On extensions and products of Boolean algebras

by

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In my paper [8] (see also [9] § 36 and § 38) I discussed the theory of σ-extensions and σ-products of Boolean algebras. The subject of this paper is to generalize this theory to the case of $\mathfrak{m}$-complete Boolean algebras, $\mathfrak{m}$ being any infinite cardinal. Topological methods applied in [8] are no longer adequate for the theory of $\mathfrak{m}$-extensions and $\mathfrak{m}$-products of Boolean algebras. They are replaced here by another argument, based on the existence of free Boolean $\mathfrak{m}$-algebras proved by Rieger [7]. (1) The existence of maximal $\mathfrak{m}$-extensions and maximal $\mathfrak{m}$-products proved in this paper is a particular case of some more general results of Kerstan [3], but it is obtained here by another method than that used in Kerstan's paper. Minimal $\mathfrak{m}$-products were earlier investigated by Christensen and Pierce [1] but were introduced in another way than in this paper. The possibility of a generalization of the theory of σ-extensions and σ-products to any cardinal $\mathfrak{m} \geqslant \sigma$ was mentioned at the end of §§ 36, 38 in [9].

The terminology and notation in this paper are the same as in [9]. The zero element of any Boolean algebra is here denoted by 0. The composite of two mappings f, g is denoted by fg.

I. $(\mathfrak{M}, \mathfrak{m})$**-extensions.** $\mathfrak{A}$ will denote a fixed Boolean algebra, $\mathfrak{m}$—a fixed infinite cardinal, and $\mathfrak{M}$ a fixed set of infinite subsets of $\mathfrak{A}$ such that, for every $S \in \mathfrak{M}$,

(1) $$\overline{\overline{S}} \leqslant \mathfrak{m}$$

and

(2) $$\bigcap_{A \in S} A \text{ exists in } \mathfrak{A}.$$

A homomorphism (isomorphism) h from $\mathfrak{A}$ into any Boolean algebra $\mathfrak{A}'$ is said to be an $\mathfrak{M}$-homomorphism ($\mathfrak{M}$-isomorphism) provided it preserves all the infinite meets (2), i.e.

$$h\Big(\bigcap_{A \in S} A\Big) = \bigcap_{A \in S} h(A) \quad \text{in} \quad \mathfrak{A}'$$

for every $S \in \mathfrak{M}$.

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(1) During the print of this paper I was informed that the results concerning extensions of Boolean algebras were independly obtained by F. M. Yaqub by te same method.

By an $(\mathfrak{M}, \mathfrak{m})$-*extension* of $\mathfrak{A}$ we shall understand any pair $\{i, \mathfrak{B}\}$ such that

(e_1) $\mathfrak{B}$ is a Boolean $\mathfrak{m}$-algebra,

(e_2) i is an $\mathfrak{M}$-isomorphism from $\mathfrak{A}$ into $\mathfrak{B}$,

(e_3) $i(\mathfrak{A})$ $\mathfrak{m}$-generates $\mathfrak{B}$.

Sometimes we shall say that $\mathfrak{B}$ itself is an $(\mathfrak{M}, \mathfrak{m})$-extension of $\mathfrak{A}$. Thus, by definition, a Boolean $\mathfrak{m}$-algebra $\mathfrak{B}$ is an $(\mathfrak{M}, \mathfrak{m})$-extension of $\mathfrak{A}$ if $\mathfrak{A}$ can be isomorphically imbedded into $\mathfrak{B}$ with the preservation of all infinite meets in $\mathfrak{M}$, and $\mathfrak{B}$ is $\mathfrak{m}$-generated by the immersion of $\mathfrak{A}$.

Instead of the preservation of meets (2) we could postulate here the preservation of any given set of infinite $\mathfrak{m}$-meets and $\mathfrak{m}$-joins. However, it is easy to see that the preservation of $\mathfrak{m}$-joins can always be reduced to the preservation of some $\mathfrak{m}$-meets, by the de Morgan formulas. Thus the restriction to $\mathfrak{m}$-meets (2) is not essential.

Let $\{i, \mathfrak{B}\}$ and $\{i', \mathfrak{B}'\}$ be two $(\mathfrak{M}, \mathfrak{m})$-extensions of $\mathfrak{A}$. Then $\{i', \mathfrak{B}'\}$ is said to be a *homomorphic image* of $\{i, \mathfrak{B}\}$ if there exists an $\mathfrak{m}$-homomorphism h from $\mathfrak{B}$ into $\mathfrak{B}'$ such that

(3) $$i' = hi\,,$$

i.e. i' is the composite of h and i. We then write

(4) $$\{i', \mathfrak{B}'\} \leqslant \{i, \mathfrak{B}\}\,.$$

Note that condition (3) is equivalent to the following condition:

(5) h is an extension of the isomorphism $i'i^{-1}$ from $i(\mathfrak{A})$ onto $i'(\mathfrak{A})$.

Hence it follows that, if the $\mathfrak{m}$-homomorphism h with the above properties exists, then it is unique. Moreover, h maps $\mathfrak{B}$ onto $\mathfrak{B}'$. The two statements are direct consequences of (e_3).

