-a|)^n \exp[b(z_0)-\varepsilon] \leqslant \exp b(z)$, if $|z-z_0| > \delta$, $z \in E$ and $n \geqslant n_0$. Thus

$$\mathrm{Log}(1/|z-a|) + \mathrm{Log}\, r + \frac{1}{n}[b(z_0)-\varepsilon] < \frac{1}{n} b(z), \quad z \in E, \ n \geqslant n_0.$$

By Lemma 6.2 there is $\lambda_0 > 0$ such that $\mathrm{Log}\,\Phi(z, E, p+\lambda q) = p(z) + \lambda q(z)$ for $z \in E$ and $|\lambda| \leqslant \lambda_0$, where $q(z) = -\mathrm{Log}|z-a|$. In view of (3.4) this implies that

$$\mathrm{Log}\,\Phi(z, E, p+b_\lambda) = p(z) + b_\lambda(z),$$

where

$$b_\lambda(z) = \lambda\{\operatorname{Log}(r/|z-a|)+\frac{1}{n}[b(z_0)-\varepsilon]\}.$$

Hence, by (3.3),

$$p(z)+\lambda\Big\{-\operatorname{Log}|z-a|+\operatorname{Log}r+\frac{1}{n}[b(z_0)-\varepsilon]\Big\} \leqslant \operatorname{Log}\Phi\Big(z, E, p+\frac{\lambda}{n}b\Big), \quad z\in E.$$

So

$$p(z_0)+\frac{\lambda}{n}[b(z_0)-\varepsilon] \leqslant \operatorname{Log}\Phi\Big(z_0, E, p+\frac{\lambda}{n}b\Big) \leqslant p(z_0)+\frac{\lambda}{n}b(z_0),$$

whence

$$b(z_0)-\varepsilon \leqslant \lim_{n\to\infty}\frac{n}{\lambda}\operatorname{Log}\Big[\Phi\Big(z_0, E, p+\frac{\lambda}{n}b\Big)/\Phi(z_0, E, p)\Big] \leqslant b(z_0),$$

since $\operatorname{Log}\Phi(z_0, E, p) = p(z_0)$. The result follows by the arbitrariness of $\varepsilon > 0$.

Remark. If $E$ is composed of a finite number of disjoint Jordan arcs or curves each of them a sum of finitely many line segments, then for every point $z_0\in E$ there is a circle $K = \{z \mid |z-a| < r\}$ such that $K \subset CE$ and $\bar{K}\cap E = \{z_0\}$.

An immediate consequence of Lemmas 7.1 and 7.2 is the following

THEOREM 7.1. *Suppose $E \subset L$ and for every $z_0\in E$ there is a circle $K = \{z \mid |z-a| < r\}$ such that $K \subset CE$ and $\bar{K}\cap E = \{z_0\}$. If $p(z) = \operatorname{Log}\sqrt{1+|z|^2}$ and $b(z)$ is an arbitrary real function continuous on $E$, then the function*

$$u(z) = \lim_{\lambda\downarrow 0}\frac{1}{\lambda}\operatorname{Log}[\Phi(z, E, p+\lambda b)/\Phi(z, E, p)], \quad z\in\bar{C},$$

*is a solution of the Dirichlet problem for every component of $CE$ with boundary values $b(z)$.*

LEMMA 7.3. *Let $E$ be a bounded closed set with $d(E) > 0$ and let $p(z) = \operatorname{Log}|(z-a_1)^{\sigma_1}\ldots(z-a_k)^{\sigma_k}|$, where $a_i\notin E$, $\sigma_i \geqslant 0$ and $\sigma = \sum_{i=1}^{k}\sigma_i < 1$. We put $p(z)\equiv 0$, if $\sigma_1 = \sigma_2 = \ldots = \sigma_k = 0$. Denote by $\Delta$ the sum of the components of $CE$ that contain at least one of the points $a_1, \ldots, a_k, \infty$ (if $p(z)\equiv 0$, then $\Delta = D(E)$). Suppose for every $z_0\in E$ there is a circle $K = \{z \mid |z-a| < r\}$ such that $K\subset\Delta$ and $\bar{K}\cap E = \{z_0\}$. Then every function $b(z)$ bounded and lower semicontinuous on $E$ belongs to $R^*(E, p)$.*

Proof. It is enough to repeat the reasoning of the proof of Lemma 7.2, using now Lemma 6.3 instead of Lemma 6.2.

As a simple consequence of Lemma 7.1 and Lemma 7.3 we obtain the following



THEOREM 7.2. *Besides the assumptions of Lemma 7.3 assume that $E\subset L$. Then the function*

$$u(z) = \lim_{\lambda\downarrow 0}\frac{1}{\lambda}\operatorname{Log}[\Phi(z, E, p+\lambda q)/\Phi(z, E, p)], \quad z\in\bar{C},$$

*is a solution of the Dirichlet problem for $CE$ and boundary values $b(z)$.*

**8. Approximation of continuous functions by harmonic functions.** Let $\Gamma$ be an arbitrary continuum with ordinary diameter $l = \sup_{z,\zeta\in\Gamma}|z-\zeta|$. An important lemma of F. Leja states (see [7]):

If $\varepsilon > 0$ and $l_0 > 0$, then there are $\delta > 0$ and $n_0 > 0$ such that for every continuum $\Gamma$ with the diameter $l > l_0$ and for every sequence of polynomials $P_n(z)$ of respective degrees $n$, $n = 1, 2, \ldots$, which are uniformly bounded on $\Gamma$ by 1,

$$|P_n(z)| \leqslant 1 \quad \text{for} \quad z\in\Gamma,\ n = 1, 2, \ldots,$$

the inequality

$$|P_n(z)| \leqslant (1+\varepsilon)^n$$

holds for $n \geqslant n_0$ and for arbitrary circle $|z-z_0| < \delta$ of the center $z_0\in\Gamma$.

Let $\eta^{(n)} = \{\eta_0^{(n)}, \ldots, \eta_n^{(n)}\}$ be an $n$-th extremal system of $\Gamma$ with respect to $b(z)\equiv 0$. Then

$$|L^{(0)}(z, \eta^{(n)})| \leqslant 1 \quad \text{for} \quad z\in\Gamma,\ i = 0, 1, \ldots, n,\ n = 1, 2, \ldots$$

This implies by the Lemma of Leja that $\operatorname{Lim}\sqrt[n]{|L^{(0)}(z, \eta^{(n)})|} = \Phi(z, \Gamma, 0) \leqslant 1+\varepsilon$, if $|z-z_0| < \delta$ and $z_0\in\Gamma$. So we obtain the following

LEMMA 8.1. *Let $l_0$ be a real positive number. If $\Gamma$ is a continuum with diameter $l \geqslant l_0$, then for every $\varepsilon > 0$ there is $\delta > 0$, $\delta$ depending only on $\varepsilon$ but not on $\Gamma$, such that*

$$\operatorname{Log}\Phi(z, \Gamma, 0) \leqslant 1+\varepsilon, \quad \textit{if} \quad |z-z_0| < \delta \textit{ and } z_0\in\Gamma.$$

