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TROISIÈME CONFÉRENCE SUR LES FONCTIONS ANALYTIQUES

CRACOVIE, 30. VIII - 4. IX. 1962

Cette conférence, de même que la précédente(1) a été organisée par les soins de l'Institut Mathématique de l'Académie Polonaise des Sciences. Le Comité d'organisation a été présidé par le Professeur F. Leja. La conférence a réuni 66 participants, dont 46 de la Pologne et les autres des deux parties de l'Allemagne, des États Unis, de Finlande, d'Hongrie, de Tchécoslovaquie et de l'URSS, avec le Professeur R. Nevanlinna, Président de l'Union Mathématique Internationale.

La majorité des communications et des discussions concernaient diverses méthodes extrémales de la théorie des fonctions d'une variable complexe et celle des fonctions de plusieurs variables complexes.

Ce fascicule de Colloquium Mathematicum est consacré entièrement aux travaux présentés ou annoncés à la Conférence et à ceux qui s'y rattachent par leur sujet. Voici la liste chronologique complète des conférences de demi-heure, des communications plus courtes, des résumés et des données bibliographiques, parvenus au Comité d'organisation.

30. VIII. 1962. Z. Charzyński (Łódź), *The method of algebraic functions*.

The idea of the method announced above is to use, instead of the full class of functions univalent in the circle, the class of functions

(1) $$a(z) = a_1 z + a_2 z^2 + \ldots, \quad |z| < 1,$$

satisfying an equation

(2) $$\Omega\big(a(z)\big) = \chi(z),$$

where $\Omega(w)$ and $\chi(z)$ are arbitrary rational functions of the form

(3) $$\Omega(w) = \frac{D_L}{w^L} + \frac{D_{L-1}}{w^{L-1}} + \ldots + \frac{D_1}{w} + D_0 + \overline{D}_0 + \overline{D}_1 w + \ldots + \overline{D}_L w^L,$$

(3') $$\chi(z) = \frac{E_L}{z^L} + \frac{E_{L-1}}{z^{L-1}} + \ldots + \frac{E_1}{z} + E_0 + \overline{E}_0 + \overline{E}_1 z + \ldots + \overline{E}_L z^L.$$

(1) qui a été tenue à Lublin, 2-6. IX. 1958 (voir Colloquium Mathematicum 7 (1959), p. 123-153).

It may be proved that any function (1) satisfying (2) is univalent and maps the unit circle $K(0,1)$ onto the domain $K(0,1)-H$, where H is the finite sum of analytic arcs along which the relation $\operatorname{Im}\{\Omega(w)\} = 0$ holds. Next it is possible to establish the conditions for the pair of functions (3), (3′) by which the function (1) analytic in the unit circle $K(0,1)$ exists and satisfies (2). These conditions can be given in a form of finitely many algebraic relations between the coefficients of the functions (3) and (3′) and the zeros of $\Omega'(w)$ and $\chi'(z)$; they are necessary and locally sufficient conditions; they define also the variation of the function (1).

Using these conditions it is possible to obtain by elementary arithmetical methods many classical results, for instance Loewner's equations, Riemann's mapping theorem and Schiffer's variation formula.

The similar theory is to be formed for functions

$$A(z) = z+A_2z^2+\ldots, \quad |z|<1, \tag{4}$$

which satisfy a simpler equation of the form

$$P\big(A(z)\big) = z, \tag{5}$$

where $P(w)$ is a polynomial of the form

$$P(w) = w+C_2w^2+\ldots+C_Lw^L. \tag{6}$$

It is interesting to notice that for the functions (4) realizing the extrema of coefficients the corresponding polynomial (6) has such a property that all zeros of its derivative $P'(w)$ are situated on the lemniscate $|P(w)| = 1$.

30. VIII. 1962. G. Szegö (Stanford, California), *On harmonic polynomials in higher Euclidean spaces.*

30. VIII. 1962. M. Marden (Milwaukee), *The critical points of a linear combination of Green's functions.*

Let R be an infinite two-dimensional region bounded by a Jordan configuration B. Let $p_k(z)$, for $k = 1, 2, \ldots, m$, be polynomials of degree n_k with all of their zeros in R and let L be the intersection of the m lemniscatic regions L_k: $|p_k(z)| > \varrho_k$, $\varrho_k > 0$, $k = 1, 2, \ldots, m$, with ϱ_k chosen so that $L \cap R \neq 0$. Let the Green's function with a pole at infinity be denoted by $G(x, y)$ for R and by $g_k(x, y)$ for L_k. Then the function

$$\Phi(x, y) = G(x, y)+\sum_{k=1}^{m}\lambda_k g_k(x, y), \qquad \lambda_k > 0,$$

has at most $N = \sum_1^m n_k$ critical points outside a certain star-shaped region containing the convex hull of B and dependent upon N, but not upon the location of the zeros of the $p_k(z)$.

30. VIII. 1962. C. Rényi (Budapest), *On some questions concerning lacunary power series* (voir ce fascicule, p. 165-171).

30. VIII. 1962. H. Grunsky (Würzburg), *Über die Umkehrung linearer Differentialoperatoren 2. Ordnung im Komplexen* (voir Archiv der Mathematik 14 (1963), p. 247-251, et H. Grunsky, *Über die Reduktion eines linearen Differentialoperators 2. Ordnung im Komplexen*, ibidem, à paraître).