If the homomorphism h with the properties mentioned is an isomorphism, then it is called an *isomorphism from* $\{i, \mathfrak{B}\}$ *onto* $\{i', \mathfrak{B}'\}$, and $\{i, \mathfrak{B}\}$, $\{i', \mathfrak{B}'\}$ are said to be *isomorphic*. Note that then h^{-1} is an isomorphism from $\{i', \mathfrak{B}'\}$ onto $\{i, \mathfrak{B}\}$.

For instance, if $\{i, \mathfrak{B}\}$ is an $(\mathfrak{M}, \mathfrak{m})$-extension of $\mathfrak{A}$ and h is an isomorphism from $\mathfrak{B}$ onto a Boolean algebra, then $\{hi, h(\mathfrak{B})\}$ is an $(\mathfrak{M}, \mathfrak{m})$-extension of $\mathfrak{A}$, isomorphic to $\{i, \mathfrak{B}\}$. In other words, any isomorph of an $(\mathfrak{M}, \mathfrak{m})$-extension of $\mathfrak{A}$ is also an $(\mathfrak{M}, \mathfrak{m})$-extension of $\mathfrak{A}$.

Let $\mathbf{K}$ be the class of all $(\mathfrak{M}, \mathfrak{m})$-extensions of $\mathfrak{A}$. It is easy to check that relation (4) is a quasi-ordering in $\mathbf{K}$. Moreover, two $(\mathfrak{M}, \mathfrak{m})$-extensions of $\mathfrak{A}$, $\{i, \mathfrak{B}\}$ and $\{i', \mathfrak{B}'\}$, are isomorphic if and only if simultaneously

$$\{i', \mathfrak{B}'\} \leqslant \{i, \mathfrak{B}\} \quad \text{and} \quad \{i, \mathfrak{B}\} \leqslant \{i', \mathfrak{B}'\}\,.$$

Sometimes it is convenient to identify isomorphic elements of $\mathbf{K}$. After this identification, $\mathbf{K}$ is partially ordered by $\leqslant$.

Let $\mathfrak{n}$ be the cardinal of $\mathfrak{A}$ and let $\mathfrak{A}_{\mathfrak{m},\mathfrak{n}}$ be the free Boolean $\mathfrak{m}$-algebra with a set of $\mathfrak{n}$ free $\mathfrak{m}$-generators. Let $\mathfrak{A}_{0,\mathfrak{n}}$ be its smallest subalgebra containing the free $\mathfrak{m}$-generators. Since $\mathfrak{A}_{0,\mathfrak{n}}$ is a free Boolean algebra with $\mathfrak{n}$ free generators, every Boolean algebra of a cardinality $\leqslant \mathfrak{n}$ is a homomorphic image of $\mathfrak{A}_{0,\mathfrak{n}}$. Since the free $\mathfrak{m}$-generators of $\mathfrak{A}_{\mathfrak{m},\mathfrak{n}}$ are free generators of $\mathfrak{A}_{0,\mathfrak{n}}$, the algebras $\mathfrak{A}_{0,\mathfrak{n}}$ and $\mathfrak{A}_{\mathfrak{m},\mathfrak{n}}$ have the property:

(e) every homomorphism from $\mathfrak{A}_{0,\mathfrak{n}}$ into any Boolean $\mathfrak{m}$-algebra $\mathfrak{A}'$ can be extended to an $\mathfrak{m}$-homomorphism from $\mathfrak{A}_{\mathfrak{m},\mathfrak{n}}$ into $\mathfrak{A}'$, the extension being unique.

Let g_0 be a fixed homomorphism from $\mathfrak{A}_{0,\mathfrak{n}}$ onto $\mathfrak{A}$, and let $\varDelta_0$ be the ideal of all elements $A \in \mathfrak{A}_{0,\mathfrak{n}}$ such that $g_0(A) = 0$.

Let $\mathbf{I}$ be the set of all $\mathfrak{m}$-ideals $\varDelta$ in $\mathfrak{A}_{\mathfrak{m},\mathfrak{n}}$ such that

(e′) $\varDelta \cap \mathfrak{A}_{0,\mathfrak{n}} = \varDelta_0$,

(e″) $\varDelta$ contains all elements $B - \bigcap_{A \in S} A$ and $\bigcap_{A \in S} A - B$ where S is any subset of $\mathfrak{A}_{0,\mathfrak{n}}$ such that $\overline{\overline{S}} \leqslant \mathfrak{m}$, $g_0(S) \in \mathfrak{M}$, and $g_0(B) = \bigcap_{A \in S} g_0(A)$ in $\mathfrak{A}_{\mathfrak{m},\mathfrak{n}}$.

For every $\varDelta \in \mathbf{I}$, let $\mathfrak{A}_\varDelta = \mathfrak{A}_{\mathfrak{m},\mathfrak{n}}/\varDelta$ and let $\mathfrak{A}_{\varDelta,0}$ be the subalgebra (of $\mathfrak{A}_\varDelta$) composed of all elements $[A]_\varDelta$ with $A \in \mathfrak{A}_{0,\mathfrak{n}}$.