We shall now prove

LEMMA 8.2. *Let $l_0$ and $R$ be fixed positive numbers and let $b(z) = |z|$. Suppose $E$ is a set contained in the circle $K = \{z \mid |z| \leqslant R\}$ and, moreover, $E = E_1\cup E_2\cup\ldots\cup E_k$, where $E_i$, $i = 1, 2, \ldots, k$, are continua. Let $0\in E_1$ and let the diameter $l$ of the continuum $E_1$ be $\geqslant l_0$. Then for every $\varepsilon > 0$ there exists a positive $\delta$ depending only on $\varepsilon$, $l_0$, $R$ but not on $E$, such that*

$$(8.1) \qquad u(z) = \lim_{\lambda\downarrow 0}\frac{1}{\lambda}\operatorname{Log}[\Phi(z, E, \lambda b)/\Phi(z, E, 0)] < \varepsilon, \quad |z| < \delta.$$

*Moreover, the function $u$ is a solution of the Dirichlet problem for $CE$ with the boundary values $b(z) = |z|$.*

Proof. By Corollary 6.1, there is $\lambda_0 > 0$ (that depends on $R$) such that $\Phi(z, K, \lambda_0 b) = \exp[\lambda_0 b(z)] = \exp[\lambda_0 |z|]$ for $z \in K$. Since $E \subset K$, this implies by (3.5) that $\Phi(z, E, \lambda_0 b) = \exp[\lambda_0|z|]$ for $z \in E$. Therefore

$$u(z) = \frac{1}{\lambda}\operatorname{Log}[\Phi(z, E, \lambda b)/\Phi(z, E, 0)] \quad \text{for} \quad 0 < \lambda \leqslant \lambda_0$$

and since $E \in L$, the function $u(z)$ is by Lemma 7.1 a solution of the Dirichlet problem for $CE$ and $b(z) = |z|$ as boundary values. The function $\Phi(z, E, 0)$ is continuous in $C$ and $\Phi(z, E, 0) \geqslant 1$ for $z \in C$. Therefore

$$u(z) \leqslant \frac{1}{\lambda_0}\operatorname{Log}\Phi(z, E, \lambda_0 b), \quad z \in C.$$

Let $\varepsilon > 0$. We shall now find $\delta > 0$ such that (8.1) holds. We may assume that $\frac{1}{2}\varepsilon = r \leqslant l_0$. Denote by $F$ the part of $E_1$ that is contained in the circle $|z| \leqslant r$; of course the diameter of $F$ is equal to $r$. Since $F \subset E$ and $\lambda_0 b(z) \leqslant \frac{1}{2}\lambda_0 \varepsilon$ for $|z| \leqslant r$, we have

$$u(z) \leqslant \frac{1}{\lambda_0}\operatorname{Log}\Phi(z, F, \lambda_0 b) \leqslant \frac{1}{\lambda_0}\operatorname{Log}\left[\left(\exp\frac{\lambda_0\varepsilon}{2}\right)\Phi(z, F, 0)\right]$$
$$= \frac{\varepsilon}{2} + \frac{1}{\lambda_0}\operatorname{Log}\Phi(z, F, 0)$$

for $z \in C$. By the Lemma of Leja there is $\delta > 0$ such that

$$\frac{1}{\lambda_0}\operatorname{Log}(z, F, 0) < \frac{\varepsilon}{2} \quad \text{for} \quad |z| < \delta,$$

$\delta$ depending only on $\lambda_0$ and $\varepsilon$. Thus

$$u(z) < \varepsilon \quad \text{for} \quad |z| < \delta,$$

where $\delta$ depends only on $\varepsilon$, $l_0$ and $R$. The proof is completed.

Suppose $E$ is a union of the boundaries of $p+1$ disjoint domains $D_0, D_1, \ldots, D_p$, $D_0 = D(E)$ being an unbounded component of $CE$. Of course our assumption does not exlude the possibility that $CE$ is a union of infinitely many components. Let $K = \{z \mid |z| \leqslant R\}$ be so large that $E \subset K$. Let

$$4l = \min_{1\leqslant i\leqslant p} l_i \quad \text{where} \quad l_i = \sup_{z,\zeta\in D_i}|z-\zeta| \quad (i = 1, 2, \ldots, p)$$

is the ordinary diameter of $D_i$. By a simple reasoning (see [6], p. 179) one can show that for every $n = 1, 2, \ldots$ there exists a set $\Gamma^{(n)} = \Gamma_0^{(n)} + \Gamma_1^{(n)} + \ldots + \Gamma_p^{(n)}$ such that $\Gamma_i^{(n)}$ is a Jordan arc of diameter $k_i \geqslant 2l$, $\Gamma_i^{(n)}$ is contained in $D_i$ and composed of a finite number of line segments. Moreover, for every $z_0 \in E$ (and arbitrarily fixed $n$) there is $z_* \in \Gamma^{(n)}$ such that $|z_0 - z_*| < 1/n$.



Let $b(z)$ be an arbitrary real function defined and continuous on $E$. By the Tietze-Urysohn extension theorem we may assume that $b(z)$ is defined and continuous in $K$. It follows from Theorem 7.2 that the function

$$(8.2) \qquad U(z, \Gamma^{(n)}, b) = \lim_{\lambda\downarrow 0}\frac{1}{\lambda}\operatorname{Log}[\Phi(z, \Gamma^{(n)}, \lambda b)/\Phi(z, \Gamma^{(n)}, 0)], \quad z \in \bar{C},$$

is a solution of the Dirichlet problem for $C\Gamma^{(n)}$ with $b(z)$ as boundary values. The convergence in (8.2) is uniform and the function $U(z, \Gamma^{(n)}, b)$ is harmonic in a neighborhood of $E$.

THEOREM 8.1. *The sequence* $U_n(z) = U(z, \Gamma^{(n)}, b)$, $n = 1, 2, \ldots$, *converges to* $b(z)$ *uniformly on* $E$ (see [6]).