Die Funktionen $p_1(z)$ und $p_2(z)$ seien in einem ganz im endlichen liegenden Gebiet der z-Ebene holomorph; wir bilden den Differentialoperator

$$L(w) = w''+p_1(z)w'+p_2(z)w.$$

L^* sei der adjungierte Operator, ω und $\tilde{\omega}$ linear unabhängige Lösungen von $L^*(\omega) = 0$. Wird der Bereich B passend gewählt, so läßt sich

$$\iint_B \bar{\psi}\omega L(w)(dz) \qquad \text{mit} \qquad \psi = \left(\frac{\tilde{\omega}}{\omega}\right)' = \omega^{-2}\exp\int p_1 dz$$

vollständig integrieren, d. h. im wesentlichen ausdrücken durch Werte von w in den „Ecken" von B. Die Wahl von B ist so zu treffen, daß bei Abbildung mittels $\mathfrak{w} = \tilde{\omega}/\omega$ ein geradlinig berandetes Polygon entsteht; dabei darf keine Nullstelle von ω in B oder auf dem Rande liegen. Von der letztgenannten Bedingung kann man sich frei machen. Die Formel leistet bei Spezialisierung auf ein „Dreieck" dasselbe, wie im Reellen die Transformation mittels der Greenschen Funktion:

$$\int_{x_1}^{x_2} G(x, \xi)L\big(y(\xi)\big)d\xi.$$

30. VIII. 1962. J. Krzyż (Lublin), *On a problem of P. Montel.*

In 1933 P. Montel suggested to investigate the class of functions $F(z)$ regular and univalent in the unit circle K which satisfy $F(0) = 0$, $F(z_0) = 1$ $(0 < |z_0| < 1)$. The determination of precise bounds of $|F(z)|$ is equivalent to the problem of estimating $|f(z_1)/f(z_2)|$ with fixed z_1, z_2 and varying univalent $f(z)$ which vanishes at the origin. The variational methods enable us to solve this problem. We have

$$k^{-1}(z_2, z_1) \leqslant |f(z_1)/f(z_2)| \leqslant k(z_1, z_2),$$

where $k(z_1, z_2)$ is defined as follows. Put

$$Q(z, \alpha) = \frac{e^{-i\alpha}(z-e^{i\alpha})^2}{z(z-z_1)(z-z_2)(1-\bar{z}_1z)(1-\bar{z}_2z)}, \qquad \alpha \text{ real},$$

$$\Omega_j(\alpha) = \int_{\Gamma_j}\sqrt{Q(\xi, \alpha)}\,d\xi = \Omega_j, \qquad j = 1, 2,$$

where Γ_j is a cycle consisting of 2 small circles with centres at 0 and z_j described in the positive direction and of a straight line segment joining both circumferences, z_k $(k \neq j)$ being possibly omitted on a small semi-circle. If $\tau(\alpha) = \mp\Omega_2(\alpha)/\Omega_1(\alpha)$, with the sign so chosen that $I(\tau(\alpha)) > 0$ and if α_0 minimizes $|\lambda(\tau(\alpha)+1)|$, where $\lambda(\tau)$ is the elliptic modular function (the Jacobian modulus), then

$$k(z_1, z_2) = |\lambda(\tau(\alpha_0)+1)|^{-1}.$$

The extremal function has the form

$$f(z) = \wp\left[\int_0^z \sqrt{Q(\xi, \alpha_0)}\, d\xi + \frac{1}{2}(\Omega_1+\Omega_2)\right] + e_1 + e_2,$$

where $e_k = e_k(\alpha_0) = \wp\left(\frac{1}{2}\Omega(\alpha_0)\right)$, $k = 1, 2$ and $\wp$ is the Weierstrass's elliptic function with periods $\Omega_1(\alpha_0)$ and $\Omega_2(\alpha_0)$. The Loewner method yields an approximate solution in terms of elementary functions:

$$\left|\frac{z_2 f(z_1)}{z_1 f(z_2)}\right| \leqslant \left(\frac{|1-z_1\bar{z}_2| + |z_1-z_2|}{1-|z_1|^2}\right)^2 \left(\frac{|1-z_1\bar{z}_2| + |z_1-z_2|}{|1-z_1\bar{z}_2| - |z_1-z_2|}\right)^{2|z_1z_2|/(1-|z_1z_2|)}.$$

Making $z_1 \to 0$, resp. $z_2 \to 0$ we obtain classical estimations of $|f(z)|$

31. VIII. 1962. S. Bergman (Stanford, Cal.), *On meromorphic functions of two complex variables.*

This lecture was based on following notes by the author: *Some properties of meromorphic functions of two complex variables*, to appear in Bulletin de la Société des Sciences et des Lettres de Łódź, 1963, and *On value distribution of meromorphic functions of two complex variables*, Studies in Mathematical Analysis and Related Topics (Essays in honor of G. Pólya), Stanford 1962.

31. VIII. 1962. J. Górski (Cracow), *Some applications of the method of extremal points in the theory of analytic functions of one complex variable* (voir ce fascicule, p. 151-156).

31. VIII. 1962. J. Siciak (Cracow), *Extremal points in the space C^n* (voir ce fascicule, p. 157-163).

31. VIII. 1962. W. Tutschke (Berlin), *Bemerkung zum Dirichletschen Prinzip in mehrfach zusammenhängenden Gebieten.*

Betrachtet werden in einem Gebiet G mit q Randkurven Lösungen des elliptischen Systems

$$\frac{\partial u^*}{\partial x_2} = a_{11}\frac{\partial u}{\partial x_1} + a_{12}\frac{\partial u}{\partial x_2}, \qquad -\frac{\partial u^*}{\partial x_1} = a_{12}\frac{\partial u}{\partial x_1} + a_{22}\frac{\partial u}{\partial x_2}.$$

Sucht man Lösungen, für die u in G eindeutig ist und u^* längs der Randkurven vorgeschriebene Perioden d_k besitzt, so wird die Dirichlet-norm

$$(1) \qquad \iint_G \sum_{i,j} a_{ij}\frac{\partial u}{\partial x_i}\cdot\frac{\partial u}{\partial x_j}\,dx_1\,dx_2$$

dann minimal, falls u auf den Randkomponenten jeweils konstante Werte c_k annimmt. Ist (A_{ik}) die Periodenmatrix und $r \leqslant q-1$

$$(C_{i_\mu i_\nu}) = (A_{i_\mu i_\nu})^{-1}_{\mu,\nu=1,\ldots,r},$$

so kann man das Dirichletintegral als Summe zweier positiv definiter quadratischer Formen schreiben, nämlich

$$(2) \qquad \sum_{\mu,\nu=1}^{r} C_{i_\mu i_\nu} d_{i_\mu} d_{i_\nu} + \sum_{k,l=r+1}^{q}\left(A_{i_l i_k} - \sum_{\mu,\nu=1}^{r} A_{i_\nu i_k} C_{i_\nu i_\mu} A_{i_l i_\mu}\right) c_{i_k} c_{i_l}.$$

Hieraus folgt: Schreibt man für u^* nur die Perioden längs r Randkomponenten vor, so wird die Norm minimal, falls $c_{i_k} = 0$ für $k = r+1, \ldots, q$ ist.