Condition (e′) means that the mapping

$$g_\varDelta([A]_\varDelta) = g_0(A) \quad \text{for} \quad A \in \mathfrak{A}_{0,\mathfrak{n}}$$

defines an isomorphism $g_\varDelta$ from $\mathfrak{A}_{\varDelta,0}$ onto $\mathfrak{A}$. Let

$$i_\varDelta = g_\varDelta^{-1}\,.$$

Condition (e″) means that $i_\varDelta$ is an $\mathfrak{M}$-isomorphism from $\mathfrak{A}$ into $\mathfrak{A}_\varDelta$. Since $\mathfrak{A}_{\varDelta,0} = i_\varDelta(\mathfrak{A})$ $\mathfrak{m}$-generates $\mathfrak{A}_\varDelta$, we get the following theorem:

(i) *For every* $\varDelta$ *in* $\mathbf{I}$, $\{i_\varDelta, \mathfrak{A}_\varDelta\}$ *is an* $(\mathfrak{M}, \mathfrak{m})$-*extension of* $\mathfrak{A}$.

Now we shall prove that, conversely,

(ii) *For every* $(\mathfrak{M}, \mathfrak{m})$-*extension* $\{i, \mathfrak{B}\}$ *of* $\mathfrak{A}$ *there exists an ideal* $\varDelta \in \mathbf{I}$ *such that* $\{i, \mathfrak{B}\}$ *is isomorphic to* $\{i_\varDelta, \mathfrak{A}_\varDelta\}$.

By (e), the homomorphism ig_0 from $\mathfrak{A}_{0,\mathfrak{n}}$ into $\mathfrak{B}$ can be extended to an $\mathfrak{m}$-homomorphism g from $\mathfrak{A}_{\mathfrak{m},\mathfrak{n}}$ onto $\mathfrak{B}$. The $\mathfrak{m}$-ideal $\varDelta$ of all $A \in \mathfrak{A}_{\mathfrak{m},\mathfrak{n}}$ such that $g(A) = 0$ has properties (e′) and (e″) and the formula

$$h([A]_\varDelta) = g(A) \quad \text{for} \quad A \in \mathfrak{A}_{\mathfrak{m},\mathfrak{n}}$$

defines the required isomorphism h from $\{i_\varDelta, \mathfrak{A}_\varDelta\}$ onto $\{i, \mathfrak{B}\}$.

It is easy to check that

(iii) *For any* $\varDelta', \varDelta'' \in \mathbf{I}$,

$$\{i_{\varDelta'}, \mathfrak{A}_{\varDelta'}\} \leqslant \{i_{\varDelta''}, \mathfrak{A}_{\varDelta''}\} \quad \textit{if and only if} \quad \varDelta'' \subset \varDelta'\,.$$

Thus $\{i_{\varDelta'}, \mathfrak{A}_{\varDelta'}\}$ is isomorphic to $\{i_{\varDelta''}, \mathfrak{A}_{\varDelta''}\}$ if and only if $\varDelta' = \varDelta''$. Hence it follows that the set $\mathbf{K}$ of all $(\mathfrak{M}, \mathfrak{m})$-extensions of $\mathfrak{A}$, partially ordered by $\leqslant$ is ordering-isomorphic to the set $\mathbf{I}$ partially ordered by the converse of the set-theoretical inclusion.

So far we do not know whether the set $\mathbf{K}$ (i.e. the set $\mathbf{I}$) is non-empty. To prove the existence of at least one $(\mathfrak{M}, \mathfrak{m})$-extension of $\mathfrak{A}$ we shall use the existence of minimal extensions of $\mathfrak{A}$ (MacNeille [4]). We recall (see Sikorski [8], or [9] § 35) that a complete Boolean algebra $\mathfrak{B}_0$ is said to be a minimal extension of $\mathfrak{A}$ provided there exists an isomorphism i_0 from $\mathfrak{A}$ into $\mathfrak{B}_0$ such that

(e*) $i_0(\mathfrak{A})$ is a regular subalgebra of $\mathfrak{B}_0$ and generates completely $\mathfrak{B}_0$.

In other words, the isomorphism i_0 from $\mathfrak{A}$ into $\mathfrak{B}_0$ preserves all infinite meets and joins in $\mathfrak{A}$, and the smallest complete subalgebra of $\mathfrak{B}_0$ which contains $i_0(\mathfrak{A})$ coincides with $\mathfrak{B}_0$.

It is known (see e.g. [9] § 35) that condition (e*) can be replaced by the following one:

(e**) $i_0(\mathfrak{A})$ is a dense subalgebra of $\mathfrak{B}_0$.

In other words, for every non-zero element $B \in \mathfrak{B}_0$ there exists a non-zero element $A \in \mathfrak{A}$ such that $i_0(A)$ is a subelement of B.