Proof. The function $b(z)$ being uniformly continuous on $K$, for every $\varepsilon > 0$ there is $M > 0$ such that

$$b(z_0) - \varepsilon - M|z - z_0| \leqslant b(z) \leqslant b(z_0) + \varepsilon + M|z - z_0| \quad \text{for} \quad z, z_0 \in K.$$

This implies that for $n = 1, 2, \ldots$

$$b(z_0) - \varepsilon - MV(z, z_0, \Gamma^{(n)}) \leqslant U(z, \Gamma^{(n)}, b) \leqslant b(z_0) + \varepsilon + MV(z, z_0, \Gamma^{(n)})$$

for $z \in C$, $z_0 \in \Gamma^{(n)}$, where for a fixed $z_0 \in \Gamma^{(n)}$ the function $V(z, z_0, \Gamma^{(n)})$ is a solution of the Dirichlet problem for $C\Gamma^{(n)}$ with boundary values $|z - z_0|$. By Lemma 8.2 there is $\delta > 0$ that does not depend on $n$ such that for every $z_0 \in \Gamma^{(n)}$

$$MV(z, z_0, \Gamma^{(n)}) < \varepsilon, \quad \text{if} \quad |z - z_0| < \delta,$$

whence

$$\text{(i)} \qquad b(z_0) - 2\varepsilon \leqslant U(z, \Gamma^{(n)}, b) \leqslant b(z_0) + 2\varepsilon, \text{ for } |z - z_0| < \delta,$$

$z_0 \in \Gamma^{(n)}$, $n = 1, 2, \ldots$ If $n$ is so large that $1/2n < \delta$, then for every $z_* \in E$ there is $z_0 \in \Gamma^{(n)}$ such that $|z_* - z_0| < 1/n$. Since $b(z)$ is uniformly continuous in $K$ and $\Gamma^{(n)} \subset K$, we have $b(z_*) - \varepsilon \leqslant b(z_0) \leqslant b(z_*) + \varepsilon$ for sufficiently large $n$, whence in view of (i) we have

$$b(z_*) - 3\varepsilon < U(z_*, \Gamma^{(n)}, b) \leqslant b(z_*) + 3\varepsilon \quad \text{for} \quad z_* \in E$$

and for sufficiently large $n$. As $\varepsilon > 0$ is arbitrary, this concludes the proof.

THEOREM 8.2. *Let $E$ be the union of the boundaries of $p+1$ ($p \geqslant 0$) domains $D_0, D_1, \ldots, D_p$ no two of which have common points. Let $D_0 = D(E)$ denote the unbounded component of $CE$. If $b(z)$ is an arbitrary real function defined and continuous on $E$, then for every $\varepsilon > 0$ there is a function*

$$q(z) = a[(z-b_1)^{\beta_1}\ldots(z-b_r)^{\beta_s}]/[(z-c_1)^{\gamma_1}\ldots(z-c_s)^{\gamma_s}],$$

*where* $b_1, \ldots, b_r$, $c_1, \ldots, c_s \in \Delta = D_0 + D_1 + \ldots + D_p$, $\beta_i \geqslant 0$, $\gamma_j \geqslant 0$ *and* $a = \text{const}$, *such that*

$$|b(z) - \operatorname{Log}|q(z)|| < \varepsilon \quad \text{for} \quad z \in E.$$

Proof. Let $\Gamma^{(n)}$ have the same meaning as in Theorem 8.1. By Tietze-Urysohn extension theorem we may assume that $b(z)$ is defined and continuous in a circle $K$ such that $E \subset K$. Given $\varepsilon > 0$, we can find, in virtue of Theorem 7.2 and Theorem 8.1, $\lambda > 0$ and $n > 0$ such that

$$(8.3)\qquad \left| b(z) - \frac{1}{\lambda}\operatorname{Log}\left|\frac{\Phi(z, \Gamma^{(n)}, \lambda b)}{\Phi(z, \Gamma^{(n)}, 0)}\right|\right| < \frac{\varepsilon}{2}, \qquad z \in E$$

Let $\eta^{(\nu)} = \{\eta_0, \eta_1, \ldots, \eta_\nu\}$ and $\xi^{(\nu)} = \{\xi_0, \xi_1, \ldots, \xi_\nu\}$ denote the $\nu$-th extremal system of $E$ with respect to $\lambda b(z)$ and 0, respectively. By Theorem 2.1, the sequences

$$\{\sqrt[\nu]{|\Phi^{(0)}(z, \eta^{(\nu)}, \lambda b)|}\} \quad \text{and} \quad \{\sqrt[\nu]{|\Phi^{(0)}(z, \xi^{(\nu)}, 0)|}\}$$

converge to $\Phi(z, \Gamma^{(n)}, \lambda b)$ and $\Phi(z, \Gamma^{(n)}, 0)$, respectively, the convergence being uniform on $E$. Hence there is $\nu$ such that

$$(8.4)\qquad \left|\operatorname{Log}\frac{\Phi(z, \Gamma^{(n)}, \lambda b)}{\Phi(z, \Gamma^{(n)}, 0)} - \frac{1}{\nu}\operatorname{Log}\frac{|\Phi^{(0)}(z, \eta^{(\nu)}, \lambda b)|}{|\Phi^{(0)}(z, \xi^{(\nu)}, 0)|}\right| < \lambda\frac{\varepsilon}{2}, \qquad z \in E.$$

Combining (8.4) and (8.3) we get

$$\left| b(z) - \frac{1}{\lambda\nu}\operatorname{Log}\frac{|\Phi^{(0)}(z, \eta^{(\nu)}, \lambda b)|}{|\Phi^{(0)}(z, \xi^{(\nu)}, 0)|}\right| < \varepsilon, \qquad z \in E,$$

whence the result follows, if we put

$$q(z) = [\Phi^{(0)}(z, \eta^{(\nu)}, \lambda b)/\Phi^{(0)}(z, \xi^{(\nu)}, 0)]^{1/\lambda\nu}.$$

Remark 8.1. If $E$ is a boundary of the domain $D_0 = D(E)$, then Theorem 8.2 may be proved by means of Runge's theorem and of Theorem 8.1. Moreover, the function $q(z)$ may be replaced by a polynomial whose zeros lie in $D_0$. Indeed, by Theorem 8.1 there is a sequence of functions $H_n(z)$ harmonic in a neighborhood of $CD_0$ such that

$$H_n(z) \to b(z), \qquad z \in E,$$

the convergence being uniform on $E$. Therefore, given $\varepsilon > 0$, there is $n_0$ such that

$$(8.5)\qquad |\exp b(z) - \exp H_{n_0}(z)| < \frac{\varepsilon}{2} \quad \text{for} \quad z \in E.$$

Let now $G$ be a neighborhood of $CD_0$ such that every component of $G$ is simply connected and $H_n(z)$ is harmonic in $G$. Let $\tilde{H}_n(z)$ be a harmonic function in $G$ conjugate with $H_n$. By the Runge theorem every function $f_n(z) = \exp[H_n(z) + i\tilde{H}_n(z)]$, $n = 1, 2, \ldots$, may be approximated by polynomials uniformly in every compact subset of $G$. In particular, there is a polynomial $q(z)$ such that



$$(8.6)\qquad |q(z) - f_{n_0}(z)| < \frac{\varepsilon}{2} \quad \text{for} \quad z \in E.$$

The function $f_{n_0}(z)$ does not vanish in $G$. Therefore the polynomial $q(z)$ can be chosen in such a way that all its zeros lie in $D_0$. From (8.5) and (8.6) we obtain $|\exp b(z) - |q(z)|| < \varepsilon$, $z \in E$, whence the result immediately follows.