Eine ausführlichere Darstellung gleichen Titels erscheint in den Monatsberichten der Deutschen Akademie der Wissenschaften, Berlin, Heft 11/12 (1962).

31. VIII. 1962. K. Stein (München), *Über holomorphe Abbildungen komplexer Räume.*

Jede eigentliche holomorphe Abbildung eines reduzierten komplexen Raumes läßt sich in natürlicher Weise als Produkt einer maximalen Abbildung, einer Überlagerungsabbildung und einer quasi-injektiven Abbildung darstellen. Es wird gezeigt, daß ähnliche Faktorisierungsaussagen für weitere Klassen holomorpher Abbildungen gelten. Insbesondere wird die Frage nach der Existenz maximaler Abbildungen behandelt, die mit vorgegebenen Abbildungen verwandt sind. Abschließend werden Anwendungen diskutiert.

31. VIII. 1962. M. M. Schiffer (Stanford, Cal.), *The coefficient problem for univalent functions* (voir P. L. Duren and M. M. Schiffer, *The theory of second variation in extremum problems for univalent functions*, Journal d'Analyse (1963), à paraître).

31. VIII. 1962. М. Я. Антоновский (Ташкент), *Открытые отображения.*

31. VIII. 1962. F. W. Gehring (Ann Arbor, Mich.), *Some distorsion theorems for conformal mappings* (voir sous le même titre F. W. Gehring and W. K. Hayman, Journal des Mathématiques, à paraître).

31. VIII. 1962. Z. Lewandowski (Lublin), *A generalization of a result of J. A. Jenkins* (voir A. Bielecki, J. Krzyż and Z. Lewandowski, *On typically-real functions with a preassigned second coefficient*, Bulletin de l'Académie Polonaise des Sciences, Série de sciences mathématiques, astronomiques et physiques 10 (1962), p. 205-208).

31. VIII. 1962. M. O. Reade (Ann Arbor, Mich.), *On sections of certain power series* (voir ce fascicule, p. 173-179).

1. IX. 1962. A. Bielecki (Lublin), *Quelques résultats récents sur les majorantes dans la théorie des fonctions holomorphes* (voir ce fascicule, p. 141-145).

1. IX. 1962. O. Tammi (Helsinki), *On a method of extremalization in the theory of univalent functions* (voir Annales Societatis Scientiarum Fennicae, Series A, I (320), p. 6.

1. IX. 1962. J. Krzyż (Lublin), *On some recent results in the theory of analytic functions* (voir ce fascicule, p. 147-150).

1. IX. 1962. Б. В. Шабат (Москва), *Квазиконформные отображения пространственных областей.*

1. IX. 1962. R. Nevanlinna (Helsinki), *On systems of complex numbers and rings* (voir Commentationes Physico-Mathematicae).

1. IX. 1962. B. Piłat (Lublin), *On a class of univalent functions.*

Let S_0 denote the class of functions regular and univalent in the unit circle $K(1)$ of the form $f(z) = a_1z+a_2z^2+\ldots$ subject to the conditions $f(z_0) = z_0$, $(0 < |z_0| < 1)$ and $\sum\limits_{n=2}^{\infty} |a_n|(n+|z_0|^{n-1}) \leqslant 1$. For $f \epsilon S_0$ the following results are true:

1. $f(z)$ is star-shaped in $K(1)$;
2. $2(2+|z_0|)^{-1} \leqslant |a_1| \leqslant 2(2-|z_0|)^{-1}$;
3. $f(K)$ contains the open disc $|w| < (2+|z_0|)^{-1}$;
4. $f\big(K(1/2)\big)$ contains the open disc $|w| < 3/4\,(2+|z_0|)^{-1}$;
5. The radius of convexity for S_0 is equal $1/2$;
6. The area A of $f\big(K(1)\big)$ satisfies the condition

$$4\pi(2+|z_0|)^{-2} \leqslant A \leqslant 6\pi(2-|z_0|)^{-2};$$

7. $2(|z|-\frac{1}{2}|z|^2)(2+|z_0|)^{-1} \leqslant |f(z)| \leqslant 2(|z|+\frac{1}{2}|z|^2)(2-|z_0|)^{-1}$;
8. $2(1-|z|)(2+|z_0|)^{-1} \leqslant |f'(z)| \leqslant 2(1+|z|)(2-|z_0|)^{-1}$.

All these bounds are sharp.

1. IX. 1962. J. Ławrynowicz (Łódź), *On the series of Meijer's functions* (voir Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 12 (1964), à paraître).