Now let $\mathfrak{A}_0$ be the smallest $\mathfrak{m}$-subalgebra of $\mathfrak{B}_0$ which contains $i_0(\mathfrak{A})$. By definition,

$$(i_0, \mathfrak{A}_0) \tag{6}$$

is an $(\mathfrak{M}, \mathfrak{m})$-extension of $\mathfrak{A}$. The $(\mathfrak{M}, \mathfrak{m})$-extension (6) of $\mathfrak{A}$ and all its isomorphs are called *minimal* $\mathfrak{m}$*-extensions of* $\mathfrak{A}$ (it is not necessary to mention here $\mathfrak{M}$ because (6) is an $(\mathfrak{M}, \mathfrak{m})$-extension of $\mathfrak{A}$ for every set $\mathfrak{M}$ satisfying conditions (1) and (2)).

(iv) *The following conditions are equivalent for every* $(\mathfrak{M}, \mathfrak{m})$*-extension* $(i, \mathfrak{B})$ *of* $\mathfrak{A}$:

(a_1) $(i, \mathfrak{B})$ *is a minimal* $\mathfrak{m}$*-extension of* $\mathfrak{A}$;

(a_2) $i(\mathfrak{A})$ *is a dense subalgebra of* $\mathfrak{B}$;

(a_3) $i(\mathfrak{A})$ *is a regular subalgebra of* $\mathfrak{B}$.

(a_1) implies (a_2). Indeed, $i_0(\mathfrak{A})$ is dense in $\mathfrak{A}_0$ by (e**). Consequently every isomorph $\{i, \mathfrak{B}\}$ of (6) also has this property.

(a_2) implies (a_3). This follows from [9], theorem 23.1.

(a_3) implies (a_1). Let $\mathfrak{B}_0$ be a minimal extension of $\mathfrak{B}$. We may suppose, for simplicity, that $\mathfrak{B} \subset \mathfrak{B}_0$, i.e. that $\mathfrak{B}$ is a regular subalgebra of the complete Boolean algebra $\mathfrak{B}_0$, and the smallest complete subalgebra (of $\mathfrak{B}_0$) containing $\mathfrak{B}$ is $\mathfrak{B}_0$ itself (see (e*)). By (a_3), $i(\mathfrak{A})$ is a regular subalgebra of $\mathfrak{B}_0$ and the smallest complete subalgebra (of $\mathfrak{B}_0$) is the whole algebra $\mathfrak{B}_0$. Thus $\mathfrak{B}_0$ is a minimal extension of $\mathfrak{A}$, i.e. $(i, \mathfrak{B})$ is of the form (6).

Since there are $(\mathfrak{M}, \mathfrak{m})$-extensions of $\mathfrak{A}$, the set $\mathbf{I}$ of $\mathfrak{m}$-ideals is not empty (ii). The intersection Δ^0 of all ideals Δ in $\mathbf{I}$ also belongs to $\mathbf{I}$. By (i),

$$\{i_{\Delta^0}, \mathfrak{A}_{\Delta^0}\} \tag{7}$$

is an $(\mathfrak{M}, \mathfrak{m})$-extension of $\mathfrak{A}$. The $(\mathfrak{M}, \mathfrak{m})$-extension (7) and all its isomorphs are called *maximal* $(\mathfrak{M}, \mathfrak{m})$*-extensions of* $\mathfrak{A}$.

(v) *In order that an* $(\mathfrak{M}, \mathfrak{m})$*-extension* $(i, \mathfrak{B})$ *of* $\mathfrak{A}$ *be maximal it is necessary and sufficient that, for every* $\mathfrak{M}$*-homomorphism* h_0 *from* $\mathfrak{A}$ *into any Boolean* $\mathfrak{m}$*-algebra* $\mathfrak{C}$, *there exist an* $\mathfrak{m}$*-homomorphism* h *from* $\mathfrak{B}$ *into* $\mathfrak{C}$ *such that*

$$h_0 = hi\,.$$

To explain better the last conditions, let us identify $\mathfrak{A}$ with $i(\mathfrak{A})$ by means of the isomorphism i. The last condition means that every $\mathfrak{M}$-homomorphism h_0 from $\mathfrak{A} = i(\mathfrak{A}) \subset \mathfrak{B}$ into any Boolean $\mathfrak{m}$-algebra $\mathfrak{C}$ can be extended to an $\mathfrak{m}$-homomorphism h from $\mathfrak{B}$ into $\mathfrak{C}$.

To prove (v), let us observe that the homomorphism $h_0 g_0$ from $\mathfrak{A}_{0,\mathfrak{n}}$ into $\mathfrak{C}$ can be extended to an $\mathfrak{m}$-homomorphism h' from $\mathfrak{A}_{\mathfrak{m},\mathfrak{n}}$ into $\mathfrak{C}$ by (e). The $\mathfrak{m}$-ideal Δ' of all $A \in \mathfrak{A}_{\mathfrak{m},\mathfrak{n}}$ such that $h'(A) = 0$ has property (e'') since h_0 is an $\mathfrak{M}$-homomorphism. Since $\Delta_0 \subset \Delta'$, we have