Let $E$ be a bounded closed set with $d(E) > 0$, $p(z)$ a real function defined and bounded on $E$ such that $\Phi(z, E, p) = \exp p(z)$ for $z \in E$. If $S$ is a subfamily of $R^*(E, p)$, then the function $b^*(z) = \sup_{b \in R^*} b(z)$ belongs to $R^*(E, p)$. This is a simple consequence of (3.1). Hence and from Theorem 8.2 we obtain

COROLLARY 8.1. *Let $E$ be the union of the boundaries of $k+1$ ($k \geq 0$) domains $D_0, D_1, \ldots, D_k$, no two of which have common points. Let $d(E) > 0$ and let $b(z)$ be an arbitrary real function defined bounded and lower semi-continuous on $E$. If $p(z) = \sqrt{1+|z|^2}$, then*

$$\lim_{\lambda \downarrow 0} \frac{1}{\lambda}\operatorname{Log}[\Phi(z, E, p+\lambda b)/\Phi(z, E, p)] = b(z) \quad \textit{for} \quad z \in E.$$

*The same equation holds if*

$$p(z) = \operatorname{Log}|(z-a_1)^{\sigma_1}\ldots(z-a_k)^{\sigma_k}|,$$

*where* $\sigma_i > 0$, $\sum_{i=1}^{k} \sigma_i < 1$, $a_i \in D_i$ $(i = 1, 2, \ldots, k)$.

THEOREM 8.3. *Let $E$ satisfy all assumptions of Corollary 8.1 and, moreover, let $E \subset L$. Let $p(z)$ denote any of the functions considered in the Corollary. Then for every real function $b(z)$ defined and continuous on $E$, the function*

$$(\mathrm{ii})\qquad u(z) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda}\operatorname{Log}[\Phi(z, E, p+\lambda b)/\Phi(z, E, p)], \qquad z \in \bar{C},$$

*is a solution of the Dirichlet problem for $CE$ and boundary values $b(z)$.*

Remark 8.2. We may allow $\lambda$ tend to 0 in (ii) through arbitrary real numbers different from zero.

**9. Dirichlet problem generalized by Kellog and Wiener.** Let $D$ be an arbitrary domain in $C$, $b(z)$ — a real function defined and continuous on the boundary of $D$. Denote by $\tilde{b}(z)$ a continuous extension of $b$ to $\bar{C}$. Let $\{D_k\}$ be a sequence of domains regular with respect to the classical Dirichlet problem and let

$$(9.1)\qquad D_k \subset D_{k+1}, \qquad D_k \nearrow D.$$

Denote by $H_k(z) = H_k(z, \tilde{b})$ the solution of the Dirichlet problem for $D_k$ and boundary values $b(z)$. A function $H(z)$ defined and harmonic in $D$ is said to be a *Kellog-Wiener solution of the Dirichlet problem* for $D$ and boundary values $b(z)$, if for every choice of the extension $\tilde{b}(z)$ and of the sequence $\{D_k\}$, the corresponding sequence of harmonic functions $\{H_n(z)\}$ is convergent to $H(z)$ uniformly in every compact subset of $D$. The purpose of this section is to prove the following

THEOREM 9.1. *If $D$ is a domain containing point $\infty$ in its interior and if the boundary $E$ of $D$ has positive transfinite diameter $d(E)$, then, given an arbitrary real function $b(z)$ defined and continuous in $E$, the function*

$$u(z) = \lim_{\substack{\lambda\to 0\\(\lambda\neq 0)}} \frac{1}{\lambda}\operatorname{Log}[\Phi(z, E, \lambda b)/\Phi(z, E, 0)], \qquad z\in\bar{C},$$

*is a Kellog-Wiener solution of the Dirichlet problem for $D$ with boundary values $b(z)$.*

Before we start the proof of the Theorem we shall prove two lemmas which are interesting also for themselves.

LEMMA 9.1. *Let $b(z)$ be a real function defined and continuous in $C$. The functional $\mu(E, b)$ defined by (4.5) is continuous with respect to $E$ in the following sense: for every $\varepsilon > 0$, there is $\delta > 0$ such that*

$$\mu(E, b) \leqslant \mu(E_\delta, b) \leqslant \mu(E, b)+\varepsilon, \tag{9.2}$$

*$E_\delta$ denoting the set of all points the distance of which from $E$ does not exceed $\delta$.*

Proof. If $c^{(n)} = \{c_1, \ldots, c_n\}$ is an arbitrary system of $n$ points of $C$ and

$$\mu_n(E) = \min_{c^{(n)}\subset E}\{\max_{z\in E}|(z-c_1)\ldots(z-c_n)|\exp[-nb(z)]\} \qquad n = 1, 2, \ldots,$$

then in accordance with (4.5)

$$\sqrt[n]{\mu_n(E)} \to \mu(E) = \mu(E, b).$$

Therefore for every $\varepsilon > 0$, there is $s$ such that

$$\mu_s^{1/s}(E) < \mu(E)+\varepsilon/2. \tag{9.3}$$

Let $K = \{z\,|\,|z|\leqslant R\}$ be so large that $E\subset K$. The function

$$\varphi(c_1, \ldots, c_s, z) = |(z-c_1)\ldots(z-c_s)|^{1/s}\exp[-b(z)]$$

is continuous for $(c_1, \ldots, c_s, z)\in K\times K\times\ldots\times K = K^{s+1}$. The set $K^{s+1}$ being closed the function $\varphi$ is uniformly continuous in it. Hence, there is $\delta > 0$ such that

$$\max_{z\in E_\delta}\varphi(c_1, \ldots, c_s, z) \leqslant \max_{z\in E}\varphi(c_1, \ldots, c_s, z)+\varepsilon/2$$



for every $(c_1, \ldots, c_s)\in K^s$. Therefore

$$\min_{c^{(s)}\subset E_\delta}\{\max_{z\in E_\delta}\varphi(c_1, \ldots, c_s, z)\} \leqslant \min_{c^{(s)}\subset E}\{\max_{z\in E}\varphi(c_1, \ldots, c_s, z)\}+\varepsilon/2,$$

because $E\subset E_\delta$. Hence

$$\mu_s^{1/s}(E_\delta) < \mu_s^{1/s}(E)+\varepsilon/2. \tag{9.4}$$

Since, as we know, $\mu_{m+n}(E_\delta) \leqslant \mu_n(E_\delta)\mu_m(E_\delta)$ for $m, n = 1, 2, \ldots$, we have $\mu_{ms}(E_\delta) \leqslant [\mu_s(E_\delta)]^m$ for $m = 1, 2, \ldots$, whence $\mu(E_\delta) \leqslant \mu_s^{1/s}(E_\delta)$. In view of (9.4) this implies that

$$\mu(E_\delta) < \mu(E)+\varepsilon.$$

It follows from (3.5) that $\Phi(z, E_\delta, b) \leqslant \Phi(z, E, b)$ for $z\in C$, because $E\subset E_\delta$, whence by (4.8) we obtain

$$\mu^{-1}(E_\delta) = \lim_{z\to\infty}\Phi(z, E_\delta, b)/|z| \leqslant \lim_{z\to\infty}\Phi(z, E, b)/|z| = \mu^{-1}(E),$$

i. e. $\mu(E) \leqslant \mu(E_\delta)$. The proof is thus completed.