The author proves on certain conditions the formula

$$(*)\qquad G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}\left(\eta\omega\,\middle|\,\begin{matrix} b_1,\ldots,b_m,c_1,\ldots,c_\sigma,b_{m+1},\ldots,b_q \\ a_1,\ldots,a_n,d_1,\ldots,d_\tau,a_{n+1},\ldots,a_p\end{matrix}\right)$$

$$= \sum_{r=0}^{\infty}\sum_{h=1}^{m}\frac{1}{r!}\,\frac{\prod\limits_{j\neq h,j=1}^{m}\Gamma(b_h-b_j)\prod\limits_{j=1}^{n}\Gamma(1-b_h+a_j)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_h+b_j)\prod\limits_{j=n+1}^{p}\Gamma(b_h-a_j)}\times$$

$$\times\left(\frac{t}{\eta}\right)^{1-b_h}{}_{p+1}F_{q-1}\left(\begin{matrix}-r,1-b_h+a_1,\ldots,1-b_h+a_p; \\ 1-b_h+b_1,\ldots * \ldots,1-b_h+b_q;\ (-1)^{p-m-n+1}t/\eta\end{matrix}\right)\times$$

$$\times G^{\mu,\nu+1}_{\sigma+1,\tau}\left(\omega t\,\middle|\,\begin{matrix} b_h-r,c_1,\ldots,c_\sigma \\ d_1,\ldots,d_\tau\end{matrix}\right)$$

by generalizing Meijer's expansions ([2]).

In the proof the well known formula concerning

$$\int G^{m,n}_{p,q}(x/\eta|{}^{-a_k}_{-b_j})\,G^{\mu,\nu}_{\sigma,\tau}(\omega x|{}^{c_k}_{d_j})\,dx$$

was applied; $G^{m,n}_{p,q}$ was expressed by ${}_pF_{q-1}$ and then, after multiplication and division of integrand by $\exp(x/t)$ (where $\mathrm{Re}\,t > 0$), the products $\exp(x/t)\,{}_pF_{q-1}\big((-1)^{p-m-n}x/\eta\big)$ were expanded into the power series with respect to x and the expressions

$$x^{-b_h+r}\exp(-x/t)\,G^{\mu,\nu}_{\sigma,\tau}(\omega x)\qquad (r = 0, 1, \ldots)$$

were integrated with result (*).

Further, another formula of type (*) was found by interchanging the role of $G^{m,n}_{p,q}$ and $G^{\mu,\nu}_{\sigma,\tau}$, and a formula concerning

$$\int_0^{\infty} G^{m,n}_{p,q}\left(\eta x\,\middle|\,\begin{matrix}a_k\\ b_j\end{matrix}\right) G^{\mu,\nu}_{\sigma,\tau}\left(\omega/x\,\middle|\,\begin{matrix}c_k\\ d_j\end{matrix}\right)\frac{dx}{x}$$

analogous to one used previously was proved, from which by the method of introducing factor $\exp(t/x)$ two more formulae of type (*) were obtained.

The results are in relation with the author's paper on phase shifts in modern physics ([3]).

([2]) C. S. Meijer, *Expansion theorems for the G-function, I-XI*, Proceedings of the Section of Sciences, Koninklijke Nederlandse Akademie van Wetenschappen 55 (1952), p. 369-379 and p. 483-487; 56 (1953), p. 43-49, p. 187-193, p. 349-357; 57 (1954), p. 77-82, p. 83-91, p. 273-279; 58 (1955), p. 243-251, p. 309-314, and 59 (1956), p. 70-82.

([3]) J. Ławrynowicz, *Schwinger's method generalized for the arbitrary quantum numbers*, Bulletin de la Société des Sciences et des Lettres de Łódź 12 (1961), nº 3, p. 1-8.

1. IX. 1962. G. Weiss (St. Louis, Missouri), *H^p-spaces on tubes.*

1. IX. 1962. J. Fuka (Prag), *Über die Carathéodorysche Grenze* (voir Colloquium Mathematicum, à paraître).

1. IX. 1962. F. Frolík (Prag), presenté par J. Fuka, *Δ-Räume und uniformisierbare Δ-Räume.*

3. IX. 1962. A. Zygmund (Chicago), *On an analogue in real variable of the area theorem in complex variable* (voir E. M. Stein and A. Zygmund, *On the differentiability of functions*, Studia Mathematica 23 (1964), p. 247-283).

3. IX. 1962. H. Royden (Stanford, Cal.), *Dual extremal problems on finite Riemann surfaces* (voir *The boundary values of analytic and harmonic functions*, Mathematische Zeitschrift 78 (1962), p. 1-24).

3. IX. 1962. L. Siewierski (Łódź), *The local solution of coefficient problem for bounded univalent functions.*

If for $|z| < 1$ the function

$$F(z) = z + A_2 z^2 + \ldots, \tag{1}$$

is univalent and

$$|F(z)| < M \quad (1 < M < \infty), \tag{2}$$

then we have the following sharp inequalities

$$|A_2| \leqslant 2\left(1 - \frac{1}{M}\right) \quad \text{for} \quad 1 < M < +\infty, \tag{3}$$

$$|A_3| \leqslant \begin{cases} \left(1 - \dfrac{1}{M^2}\right) & \text{for} \quad 1 < M \leqslant e, \\ 2\lambda^2 + 1 - \dfrac{4\lambda}{M} + \dfrac{1}{M^2} & \text{for} \quad e \leqslant M < +\infty, \end{cases} \tag{4}$$

where λ is the greater root of the equation $\lambda \log \lambda = -1/M$.

The general formulation of a hypothesis analogous to that of Bieberbach in this case seems to be difficult and is so far unknown. However, inequalities (3) and (4) suggest another conjecture, namely that there exists some uniform estimate for bounded univalent functions near the identity. And so a hypothesis due to Z. Charzyński says that, given arbitrary $N \geqslant 2$, when $M > 1$ is sufficiently near unity, then for every function satisfying condition (2) the precise estimate

$$|A_N| \leqslant \frac{2}{(N-1)}\left(1 - \frac{1}{M^{N-1}}\right) \tag{5}$$

holds true.