$$(\Delta' \cap \Delta^0) \cap \mathfrak{A}_{0,\mathfrak{n}} = \Delta' \cap (\Delta^0 \cap \mathfrak{A}_{0,\mathfrak{n}}) = \Delta' \cap \Delta_0 = \Delta_0\,,$$

i.e. the ideal $\Delta' \cap \Delta^0$ satisfies (e'). Since the ideal $\Delta' \cap \Delta^0$ also satisfies (e''), it belongs to $\mathbf{I}$. Hence it follows that $\Delta^0 \cap \Delta' \supset \Delta^0$, i.e. $\Delta' \supset \Delta^0$. Consequently, the formula

$$h([A]_{\Delta^0}) = h'(A) \quad \text{ for } \quad A \in \mathfrak{A}_{\mathfrak{m},\mathfrak{n}}$$

defines an $\mathfrak{m}$-homomorphism from $\mathfrak{A}_{\Delta^0}$ into $\mathfrak{C}$, and $h_0 = hi_{\Delta^0}$.

Thus we have proved that (7) has the extension property mentioned in (v). Consequently, every isomorph of (7), i.e. every maximal $(\mathfrak{M}, \mathfrak{m})$-extension of A, has this property.

On the other hand, it is easy to check that all $(\mathfrak{M}, \mathfrak{m})$-extensions of $\mathfrak{A}$ with the extension property are isomorphic. Since (7) has the extension property, all $(\mathfrak{M}, \mathfrak{m})$-extensions with the extension property are isomorphic to (7), i.e. are maximal.

Identifying isomorphic elements in $\mathbf{K}$, we get the following theorem, which justifies the names "maximal" and "minimal".

(vi) *The maximal* $(\mathfrak{M}, \mathfrak{m})$*-extension* (7) *of* $\mathfrak{A}$ *is the greatest element in* $\mathbf{K}$. *The minimal* $\mathfrak{m}$*-extension* (6) *of* $\mathfrak{A}$ *is a minimal element in* $\mathbf{K}$.

The first part of (vi) easily follows from the extension property mentioned in (v). To prove the second part, is suffices to show that if h is an $\mathfrak{m}$-homomorphism (having property (3)) from (6) onto another $(\mathfrak{M}, \mathfrak{m})$-extension $(i, \mathfrak{B})$ of $\mathfrak{A}$, then h is an isomorphism. Indeed, h coincides with the isomorphism ii_0^{-1} on $i_0(\mathfrak{A})$; thus $h(A) \neq 0$ for every element $A \neq 0$, $A \in i_0(\mathfrak{A})$. Since $i_0(\mathfrak{A})$ is dense in $\mathfrak{A}_0$, we infer that $h(A) \neq 0$ for every $A \neq 0$, $A \in \mathfrak{A}_0$. This proves that h is an isomorphism.

It is not known whether the minimal $\mathfrak{m}$-extension (6) is the smallest element of $\mathbf{K}$. The answer is positive if $\mathfrak{A}$ has some additional properties. To explain this, let us introduce the following definition.

A Boolean $\mathfrak{m}$-algebra $\mathfrak{B}$ is said to have the *weak* $\mathfrak{m}$-*extension property* if, for every Boolean $\mathfrak{m}$-algebra $\mathfrak{B}'$ and for every its subalgebra $\mathfrak{B}''$ $\mathfrak{m}$-generating $\mathfrak{B}'$, every homomorphism h from $\mathfrak{B}''$ into $\mathfrak{B}$ such that

$$\bigcap_{A\in S} A = 0 \text{ in } \mathfrak{B}',\ S\subset\mathfrak{B}'',\ \overline{\overline{S}}\leqslant\mathfrak{m} \text{ imply } \bigcap_{A\in S} h(A) = 0 \text{ in } \mathfrak{B}$$

can be extended to an $\mathfrak{m}$-homomorphism from $\mathfrak{B}'$ into $\mathfrak{B}$.

Every $\mathfrak{m}$-distributive Boolean $\mathfrak{m}$-algebra has the weak $\mathfrak{m}$-extension property (this follows easily from theorems 34.2 and 24.6 in Sikorski [9]). Dubins [2] proved that every measure algebra has the weak σ-extension property. Recently Matthes [5] has proved that every weakly $\mathfrak{m}$-distributive $\mathfrak{m}$-algebra has the weak $\mathfrak{m}$-extension property.

(vii) *If* $(i,\mathfrak{B})\in\mathbf{K}$ *and* $\mathfrak{B}$ *has the weak* $\mathfrak{m}$-*extension property, then* $(i,\mathfrak{B})$ *is the minimal* $\mathfrak{m}$-*extension of* $\mathfrak{A}$ *and the smallest element of* $\mathbf{K}$.

In fact, let $(i',\mathfrak{B}')$ be any element of $\mathbf{K}$. Since $\mathfrak{B}$ has the weak $\mathfrak{m}$-extension property, the isomorphism ii'^{-1} from $i'(A)$ into $\mathfrak{B}$ can be extended to an $\mathfrak{m}$-homomorphism from $\mathfrak{B}'$ into $\mathfrak{B}$. This proves that $(i,\mathfrak{B})\leqslant(i',\mathfrak{B}')$. Thus $(i,\mathfrak{B})$ is the smallest element in K and, consequently, it is the minimal $\mathfrak{m}$-extension of $\mathfrak{A}$.