COROLLARY 9.1. *If $E_{n+1}\subset E_n$, $n = 1, 2, \ldots$, and $E_n\to E$, then*

$$\mu(E_n, b) \to \mu(E, b).$$

Let $D$ be an arbitrary domain containing point $\infty$ in its interior such that the boundary $E$ of $D$ has the positive transfinite diameter. Let $\{D_n\}$ be a sequence of domains regular with respect to the Dirichlet problem and such that

$$\bar{D}_n \subset D_{n+1}, \qquad \infty\in D_n, \qquad n = 1, 2, \ldots, D_n\to D.$$

Put $E_n = \bar{D}-D_n$. The sets $E_n$ $(n = 1, 2, \ldots)$ are bounded and closed and, moreover, $E_{n+1}\subset E_n$ and $E_n\to E$. We shall now prove the following

LEMMA 9.2. *If $b(z)$ is a continuous real function defined in $C$, then*

$$\Phi(z, E, b) = \lim_{n\to\infty}\Phi(z, E_n, b), \qquad z\in C,$$

*the convergence being uniform in every compact subset of $D$.*

Proof. The functions $R_n(z) = \operatorname{Log}\Phi(z, E_n, b)-\operatorname{Log}\Phi(z, E, b)$ are by (3.10) harmonic in $D_n$. Moreover, since $E\subset E_{n+1}\subset E_n$, we have, by (3.5),

$$R_n(z) \leqslant R_{n+1}(z) \leqslant 0 \quad \text{for} \quad z\in C \text{ and } n = 1, 2, \ldots$$

Observe that, by Lemma 9.1, $R_n(\infty) = \operatorname{Log}[\mu(E, b)/\mu(E_n, b)] \to 0$. The result follows by the Harnack principle.

If $b(z)\equiv 0$, then by (3.14) $\operatorname{Log}\Phi(z, E_n, 0)$ is the classical Green function of $D_n$ with its logarithmic pole at infinity. This implies by Lemma 9.2 the following

COROLLARY 9.1. (see [2]). *If $D$ is a domain containing $\infty$ in its interior and the boundary $E$ of $D$ has the positive transfinite diameter, then $\operatorname{Log}\Phi(z, E, 0)$ is a generalized Green function for $D$ with a pole at $\infty$.*

Proof of Theorem 9.1. By the Urysohn extension theorem we may assume $b(z)$ to be defined and continuous in $C$. In the sequel we shall use the notation of Lemma 9.2. Let $F_n$ denote the boundary of $D_n$. By Theorem 8.3 the function

$$u(z, F_n, b) = \lim_{\lambda\downarrow 0}\frac{1}{\lambda}\operatorname{Log}[\Phi(z, F_n, b)/\Phi(z, F_n, 0)], \qquad z\in C,$$

is a solution of the Dirichlet problem for $D_n$ with boundary values $b(z)$. Therefore the Theorem will be proved, if we show that

$$u(z, F_n, b)\to u(z, E, b)$$

uniformly in every compact subset of $D$.

Let $\varepsilon>0$. By Theorem 8.2, there is a function

$$q(z) = [a(z-b_1)^{\beta_1}\dots(z-b_r)^{\beta_r}/(z-c_1)^{\lambda_1}\dots(z-c_s)^{\lambda_s}]$$

with $b_1, \dots, b_r, c_1, \dots, c_s\in D_n$, $\beta_k\geqslant 0$, $\gamma_k\geqslant 0$ and $a = \mathrm{const}$ such that

$$(9.5)\qquad \big|b(z)-\operatorname{Log}|q(z)|\big| < \varepsilon, \qquad z\in E.$$

The functions $b(z)$ and $\operatorname{Log}|q(z)|$ being continuous in a neighborhood of $E$, there is $\delta>0$ such that

$$(9.6)\qquad \big|b(z)-\operatorname{Log}|q(z)|\big| < 2\varepsilon \quad \text{for} \quad z\in E_\delta,$$

where $E_\delta = \bigcup_{z_0\in E}\{z\,\big|\,|z-z_0|\leqslant\delta\}$. There is $n_0$ such that $E_n\subset E_\delta$ if $n\geqslant n_0$. Let $b_1(z)$ denote an arbitrary real function defined and continuous in $C$ such that $b_1(z) = \operatorname{Log}|q(z)|$ for $z\in E_\delta + CD$. By Theorem 6.2 there is $\lambda_0>0$ such that

$$u(z, F_n, b_1)\equiv u_\lambda(z, F_n, b_1), \quad \text{if} \quad 0<\lambda\leqslant\lambda_0.$$

Observe that by the maximum principle all extremal points of $CD_n$ $(n\geqslant n_0)$ with respect to $b_1(z)$ $(=\operatorname{Log}|q(z)|)$ lie in $F_n$. Hence, since $F_n\subset E_n\subset CD_n$, we have $\Phi(z, F_n, \lambda b_1) = \Phi(z, E_n, \lambda b_1)\equiv\Phi(z, CD_n, \lambda b_1)$ for $\lambda\geqslant 0$, whence

$$u(z, F_n, b_1) = \frac{1}{\lambda_0}\operatorname{Log}\frac{\Phi(z, F_n, \lambda_0 b_1)}{\Phi(z, F_n, 0)} \equiv \frac{1}{\lambda_0}\operatorname{Log}\frac{\Phi(z, E_n, \lambda_0 b_1)}{\Phi(z, E_n, 0)} = u(z, E_n, b_1).$$



By Lemma 9.2, $u(z, F_n, b_1) = u(z, E_n, b_1)\to u(z, E, b_1)$ uniformly in every compact subset of $D$. Therefore, given an arbitrary compact subset $\Delta$ of $D$, we have

$$\text{(a)}\qquad |u(z, E, b_1)-u(z, F_n, b_1)| < \varepsilon, \qquad z\in\Delta,\ n\geqslant n_1 = \mathrm{const}.$$

If $n\geqslant n_0$, then by (9.6)

$$\text{(b)}\qquad |u(z, F_n, b)-u(z, F_n, b_1)| < 2\varepsilon \quad \text{for} \quad z\in D_n.$$

In virtue of (9.5) and of (3.3) we have

$$\text{(c)}\qquad |u(z, E, b)-u(z, E, b_1)| < \varepsilon, \qquad z\in C.$$

If $n$ is sufficiently large the inequalities (a), (b) and (c) imply the inequality

$$|u(z, E, b)-u(z, F_n, b)| < 4\varepsilon, \qquad z\in\Delta.$$

Because $\varepsilon$ and $\Delta$ are arbitrary, the proof is concluded.