The author has presented a proof of Charzyński's hypothesis for odd N:

THEOREM. *For odd natural $N \geqslant 3$ there exists an $M_N > 1$ such that for every $M \leqslant M_N$ and for every bounded univalent function* (1) *the sharp inequality in* (5) *holds; equality in* (5) *is attained for a function of Koebe's type of the form*

$$H^*(z) = Mh^*(z) = z + \frac{2}{N-1}\left(1 - \frac{1}{M^{N-1}}\right)(e^{i\alpha})^{N-1} z^N + \ldots, \tag{6}$$

where

$$\frac{h^*(z)}{\left[1 - \left(e^{i\alpha} h^*(z)\right)^{N-1}\right]^{2/(N-1)}} = \frac{1}{M} \frac{z}{\left[1 - (e^{i\alpha} z)^{N-1}\right]^{2/(N-1)}}, \tag{7}$$

$$|z| < 1, \qquad |h^*(z)| < 1.$$

3. IX. 1962. J. Śladkowska (Łódź), *Bounds of analytic functions of two complex variables in domain with the Bergman-Shilov boundary.*

Let B^4 be an analytic polyhedron ([4]). The boundary b^3 consists of n segments e_k^3 of analytic hypersurfaces. Each of the segments can be expressed in the form

$$z_\varkappa = h_{\varkappa k}(z_k, \lambda_k), \qquad \varkappa = 1, 2, \qquad |z_k| \leqslant 1, \qquad \lambda_k \in \langle 0, 2\pi \rangle, \tag{$*$}$$

where $h_{\varkappa k}$ are continuously differentiable functions. The set ($*$) for fixed λ_k is called *lamina* $J_k^2(\lambda_k)$. F^2 is the Bergman-Shilov boundary of B^4. Let G_0^2 be an analytic surface which intersects B^4. Let $G^2 = G_0^2 \cap B^4$ anb $g^1 = G_0^2 \cap b^3$.

Let f be a function of the complex variables, z_1, z_2, regular in $\bar{B}^4 - F^2$, continuous in $\bar{B}^4 - F^2 \cup g^1 \cap F^2$. We write $f \in F_B(G_0^2, P)$ for $P > 0$, if

(1) $f(z_1, z_2) \neq 0$ in $B^4 \cup g^1 \cap F^2$,

(2) in every $J_k^2(\lambda_k)$, which is intersected by G_0^2, f is mean-multivalent of the order $p_k(\lambda_k)$,

(3) $\int p_k^2(\lambda_k)\, d\lambda_k < \infty$ and $\sum\limits_k \left(\dfrac{1}{2\pi}\int p_k^2(\lambda_k)\, d\lambda_k\right)^{1/2} \leqslant JP$, where J denotes the number of segments e_k^3 intersected by G_0^2.

The author gives the bounds for $|f|$ in G^2 for every $f \in F_B(G_0^2, P)$.

3. IX. 1962. L. Mikołajczyk (Łódź), *The variability regions of the coefficients A_2 and A_3 of univalent, symmetrical and bounded functions.*

Let $S(M)$ denote the class of univalent functions of the form

$$F(z) = z + A_2 z^2 + A_3 z^3 + \ldots,$$

where $|z| < 1$, $|F(z)| < M$, $M = \text{const} > 1$ and $A_2, A_3, \ldots$ are real numbers.

([4]) See S. Bergman, *Über eine in gewissen Bereichen mit Maximumfläche gültige Integraldarstellung der Funktionen zweier komplexer Variablen*, Mathematische Zeitschrift 39 (1935), p. 76-94.

To each function $F(z) \in S(M)$ we assign the point (A_2, A_3). The set of all such points shall be denoted by B_{3M} and called the *variability region* of functions of the class in question. The author proves that

(1) B_{3M} is a closed and bounded region containing the point $(0,0)$ in its interior.

(2) If (A_2^*, A_3^*) is an arbitrary point of the boundary of B_{3M}, then there exists a function $F^*(z) \in S(M)$ such that

$$F^*(z) = z + A_2^* z^2 + A_3^* z^3 + \ldots$$

(3) The boundary of B_{3M} is the union of the four sets K_1, K_2, K_1', K_2' defined by the following parametrical equations:

$$K_1: \begin{cases} A_2^* = 2T - 2\cos\nu + 2\cos\nu\log\cos\nu, \\ A_3^* = A_2^{*2} + 2A_2^*\cos\nu - 4T\cos\nu + 2\cos^2\nu + 1 + T^2, \end{cases}$$
$$T \leqslant \cos\nu \leqslant 1;$$

$$K_1': \begin{cases} A_2^* = -2T - 2\cos\nu + 2\cos\nu\log(-\cos\nu), \\ A_3^* = A_2^{*2} + 2A_2^*\cos\nu + 4T\cos\nu + 2\cos^2\nu + 1 + T^2, \end{cases}$$
$$-1 \leqslant \cos\nu \leqslant -T;$$

$$K_2: \; A_2^* = 2\cos\nu\log T, \quad A_3^* = A_2^{*2} + 2A_2^*\cos\nu - T^2 + 1,$$
$$0 \leqslant \cos\nu \leqslant T;$$

$$K_2': \; A_2^* = 2\cos\nu\log T, \quad A_3^* = A_2^{*2} + 2A_2^*\cos\nu - T^2 + 1,$$
$$-T \leqslant \cos\nu \leqslant 0.$$

T denotes a constant from the interval $(0,1)$. The proof is based on the method of extremal functions.

3. IX. 1962. J. J. Kohn (Princeton, N. J.), *Potential theoretic methods in several complex variables* (voir J. J. Kohn, *Harmonic integrals on strongly pseudo-convex manifolds*, *I* and *II*, Annals of Mathematics, à paraître).

Let M be a strongly pseudo-convex manifold, $\mathscr{A}^{(p,q)}$ the space of C^∞ forms of type (p,q), $\mathscr{L}$ the space of square-integrable forms, $\bar\partial: \mathscr{A}^{(p,q)} \to \mathscr{A}^{(p+1,q)}$ the complex gradient, ϑ the formal adjoint of $\bar\partial$ and $\Box = \bar\partial\vartheta + \gamma\bar\partial$ the complex Laplace operator.