(viii) *If* $\mathfrak{A}$ *is* $\mathfrak{m}$-*distributive, then the minimal* $\mathfrak{m}$-*extension of* $\mathfrak{A}$ *is the smallest element in* $\mathbf{K}$.

Pierce [6] proved that if $\mathfrak{A}$ is $\mathfrak{m}$-distributive, so is the minimal extension of $\mathfrak{A}$ and, consequently, the algebra $\mathfrak{B}_0$ in (6). Thus theorem (viii) is an immediate consequence of (vii).

Traczyk [11] proved that, if $\mathfrak{A}$ is weakly $\mathfrak{m}$-distributive and satisfies the $\mathfrak{m}$-chain condition, then its minimal extension is also weakly $\mathfrak{m}$-distributive. Hence it follows that the $\mathfrak{m}$-distributivity can be replaced in (viii) by the hypothesis that $\mathfrak{A}$ is weakly $\mathfrak{m}$-distributive and satisfies the $\mathfrak{m}$-chain condition.

Now let $\mathfrak{M}$ be the class of all sets S satisfying conditions (1) and (2). In that case $(\mathfrak{M},\mathfrak{m})$-extensions will be called $\mathfrak{m}$-*extensions*. Thus $\{i,\mathfrak{B}\}$ is an $\mathfrak{m}$-extension of $\mathfrak{A}$ if $\mathfrak{B}$ is an $\mathfrak{m}$-algebra, i is an isomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ such that $i(\mathfrak{A})$ is an $\mathfrak{m}$-regular subalgebra of $\mathfrak{B}$, and $i(\mathfrak{A})$ $\mathfrak{m}$-generates $\mathfrak{B}$.

Let $\mathbf{K}_r$ denote the class of all $\mathfrak{m}$-extensions $(i,\mathfrak{B})$ such that $\mathfrak{B}$ is $\mathfrak{m}$-representable. If $\mathfrak{A}$ is not $\mathfrak{m}$-representable, then $\mathbf{K}_r$ is empty. For if $(i,\mathfrak{B})\in\mathbf{K}_r$, then $i(\mathfrak{B})$ is $\mathfrak{m}$-representable, and by isomorphism $\mathfrak{A}$ is also representable. On the other hand, if $\mathfrak{A}$ is representable, then $\mathbf{K}_r$ is not empty. For let X be the Stone space of $\mathfrak{A}$, let s be the Stone isomorphism of $\mathfrak{A}$ onto the field $\mathfrak{F}$ of all both open and closed subsets of X, let $\mathfrak{F}_\mathfrak{m}$ be the smallest $\mathfrak{m}$-field containing $\mathfrak{F}$, and let $\Delta_\mathfrak{m}$ be the $\mathfrak{m}$-ideal of all sets $A\in\mathfrak{F}_\mathfrak{m}$ which are of the $\mathfrak{m}$-category. Then the formula

$$i^*(A) = [s(A)]_{\Delta_\mathfrak{m}}$$

defines an $\mathfrak{m}$-isomorphism from $\mathfrak{A}$ into $\mathfrak{F}_\mathfrak{m}/\Delta_\mathfrak{m}$, i.e.

$$\{i^*(A),\mathfrak{F}_\mathfrak{m}/\Delta_\mathfrak{m}\} \tag{8}$$

is an $\mathfrak{m}$-extension of $\mathfrak{A}$. (8) and all of its isomorphs will be called *maximal representable* $\mathfrak{m}$-*extensions of* $\mathfrak{A}$.

Let $\mathbf{I}_r$ be the class of all ideals Δ in $\mathfrak{F}_\mathfrak{m}$ such that

(e_*) $\Delta_\mathfrak{m}$ is a subset of Δ;

(e_{**}) no open non-empty set belongs to Δ.

Thus if Δ is in $\mathbf{I}_r$, then the formula

$$i^\Delta(A) = [s(A)]_\Delta$$

defines an $\mathfrak{m}$-isomorphism from $\mathfrak{A}$ into $\mathfrak{F}_\mathfrak{m}/\Delta$ and $i^\Delta(\mathfrak{A})$ $\mathfrak{m}$-generates $\mathfrak{F}_\mathfrak{m}/\Delta$. In other words,

(ix) *For every* $\Delta\in\mathbf{I}_r$, $\{i^\Delta,\mathfrak{F}_\mathfrak{m}/\Delta\}$ *is an* $\mathfrak{m}$-*extension of* $\mathfrak{A}$.

The proof of the following theorem is similar to that of theorem 36.4 in [9].