**10. Dirichlet problem generalized by Perron.** Given an arbitrary real function $h(z)$ defined and bounded in a set $F$, we define $\underline{h}(z)$ and $\overline{h}(z)$ by

$$(10.1)\qquad \begin{aligned}\underline{h}(z) &= \lim_{\delta\to 0}\{\inf_{|z-z_0|<\delta, z_0\in F} h(z_0)\},\\ \overline{h}(z) &= \lim_{\delta\to 0}\{\sup_{|z-z_0|<\delta, z_0\in F} h(z_0)\},\end{aligned} \qquad (z\in\overline{F}).$$

It is easy to check that $\underline{h}(z)$ is lower and $\overline{h}(z)$ is uppersemicontinuous in $\overline{F}$. Moreover,

$$(10.2)\qquad \underline{h}(z)\leqslant h(z)\leqslant\overline{h}(z), \qquad z\in F.$$

Suppose $b(z)$ is a real function defined and bounded on the boundary $E$ of a domain $D$. Function $U(z)$ defined and harmonic in $D$ is said to be a *Perron solution of the Dirichlet problem* for $D$ with boundary values $b(z)$, if

$$(10.3)\qquad \underline{b}(z)\leqslant\underline{U}(z)\leqslant\overline{U}(z)\leqslant\overline{b}(z), \qquad z\in E.$$

We shall prove the following

THEOREM 10.1. *If $D$ is a domain containing $\infty$ in its interior and the boundary $E$ of $D$ belongs to $L$ (i. e. $D$ is regular with respect to the Dirichlet problem), then the function $u_0(z) = u(z, E, b)$ is the least and the function $u^0(z) = -u(z, E, -b)$ is the greatest Perron solution of the Dirichlet problem for $D$ and boundary values $b(z)$.*

Proof. First of all observe that

$$(10.4)\qquad u(z, E, \underline{b}) = u(z, E, b)\leqslant -u(z, E, -b) = -u(z, E, -\overline{b}).$$

Indeed, the equalities follow immediately from corollary 2.1. To show the inequality observe that by (3.7) we have $\Phi(z, E, \lambda b)\Phi(z, E, -\lambda b) \leqslant \Phi^2(z, E, 0)$, whence the result follows by (7.1).

We shall now prove that $u_0$ and $u^0$ are Perron solution of the Dirichlet problem for $D$ with boundary values $b(z)$. Indeed, by the theorem of Baire there are two sequences of continuous functions $\{b_\nu(z)\}$ and $\{B_\nu(z)\}$ such that $b_\nu(z) \nearrow \underline{b}(z)$ and $B_\nu(z) \searrow \bar{b}(z)$. In view of Theorem 7.2. this implies that $u(z, E, b) = \underline{b}(z)$ and $u(z, E, -b) = -\bar{b}(z)$ for $z \in E$. In virtue of the inequalities $b_\nu(z) \leqslant \underline{b}(z)$ and $\bar{b}(z) \leqslant B_\nu(z)$ and of (10.4) we have

$$u(z, E, b_\nu) \leqslant u(z, E, b) \leqslant -u(z, E, -b) \leqslant u(z, E, B_\nu), \qquad \nu = 1, 2, \ldots,$$

whence the result follows, because $\{u(z, E, b_\nu)\}$ is increasing and $\{u(z, E, B_\nu)\}$ is decreasing. To prove that $u_0$ is the least Perron solution, let $v(z)$ be an arbitrary real function harmonic in $D$ such that

$$\underline{b}(z) \leqslant \underline{v}(z) \leqslant \bar{v}(z) \leqslant \bar{b}(z) \quad \text{for} \quad z \in E. \tag{10.5}$$

We must prove that $u_0(z) \leqslant v(z)$ for $z \in D$. It follows from (10.5) that

$$u_0(z) = u(z, E, \underline{b}) \leqslant u(z, E, \underline{v}), \quad z \in \bar{C}.$$

Put $\varphi_\lambda(z) = \mathrm{Log}[\Phi(z, E, 0)\exp\lambda\underline{v}(z)]$. The function $\varphi_\lambda(z)$ is harmonic in $D - \{\infty\}$ and it has a logarithmic pole of order one at $\infty$. Therefore by the maximum principle for analytic functions all extremal points of $\bar{D}$ with respect to $\varphi_\lambda(z)$ lie on $E$, whence $\Phi(z, \bar{D}, \varphi_\lambda) \equiv \Phi(z, E, \varphi_\lambda)$. Therefore, by (3.1), $\Phi(z, E, \lambda\underline{v}) \leqslant \Phi(z, E, 0)\exp\lambda v(z)$ for $z \in \bar{D}$, whence $u(z, E, \underline{v}) \leqslant \underline{v}(z)$ for $z \in \bar{D}$. Therefore $u_0(z) \leqslant v(z)$, $z \in D$, since $\underline{b}(z) \leqslant \underline{v}(z)$ for $z \in E$. In the same way we may prove that $v(z) \leqslant u^0(z)$ for $z \in D$. Therefore $u_0$ is the least and $u^0$ is the greatest Perron solution of the Dirichlet problem for $D$ with boundary values $b(z)$.

We shall conclude this section by the following

Remark. A necessary and sufficient condition that $u_0(z) = u^0(z)$ for $z \in D$ is that

$$\lim_{\lambda \downarrow 0} \frac{1}{\lambda} \mathrm{Log} \frac{\mu^2(E, 0)}{\mu(E, \lambda b)\mu(E, -\lambda b)} = 0.$$

Indeed, by (4.8),

$$\lim_{z\to\infty} \frac{1}{\lambda} \mathrm{Log}[\Phi(z, E, \lambda\underline{b})/\Phi(z, E, 0)] = -\frac{1}{\lambda}\mathrm{Log}[\mu(E, \lambda\underline{b})/\mu(E, 0)],$$

whence

$$u(\infty, E, \underline{b}) = -\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\mathrm{Log}\frac{\mu(E, \lambda b)}{\mu(E, ( )}.$$



Analogously

$$u(\infty, E, -\bar{b}) = \lim_{\lambda\downarrow 0}\frac{1}{\lambda}\mathrm{Log}[\mu(E, 0)/\mu(E, -\lambda\bar{b})].$$

Therefore

$$u_0(\infty) - u^0(\infty) = u(\infty, E, \underline{b}) + u(\infty, E, -\bar{b})$$

$$= \lim \frac{1}{\lambda}\mathrm{Log}[\mu^2(E, 0)/\mu(E, \lambda\underline{b})\mu(E, -\lambda\bar{b})].$$

The result follows now by the inequality $u_0(z) - u^0(z) \leqslant 0$ and by the maximum principle for harmonic functions.

**11. Remarks on the effectiveness of the method of extremal points.** We would like to show in this section that the method of extremal points is, at least theoretically, effective (computable). We shall describe a procedure which will enable us to compute $\mu(E, b)$ or $\Phi(z, E, b)$ as limits of some sequences $\{\mu_n^*(E, b)\}$ or $\{\Phi_n^*(z, E, b)\}$, respectively, such that, for a given $n$, $\mu_n^*$ and $\Phi_n^*$ may be found by a finite number (although very large in general) of simple and realizable operations.