THEOREM. *There exists a bounded self-adjoint operator N on $\mathscr{L}$ such that*

$$\mathscr{A}^{(p,q)} \cap \mathscr{L} = \bar\partial\vartheta N(\mathscr{A}^{(p,q)} \cap \mathscr{L}) + \vartheta\bar\partial N(\mathscr{A}^{(p,q)} \cap \mathscr{L}) + \varkappa^{(p,q)},$$

where $\varkappa^{(p,q)}$ is a subspace of $\mathscr{A}^{(p,q)} \cap \mathscr{L}$ which is annihilated by $\Box$. Furthermore, $\bar\partial N = N\bar\partial$ and, if $q \neq 0$, N is completely continuous.

Let T be the closure of $\bar\partial$, let T^* be the Hilbert space adjoint of T and let $L = TT^* + T^*T$. The theorem is proved by showing that L has a closed range, and this follows from various *a priori* estimates on the form $(T_\varphi, T_\varphi) + (T_\varphi^*, T_\varphi^*)$.

The above theorem gives a representation of the $\bar\partial$-cohomology by harmonic forms. The main use of the theorem is in establishing the existence of various holomorphic functions and mappings and in obtaining explicit formulae for them in terms of the operator N.

3. IX. 1962. S. Rolewicz (Warsaw), *On formulas for coefficients of some classes of analytic functions of several variables* (voir *On Cauchy-Hadamard formulas for Köthe power spaces*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques 10 (1962), p. 211-216).

3. IX. 1962. W. Kleiner (Cracow), *A Dini-Lipschitz condition in potential theory* (voir *Une condition de Dini-Lipschitz dans la théorie du potentiel*, Annales Polonici Mathematici 14 (1963), p. 117-130).

Let σ be a signed measure on a plane curve $J \in C^1$; we define $[\sigma] = \sup(|\sigma(L)|,\ L$ arc of $J)$; then

$$|U^\sigma(z)| = \left|\int \log 1/|z-t|\, d\sigma(t)\right| \leqslant [\sigma]V(z), \qquad V(z) = \operatorname*{Var}_{t \in J} \log|z-t|.$$

Let ν be a fixed positive measure with continuous potential U^ν. Measures σ: $|\sigma(E)| \leqslant \nu(E)$, $\sigma(J) = 0$, satisfy an inequality implying for bounded density $\nu' \leqslant M$ that

$$[\sigma] \leqslant c_1(J, M)\|\sigma\|\log 1/\|\sigma\| \quad \text{if} \quad \|\sigma\|^2 = \int U^\sigma d\sigma \leqslant 1/c_2(J, M).$$

This inequality forms a base for subsequent estimations.

3. IX. 1962. W. Kleiner (Cracow), *Degree of approximation in Leja's extreme points method and its modification.*

Let $\sum_{j \neq k} \log|z_{jn} - z_{kn}|^{-1}$, $j, k = 1, \ldots, n$, $z_{kn} \in J$ (J a curve) be least for $z_{jn} = \eta_{jn}$. Leja proved that

$$f_n(z) = \sqrt[n]{(z-\eta_{1n})\ldots(z-\eta_{nn})} \to f(z),$$

$f(\mathrm{ext} J) = K = (w\colon |w| > d)$, d = transfinite diameter of J, $f'(\infty) = 1$. We prove for $J \in C^2$ (see above) that

$$[\eta_n - \lim \eta_n] \leqslant c_3(J) n^{-1/2} \log n$$

(η_n positive measure, $\eta_n(\{\eta_{kn}\}) = 1/n$, $\eta_n(\text{plane}) = 1$). Hence 1° $f_n(z)$ converge as $V(z) n^{-1/2} \log n$, 2° a similar result holds for Leja's method in Dirichlet problem, 3° polynomials $S_{n-1}(w)$ in $1/w$, $S_{n-1}(w_{kn}) = \eta_{k,n-1} - w_{kn}$, $w_{kn} = d \cdot e^{2\pi ik/n}$, converge to $f^{-1}(w) - w$ uniformly in $\bar{K}$, 4° n among n^2 equally spaced points of J approximate η_{kn} sufficiently.

Place in n points $a_{kn} \in J$ $(|a_{kn} - a_{k-1,n}| \approx |J|/n)$ masses m_{kn} minimizing $\sum_{jk} m_{jn} m_{kn} \log_n |a_{jn} - a_{kn}|^{-1}$, $\log_n r^{-1} = \min(\log r^{-1}, \log 2n/|J|)$ $(0 \leqslant r)$. Effective approximation results in equilibrium distribution, conformal mapping, and in similar manner in Dirichlet's problem with boundary values belonging to C^1 (no iterated limit). Convergence as $V(z) n^{-1/2} \log n$.

Five papers on these subjects are to be published in the Annales Polonici Mathematici, starting from the volume 14 (1963).

4. IX. 1962. R. Finn (Stanford, Cal.), *On the inclination of a minimal surface $\varphi(x, y)$* (voir ce fascicule, p. 195-201).

4. IX. 1962. F. Leja (Cracow), *Quelques problèmes concernant certaines fonctions extrémales de plusieurs variables complexes* (voir F. Leja, *Sur certaines suites de fonctions extrémales de plusieurs variables complexes*, Annales Polonici Mathematici 12 (1962), p. 105-114).

4. IX. 1962. J. Górski (Cracow), *Remark on a certain theorem of H. J. Bremermann.*

Bremermann [5] proved the following theorem:

Let D be a bounded pseudo-convex domain in the space of n complex variables $z = (z_1, z_2, \ldots, z_n)$ of the form $D = (z: V(z) < 0)$, where $V(z)$ is continuous, plurisubharmonic in a neighbourhood of D. Then the generalized Dirichlet's problem is possible for the upper envelope $\Phi(z)$ of the class $L(D, b(z))$ of functions which are plurisubharmonic in a neighbourhood of D and $\leqslant b(z)$, where $b(z)$ is an arbitrary continuous function on the boundary of D, if and only if $b(z)$ is prescribed on the Shilov boundary $S(D)$ of D.