(x) *In order that an* $\mathfrak{m}$-*extension* $(i,\mathfrak{B})\in\mathbf{K}_r$ *be maximal representable it is necessary and sufficient that, for every* $\mathfrak{m}$-*homomorphism* h_0 *from* $\mathfrak{A}$ *into any* $\mathfrak{m}$-*representable* $\mathfrak{m}$-*algebra* $\mathfrak{C}$, *there exist an* $\mathfrak{m}$-*homomorphism* h *from* $\mathfrak{B}$ *into* $\mathfrak{C}$ *such that* $h_0 = hi$.

In other words, $\{i,\mathfrak{B}\}\in\mathbf{K}_r$ is $\mathfrak{m}$-representable if every $\mathfrak{m}$-homomorphism from $i(\mathfrak{A})$ into any $\mathfrak{m}$-representable $\mathfrak{m}$-algebra $\mathfrak{C}$ can be extended to an $\mathfrak{m}$-homomorphism from $\mathfrak{B}$ into $\mathfrak{C}$.

(xi) *For every* $\{i,\mathfrak{B}\}\in\mathbf{K}_r$ *there exists an ideal* $\Delta\in\mathbf{I}_r$ *such that* $\{i,\mathfrak{B}\}$ *is isomorphic to* $\{i^\Delta,\mathfrak{F}_\mathfrak{m}/\Delta\}$.

The proof of (xi) is similar to that of theorem 36.5 in [9].

It is easy to check that

(xii) *For any* Δ',Δ'' *in* $\mathbf{I}_r$,

$$\{i^{\Delta'},\mathfrak{F}_\mathfrak{m}/\Delta'\}\leqslant\{i^{\Delta''},\mathfrak{F}_\mathfrak{m}/\Delta''\} \quad \textit{if and only if} \quad \Delta''\subset\Delta'.$$

Thus $\{i^{\Delta'},\mathfrak{F}_\mathfrak{m}/\Delta'\}$ is isomorphic to $\{i^{\Delta''},\mathfrak{F}_\mathfrak{m}/\Delta''\}$ if and only if $\Delta'=\Delta''$. Hence it follows that the set $\mathbf{K}_r$ partially ordered by $\leqslant$ is ordering-isomorphic to the set $\mathbf{I}_r$ partially ordered by the converse of the set-theoretical inclusion. The maximal representable $\mathfrak{m}$-extension ist the greatest element of $\mathbf{K}_r$.

Observe also that

(xiii) *If* $\{i,\mathfrak{B}\}\in\mathbf{K}_r$, $\{i',\mathfrak{B}'\}\in\mathbf{K}$ *and* $\{i',\mathfrak{B}'\}\leqslant\{i,\mathfrak{B}\}$, *then* $\{i',\mathfrak{B}'\}\in\mathbf{K}_r$.

It is not known whether the minimal $\mathfrak{m}$-extension is in $\mathbf{K}_r$. It is surely in $\mathbf{K}_r$ by (xiii) if it is the smallest element of $\mathbf{K}$.

In the case where $\mathfrak{m} = \sigma$ we have $\mathbf{K} = \mathbf{K}_r$ since every Boolean algebra is σ-representable.

II. Pseudo-$\mathfrak{m}$-products and $\mathfrak{m}$-products of Boolean algebras. In this section we shall consider a fixed set

(9) $$\{\mathfrak{A}_t\}_{t \in T}$$

of non-degenerate Boolean algebras.

Following [9] § 13, by a Boolean product of $\{\mathfrak{A}_t\}_{t \in T}$ we shall mean any pair

(10) $$\{\{i_t^0\}_{t \in T}, \mathfrak{A}^0\}$$

such that

(p_1^0) $\mathfrak{A}^0$ is a Boolean algebra;

(p_1^0) for every $t \in T$, i_t^0 is an isomorphism from $\mathfrak{A}_t$ into the Boolean algebra $\mathfrak{A}^0$;

(p_2^0) the subalgebras $i_t^0(\mathfrak{A})$ are independent;

(p_∞^0) the union of all subalgebras $i_t^0(\mathfrak{A})$ $(t \in T)$ generates $\mathfrak{A}^0$.

We quote without proofs a few fundamental properties of products of $\{\mathfrak{A}_t\}_{t \in T}$ (the proofs can be found in [9] § 13).

All products of $\{\mathfrak{A}_t\}_{t \in T}$ are isomorphic in the following sense: if $\{\{i_t^0\}_{t \in T}, \mathfrak{A}^0\}$ and $\{\{i_t'\}_{t \in T}, \mathfrak{A}'\}$ are Boolean products of (9), then there is an isomorphism h from $\mathfrak{A}^0$ onto $\mathfrak{A}'$ such that $i_t' = hi_t^0$ for every $t \in T$ (in other words, the isomorphism h is a common extension of all the isomorphisms $i_t'(i_t^0)^{-1}$). On the other hand, if (10) is a product of (9) and h is an isomorphism from $\mathfrak{A}^0$ onto a Boolean algebra, then $\{\{hi_t^0\}_{t \in T}, h(\mathfrak{A}^0)\}$ is also a product of (9). Thus the Boolean product of (9) is determined by (9) uniquely, up to isomorphism.