Let $E$ be a boundary of a domain $D = D(E)$ containing point $\infty$ in its interior. Let $d(E) > 0$. Suppose $b(z)$ is defined and lower semicontinuous in $E$. If

$$x^{(n)} = \{x_1^{(n)}, \ldots, x_n^{(n)}\}, \qquad n = 1, 2, \ldots,$$

is a triangular sequence of points of $C$, we define

$$\omega(z, x^{(n)}) = (z - x_1^{(n)})\ldots(z - x_n^{(n)}), \qquad n = 1, 2, \ldots$$

LEMMA 11.1. *Let $x^{(n)} = \{x_1^{(n)}, \ldots, x_n^{(n)}\}$ be a triangular sequence of points of $E$. If for every point $z \in D$*

$$\sqrt[n]{|\omega(z, x^{(n)})|} \to \mu(E, b)\Phi(z, E, b), \tag{11.1}$$

*then for every real function $B(z)$ defined and continuous in $E$ we have*

$$B_0 = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} B(x_i^{(n)}) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} B(\eta_i^{(n)}),$$

*where $\eta^{(n)} = \{\eta_0^{(n)}, \eta_1^{(n)}, \ldots, \eta_n^{(n)}\}$ is an extremal system of $E$ with respect to $b(z)$.*

Proof. By Remark 8.1, for $\varepsilon > 0$ there is a polynomial $q(z) = a(z - a_1)\ldots(z - a_k)$ such that $a_i \in D$ and

$$-\varepsilon + \mathrm{Log}|q(z)| \leqslant B(z) \leqslant \log|q(z)| + \varepsilon, \qquad z \in E. \tag{11.2}$$

Observe that in view of (11.1)

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\operatorname{Log}|a_j-x_i^{(n)}| = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log|a_j-\eta_i^{(n)}|$$
$$= \operatorname{Log}[\mu(E,b)\Phi(a_j,E,b)],$$

whence the sequences

$$\left\{\frac{1}{n}\sum_{i=1}^{n}[\operatorname{Log}|q(x_i^{(n)})|+\varepsilon]\right\} \quad\text{and}\quad \frac{1}{n}\sum_{i=1}^{n}[\operatorname{Log}|q(\eta_i^{(n)})|+\varepsilon]$$

converge to the same limit equal to

$$\varepsilon+\operatorname{Log}|a|+k\operatorname{Log}\mu(E,b)+\operatorname{Log}\prod_{i=1}^{k}\Phi(a_i,E,b).$$

The result follows now from (11.2) due to the arbitrariness of $\varepsilon>0$.

LEMMA 11.2 *If*

$$\max_{z\in E}|\omega(z,x^{(n)})\exp[-nb(z)]|^{1/n}\to\mu(E,b),$$

*then*

$$\sqrt[n]{|\omega(z,x^{(n)})|}\to\mu(E,b)\Phi(z,E,b) \tag{11.3}$$

*uniformly in every compact subset of CE.*

Proof. The functions

$$\frac{1}{n}\operatorname{Log}|\omega(z,x^{(n)})|$$

being harmonic and loco uniformly bounded outside of $E$, it suffices to prove the pointwise convergence of (11.3). If $z\in D$, the convergence may be proved by the repetition of the reasoning used in the proof of Theorem 4.1. Let now $z_0$ be a fixed finite point of $CE$. The function $B(z) = \operatorname{Log}|z_0-z|$ is continuous on $E$. By Lemma 11.1,

$$\lim_{n\to\infty}\frac{1}{n}\operatorname{Log}|\omega(z_0,x^{(n)})| = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\operatorname{Log}|z_0-x_i^{(n)}| = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\operatorname{Log}|z_0-\eta_i^{(n)}|.$$

But

$$\lim\frac{1}{n}\sum_{i=1}^{n}\operatorname{Log}|z_0-\eta_i^{(n)}| = \operatorname{Log}[\mu(E,b)\Phi(z_0,E,b)];$$

hence the result.



Let $\{\varepsilon_n\}$ be a sequence of real numbers and $\{E_n\}$ a sequence of subsets of $E$ such that:

1° $\sqrt[n]{\varepsilon_n}\to 0$ and $0<\varepsilon_n<1$, $n=1,2,\ldots$

2° Every set $E_n$ contains only a finite number of points

$$E_n=\{e_1^{(n)},\ldots,e_{n^*}^{(n)}\}$$

such that for any $z\in E$ there exists a point $e_n(z)$ of $E_n$ such that

$$z-e_n(z)=\hat{\varepsilon}_n,$$

where $\hat{\varepsilon}_n$ denotes a complex number satisfying the inequality $|\hat{\varepsilon}_n|<\varepsilon_n$.

3° $E_n\subset E_{n+1}$, $n=1,2,\ldots$

Example 1. Let $E$ be contained in the square $Q=\{z\mid |\operatorname{Re}z|\leqslant R, |\operatorname{Im}z|\leqslant R\}$. Put $\varepsilon_n=\sqrt{2}R/2^{n^2}$, $n=1,2,\ldots$ Devide $Q$ into small closed squares $Q_i^{(n)}$, $i=1,\ldots,4^{n^2+1}$ by means of the lines

$$\operatorname{Re}z=k\frac{R}{2^{n^2}},\quad \operatorname{Im}z=l\frac{R}{2^{n^2}},\quad k,l=\pm1,\pm2,\ldots,\pm2^{n^2}.$$

Denote by $q_1^{(n)},q_2^{(n)},\ldots,q_{n_*}^{(n)}$ all the small squares so obtained such that $q_k^{(n)}\cap E\neq\emptyset$, $k=1,2,\ldots,n_*$. Take an arbitrary point $e_k^{(n)}$ of $q_k^{(n)}\cap E$ and form $F_n=\{e_1^{(n)},\ldots,e_{n*}^{(n)}\}$. Define $E_n=\bigcup_{k=1}^{n}F_k$. The sequences $\{\varepsilon_n\}$ and $\{E_n\}$ satisfy 1°, 2° and 3°.

Example 2. Let $E\colon z=z(t)$, $0\leqslant t\leqslant 1$, be a rectifiable plane curve of length $L$. Let $0<t_1^{(n)}<t_2^{(n)}<\ldots<t_{n_*}^{(n)}=1$ $(n_*=2^{n^2})$ be chosen in such a way that

$$\int_{t_i^{(n)}}^{t_{i+1}^{(n)}}|z'(t)|\,dt=L\cdot2^{-n^2},\quad i=1,2,\ldots,n_*.$$

The sets $E_n=\{z(t_1^{(n)}),\ldots,z(t_{n*}^{(n)})\}$ and numbers $\varepsilon_n=L/2^{n^2}$ have the properties 1°, 2° and 3°. Observe that the selection of the sets $E_n$ is especially easy if $E$ is composed of a finite number of line segments, which is practically the most important case.