The following is proved:

Let D be a bounded domain in the space of n complex variables regular with respect to the ordinary Dirichlet problem. Let $\tilde{S}(D)$ be the Shilov boundary of D with respect to the class M of all plurisubharmonic functions $\psi(z)$, $z \in D$, continuous in $\bar{D} = D + D^{\cdot}$ and $\leqslant b(z)$ on $D^{\cdot}$, where $b(z)$ is an arbitrary continuous function defined on the boundary $D^{\cdot}$. Then there exists in D the bounded upper envelope $\Phi(z)$ of the class M and $\overline{\lim\limits_{z \to z_0 \in \tilde{S}(D)}} \Phi(z) = b(z_0)$.

4. IX. 1962. W. Bach (Cracow), *A solution of the problem of four limits.*

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[5] H. J. Bremermann, *On a generalized Dirichlet problem for plurisubharmonic functions and pseudoconvex domains*, Transactions of the American Mathematical Society 91 (1959), p. 246-276.

Let R be the z-plane, $p^{(n)} = \{p_0, p_1, \ldots, p_n\}$ a system of $n+1$ points of a set E and $U_{jk} = U_{jk}(u, p^{(n)})$ the function defined as follows:

$$U_{jk} = \frac{z - p_k}{p_j - p_k} \quad \text{for} \quad j \neq k, \; U_{jj} = 1.$$

Denote by $A_n(z)$, $B_n(z)$, $C_n(z)$, and $D_n(z)$ the infima of the products

$$A_n(z; p^{(n)}) = \prod_{j=0}^{n} \prod_{k=0}^{n} |U_{jk}|, \qquad B_n(z; p^{(n)}) = \max_{(j)} \prod_{k=0}^{n} |U_{jk} U_{kj}|,$$

$$C_n(z; p^{(n)}) = \max_{(j)} \prod_{k=0}^{n} |U_{jk}|, \qquad D_n(z; p^{(n)}) = \max_{(j)} \prod_{k=0}^{n} |U_{kj}|,$$

when the system $p^{(n)}$ varies in E.

Leja [6] proved that if the capacity of E is positive, then the sequences $\sqrt[n(n+1)]{A_n(z)}$, $\sqrt[2n]{B_n(z)}$, $\sqrt[n]{C_n(z)}$, $\sqrt[n]{D_n(z)}$ are convergent respectively to $A(z)$, $B(z)$, $C(z)$, $D(z)$ and $A(z) \equiv B(z)$. Leja formulated the problem: prove that $A(z) = C(z) = D(z)$ or that at least one of these identities does not hold [7]. The author proves that $A(z) < C(z)$ and $C(z) = D(z)$, except the points of a certain set of capacity zero.

4. IX. 1962. B. Szafirski (Cracow), *Uniqueness of the extremal pseudopolynomials and their connection with the generalized ecart of a set* (voir Zeszyty Naukowe Uniwersytetu Jagiellońskiego 1964, à paraître).

Let $\omega(x, y)$ be a continuous, non-negative and symmetric function for $x, y \in R^m$, where R^m is m-dimensional Euclidean space. The expression

$$\prod_{i=1}^{n} \omega(x, p_i), \qquad p^{(n)} = (p_1, \ldots, p_n) \subset R^m$$

is called the *pseudopolynomial of degree n*. The function $\prod_{i=1}^{n} \omega(x, t_i)$ such that

$$\max_{x \in E} \prod_{i=1}^{n} \omega(x, t_i) = \inf_{p^{(n)} \subset R^m} \max_{x \in E} \prod_{i=1}^{n} \omega(x, p_i),$$

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[6] See F. Leja, *Sur certaines limites relatives aux polynômes de Lagrange et aux ensembles fermés*, Bulletin International de l'Académie Polonaise des Sciences et des Lettres 1933, nº 7A, p. 281-289; *Sur la définition du diamètre et de l'écart transfini d'un ensemble*, Annales de la Société Polonaise de Mathématique 12 (1934), p. 29-34, et *Sur une suite de fonctions liée aux ensembles plans fermés*, ibidem 13 (1935), p. 53-58.

[7] F. Leja, *Problèmes*, Annales de la Société Polonaise de Mathématique 17 (1938), p. 130.

where E is a compact subset of E^m, is called *Tchebycheff's extremal pseudopolynomial* of E with respect to $\omega(x, y)$. Under some assumptions on $\omega(x, y)$ and E the uniqueness of these extremal pseudopolynomials and their connection with the generalized ecart can be proved.

4. IX. 1962. F. Bierski (Cracovie), *Certaines généralisations de l'écart restreint* (voir F. Bierski, *Les écarts restreints: arithmétique, géométrique et harmonique*, Zeszyty Naukowe Akademii Górniczo-Hutniczej, Kraków 1964, à paraître).

Soient R un espace topologique, p un nombre entier au moins égal à 2, $z_1, z_2, \ldots, z_p$ un système de p points de R et $\omega(z_1, z_2, \ldots, z_p)$ une fonction réelle continue de p variables $z_1, \ldots, z_p$ dans R satisfaisant aux conditions:

$$(1)\quad \begin{cases} \omega(z_1, z_2, \ldots, z_p) \geqslant 0, \ \omega(z_1, z_2, \ldots, z_p) = 0 \text{ lorsque } z_i = z_k \text{ pour } i \neq k, \\ \omega(z_1, z_2, \ldots, z_p) = \omega(z_{i_1}, z_{i_2}, \ldots, z_{i_p}) \quad \text{lorsque } (i_1, i_2, \ldots, i_p) \text{ est} \\ \qquad \text{une permutation quelconque des nombres } (1, 2, \ldots, p). \end{cases}$$

D'autre part, soit E la somme de deux ensembles E_1 et E_2 compacts disjoints de l'espace R, α un nombre quelconque appartenant à l'intervalle ouvert $(0, 1)$ et n un entier au moins égal à p. Désignons par n_α le nombre entier satisfaisant à l'inégalité $n\alpha - 1 < n_\alpha \leqslant n\alpha$ et par $z^{(n)} = \{z_1, z_2, \ldots, z_{n_\alpha}, z_{n_\alpha+1}, \ldots, z_n\}$ un système de n points de l'ensemble E satisfaisant à la condition $W(n, \alpha)$, c'est-à-dire tel que n_α points initiaux $z_1, z_2, \ldots, z_{n_\alpha}$ du système $z^{(n)}$ appartiennent à l'ensemble E_1 et les autres points à E_2.

Si le système $z^{(n)}$ $(n \geqslant p)$ satisfait à la condition $W(n, \alpha)$, nous désignerons par $a(z^{(n)}, \alpha)$, $g(z^{(n)}, \alpha)$ et $h(z^{(n)}, \alpha)$ les moyennes suivantes:

$$(2)\quad \begin{cases} a(z^{(n)}, \alpha) = \displaystyle\sum_{1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n} \omega(z_{i_1}, z_{i_2}, \ldots, z_{i_p}) \Big/ \binom{n}{p}, \\ g(z^{(n)}, \alpha) = \Big\{ \displaystyle\prod_{1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n} \omega(z_{i_1}, z_{i_2}, \ldots, z_{i_p}) \Big\}^{1/\binom{n}{p}}, \\ h(z^{(n)}, \alpha) = \binom{n}{p} \Big/ \displaystyle\sum_{1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n} [\omega(z_{i_1}, z_{i_2}, \ldots, z_{i_p})]^{-1} \end{cases}$$

et par

$$a_n(E_1, E_2, \alpha), \quad g_n(E_1, E_2, \alpha), \quad h_n(E_1, E_2, \alpha) \quad (n = p, p+1, \ldots)$$

les bornes supérieures des moyennes (2) lorsque le système $z^{(n)}$ varie arbitrairement dans $E = E_1 + E_2$ en satisfaisant toujours à la condition $W(n, \alpha)$.

On a les théorèmes:

I. *Les suites*

$$\{a_n(E_1, E_2, \alpha)\}, \quad \{g_n(E_1, E_2, \alpha)\}, \quad \{h_n(E_1, E_2, \alpha)\}$$

tendent vers des limites finies

$$a(E_1, E_2, \alpha), \quad g(E_1, E_2, \alpha), \quad h(E_1, E_2, \alpha),$$

dites écarts restreints arithmétique, géometrique et harmonique (dans le rapport α) des ensembles E_1 et E_2 respectivement;

II. *Les fonctions*

$$a(E_1, E_2, \alpha), \quad g(E_1, E_2, \alpha), \quad h(E_1, E_2, \alpha)$$

sont continues dans l'intervalle ouvert $(0, 1)$.

4. IX. 1962. W. Żelazko (Warsaw), *Analytic functions in p-normed algebras* (voir Studia Mathematica 21 (1962), p. 345-350).

4. IX. 1962. W. Żelazko (Warsaw), *Entire functions in B_0-algebras* (voir sous le même titre B. Mitiagin, S. Rolewicz and W. Żelazko, Studia Mathematica 21 (1962), p. 291-306).

4. IX. 1962. S. Balcerzyk (Toruń), *Power series as orthogonal expansions* (voir Studia Mathematica 23 (1963), p. 31-39).

Consider Hilbert space H_2' consisting of functions f, analytic in the unit circle D $(|z| < 1)$, for which

$$\iint_D |f(z)|^2 dx\,dy < \infty.$$

The scalar product in H_2' is defined by

$$(f, g) = \iint_D f(z)\overline{g(z)}\,dx\,dy.$$

The power series of a function f is its orthogonal expansion with respect to the system $\{z^n\}$.

The following theorems hold:

THEOREM 1. *If a system of bounded analytic functions $\{u_n\}$ is orthogonal and complete in H_2' and if it satisfies the condition*

(i) *for any pair of indices n, m there exist an index k and a complex number $\lambda_{n,m}$ such that $u_n \cdot u_m = \lambda_{n,m} u_k$,*

then the system $\{u_n\}$ differs from $\{z^n\}$ at most by ordering and numerical coefficients.

THEOREM 2. *If a system of functions $\{u_n\}$, analytic in the closed circle $|z| \leqslant 1$, is orthogonal and complete in H_2' and if it satisfies the condition*

(ii) *for any index n there exist an index k and a complex number λ_n such that $u_n' = \lambda_n u_k$,*

then the system $\{u_n\}$ differs from $\{z^n\}$ at most by ordering and numerical coefficients.

4. IX. 1962. E. Złotkiewicz (Lublin), *On the precise bounds of the argument of $f(z)/z$ and $zf'(z)/f(z)$ for close-to-convex functions.*

In connection with some results due to Biernacki [8] the following sharp bounds for the principal values of the argument of $f(z)/z$, $zf'(z)/f(z)$, where f ranges over the class L of close-to-convex functions, can be given:

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \leqslant \arcsin\frac{8r-9r^3}{4-3r^2}, \qquad \left|\arg\frac{f(z)}{z}\right| \leqslant \arcsin\frac{8r-9r^3}{4-3r^2}, \qquad (r = |z|).$$

The sign of equality occurs only for $f(z)$ mapping the unit circle onto the w-plane slit along a ray whose prolongation does not contain the origin.

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(8) M. Biernacki, *Sur la représentation conforme des domaines linéairement accessibles*, Prace Matematyczno-Fizyczne 44 (1936), p. 293-314.

TABLE DES MATIÈRES DU VOLUME XI

COMMUNICATIONS