THEOREM 11.1. *Let $E$ be a bounded closed set and let $b(z)$ be a real function continuous in $E$. Suppose $\{\varepsilon_n\}$ and $\{E_n\}$ satisfy 1°, 2° and 3°. Define $\mu_n^*(E,b)$ by*

$$\mu_n^*(E,b)=\min_{x^{(n)}\subset E_n}\{\max_{z\in E_n}|\omega(z,x^{(n)})|\exp[-nb(z)]\},$$

*where $\omega(z,x^{(n)})=(z-x_1^{(n)})\ldots(z-x_n^{(n)})$. Then*

$$\operatorname*{Lim}_{n\to\infty}\sqrt[n]{\mu_n^*(E,b)}=\mu(E,b).$$

Proof. At first we shall prove that for every system of points $x^{(n)} = \{x_1, \ldots, x_n\}$ we have

(i) $$\max_{z\in E_n} |\omega(z, x^{(n)}) \exp[-nb(z)]| \leqslant \max_{z\in E} |\omega(z, x^{(n)}) \exp[-nb(z)]|$$
$$\leqslant \{\max_{z\in E_n} |\omega(z, x^{(n)}) \exp[-nb(z)]| + \varepsilon_n (1+M)^n\} \exp(n\delta_n),$$

where

$$M = \sup_{z,\zeta\in E} \{|z-\zeta| \exp[-b(z)]\} \quad \text{and} \quad \delta_n = \max_{z\in E} |b(z) - b(e_n(z))|.$$

Indeed, the first inequality is obvious because $E_n \subset E$. To prove the second one, observe that by 2° for every $z\in E$ there is $e_n = e_n(z) \in E_n$ such that $z = e_n + \hat{\varepsilon}_n$, whence

$$|\omega(z, x^{(n)}) \exp[-nb(z)]| = |\omega(e_n + \hat{\varepsilon}_n, x^{(n)})| \exp[-nb(e_n)] \exp[nb(e_n) - nb(z)]$$
$$\leqslant \{|\omega(e_n, x^{(n)})| \exp[-nb(e_n)] + (M+\varepsilon_n)^n - M^n\} \exp n\delta_n,$$

and this implies the second inequality of (i). It follows from (i) that

$$\mu_n^*(E, b) \leqslant \mu_n(E, b) \leqslant [\mu_n^*(E, b) + \varepsilon_n (M+1)^n] \exp(n\delta_n), \qquad n = 1, 2, \ldots$$

By the continuity of $b(z)$, $\delta_n \to 0$. Hence and from 1° we obtain the result.

Observe that $E_n$ contains only a finite number of points and therefore $\mu_n^*(E, b)$ may be found by a finite number of trials.

Take now the points $\xi^{(n)} = \{\xi_0^{(n)}, \xi_1^{(n)}, \ldots, \xi_n^{(n)}\}$ defined by

$$\mu_n^*(E, b) = \min_{x^{(n)}\subset E_n} \{\max_{z\in E_n} |\omega(z, x^{(n)})| \exp[-nb(z)]\}$$
$$= |\omega(\xi_0^{(n)}, \xi^{(n)})| \exp[-nb(\xi_0^{(n)})].$$

In virtue of Theorem 11.1 and (i) we have

$$\lim_{n\to\infty} \{\max_{z\in E_n} |\omega(z, \xi^{(n)})| \exp[-nb(z)]\}^{1/n}$$
$$= \lim_{n\to\infty} [\max_{z\in E} |\omega(z, \xi^{(n)}) e^{-nb(z)}|]^{1/n} = \mu(E, b).$$

Therefore by Lemma 11.2 we have

$$|\omega(z, \xi^{(n)}) e^{-nb(z)}|^{1/n} \to \mu(E, b) \Phi(z, E, b)$$

uniformly in every compact subset of $CE$.

Remarks. 1° Let $y^{(n)} = \{y_1^{(n)}, \ldots, y_n^{(n)}\}$ be an $n$-th extremal system of $E_n$ with respect to $b(z)$. By means of (i) one can easily prove that $\{\sqrt[n]{|\Phi^{(0)}(z, y^{(n)}, b)|}\}$ converges to $\Phi(z, E, b)$ uniformly in every compact subset of $CE$.



2° Let $a_0$ be a fixed point of $E$. Assuming that $a_0, a_1, \ldots, a_n$ are given, we define $a_{n+1}$ by

$$R_{n+1} = |(a_{n+1} - a_0) \ldots (a_{n+1} - a_n)| = \max_{z\in E} |(z-a_0) \ldots (z-a_n)|$$

Leja has proved in [16] that $\sqrt[n]{R_n} \to d(E)$.

Let $x_0$ be a fixed point of $E$. Assuming that $x_0, x_1, \ldots, x_n$, $n \geqslant 0$, are given, we define $x_{n+1}$ by

$$R_{n+1}^* = |(x_{n+1} - x_0) \ldots (x_{n+1} - x_n)| = \max_{z\in E_{n+1}} |(z-x_0) \ldots (z-x_n)|.$$

Using inequalities (i) and the result of Leja we can easily prove that $\sqrt[n]{R_n^*} \to d(E)$.

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# *ON SOME EXTREMAL FUNCTIONS OF LEJA IN THE SPACE*

BY

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Let $R^m$ be $m$-dimensional Euclidean space, $m \geqslant 2$, $E$ a closed and bounded set in $R^m$ and

$$\omega(p, q) \overset{\mathrm{df}}{=} \begin{cases} |p-q| & \text{for} \quad m = 2, \\ e^{-|p-q|^{2-k}} & \text{for} \quad m \geqslant 3. \end{cases}$$

Let $q^{(n)} = \{q_0, q_1, \ldots, q_n\}$ be an $n$-th extremal system of $E$ with respect to $\omega(p, q)$ (see [2, 4]) and $p^{(n)} = \{p_0, p_1, \ldots, p_n\}$ an arbitrary system of $n+1$ different points of $E$. We put

$$A_n(r) \overset{\mathrm{df}}{=} \max_{(j)} \prod_{\substack{k=0 \\ k\neq j}}^{n} \frac{\omega(r, q_k)}{\omega(q_j, q_k)}, \qquad B_n(r) \overset{\mathrm{df}}{=} \inf_{p^{(n)}\subset E} \left\{\max_{(j)} \prod_{\substack{k=0 \\ k\neq j}}^{n} \frac{\omega(r, p_j)}{\omega(p_k, p_j)}\right\}.$$

It is known [2, 4, 5] that if $m = 2$ and $r \notin E$, then

$$\lim_{n\to\infty} \frac{1}{n} \log A_n(r) = G(r), \tag{1}$$

$$\lim_{n\to\infty} \frac{1}{n} \log B_n(r) = G(r), \tag{2}$$

where $G(r) = I - u(r)$ (see below).

Let $D_\infty$ be the component of the complement of $E$ containing the point $r = \infty$ and $F_\infty$ the boundary of $D_\infty$. If $m = 2$ and $r \in D_\infty$, then $G(r)$ is the Green function for $D_\infty$ with a pole at infinity and for $r \notin D_\infty$ we have $G(r) = 0$ excepting a set of capacity zero.

The object of this paper is to prove (1) and (2) in general case (for $m \geqslant 2$).

Denote by $M$ the class of all positive Radon measures $\nu$ such that $\nu(E) = 1$ and $\nu(e \cap E) = 0$ if $e \cap E = \emptyset$.