The Boolean product of (9) always exists. For let X_t be the Stone space of $\mathfrak{A}_t$, let s_t be the Stone isomorphism of $\mathfrak{A}_t$ onto the field of all both open and closed subsets of X_t, and let $X = P_{t \in T} X_t$ be the product of all the spaces X_t with the usual topology. For every set $A \subset X_t$, let A^* be the set of all points in X whose t^{th} coordinate is in A. The mapping

$$i_t^0(A) = s_t(A)^* \quad \text{for} \quad A \in \mathfrak{A}_t$$

is an isomorphism from $\mathfrak{A}_t$ into the Boolean algebra $\mathfrak{A}$ of all both open and closed subsets of X, and $\{\{i_t^0\}_{t \in T}, \mathfrak{A}^0\}$ is a Boolean product of (9).

It follows directly from the above set-theoretical representation of the Boolean product that

(xiv) *In any Boolean product* (10) *of* (9), *all the isomorphisms* i_t^0 *from* $\mathfrak{A}_t$ *into* $\mathfrak{A}^0$ *are complete.*

In other words, every isomorphism i_t^0 transforms infinite joins and meets in $\mathfrak{A}_t$ onto corresponding infinite joins and meets in $\mathfrak{A}^0$.

In what follows we shall consider a fixed Boolean product (10) of (9), which will be an auxiliary tool to examine other notions, called Boolean $\mathfrak{m}$-products, and pseudo-$\mathfrak{m}$-products, $\mathfrak{m}$ being a fixed infinite cardinal. By $\mathfrak{M}$ we shall denote the class of all sets $i_t^0(S)$ where t is any element of T, and S is any subset of $\mathfrak{A}_t$ such that $\overline{\overline{S}} \leqslant \mathfrak{m}$ and the meet $\bigcap_{A \in S} A$ exists in $\mathfrak{A}_t$. By (xiv), the meet $\bigcap_{A \in S} i_t^0(A)$ exists in $\mathfrak{A}^0$. Hence it follows that the class $\mathfrak{M}$ of subsets of $\mathfrak{A}^0$ satisfies conditions (1) and (2). Consequently the notion of $(\mathfrak{M}, \mathfrak{m})$-extensions of $\mathfrak{A}^0$ is well defined.

By a *Boolean $\mathfrak{m}$-product* of an indexed set $\{\mathfrak{A}_t\}_{t \in T}$ of non-degenerate Boolean algebras we shall understand any pair

(11) $$\{\{i_t\}_{t \in T}, \mathfrak{B}\}$$

such that

(p_0) $\mathfrak{B}$ is a Boolean $\mathfrak{m}$-algebra;

(p_1) for every $t \in T$, i_t is an $\mathfrak{m}$-isomorphism from $\mathfrak{A}_t$ into $\mathfrak{B}$;

(p_2) the subalgebras $i_t(\mathfrak{A}_t)$ are $\mathfrak{m}$-independent;

(p_3) the union of all the subalgebras $i_t(\mathfrak{A}_t)$ $(t \in T)$ $\mathfrak{m}$-generates $\mathfrak{B}$.

Condition (p_2) means that

(12) $$\bigcap_{t \in T'} B_t \neq 0$$

for any non-zero elements $B_t \in i_t(\mathfrak{A}_t)$, $T' \subset T$, $\overline{\overline{T'}} \leqslant \mathfrak{m}$.

If (11) satisfies conditions (p_0), (p_1), (p_3) and the following condition (which is evidently weaker than (p_2))

(p_2') the subalgebras $i_t(\mathfrak{A}_t)$ are independent,

then (11) is said to be a *Boolean pseudo-$\mathfrak{m}$-product* of (9).

Sometimes we call $\mathfrak{B}$ a Boolean $\mathfrak{m}$-product or pseudo-$\mathfrak{m}$-product of (9), respectively.

The class of all Boolean $\mathfrak{m}$-products (11) of (9) will be denoted by $\mathbf{L}$. The class of all Boolean pseudo-$\mathfrak{m}$-products (11) of (9) will be denoted by $\mathbf{L}_p$. By definition, $\mathbf{L} \subset \mathbf{L}_p$.

Let (11) and

(13) $$\{\{i_t\}_{t \in T}, \mathfrak{B}'\}$$

be pseudo-$\mathfrak{m}$-products of (9). Then $\{\{i_{t \in T}'\}, \mathfrak{B}'\}$ is said to be a homomorphic image of (11) if there exists an $\mathfrak{m}$-homomorphism h of $\mathfrak{B}$ into $\mathfrak{B}'$ such that

(14) $$i_t' = hi_t \quad \text{for every} \quad t \in T.$$

We then write

(15) $$\{\{i_t'\}_{t \in T}, \mathfrak{B}'\} \leqslant \{\{i_t\}_{t \in T}, \mathfrak{B}\}.$$

Note that condition (14) is equivalent to the following one:

(16) h is a common extension of all the isomorphisms $i'_t i_t^{-1}$ from $i_t(\mathfrak{A}_t)$ onto $i'_t(\mathfrak{A}_t)$, $t \in T$.

Hence it follows that, if the $\mathfrak{m}$-homomorphism h with the above properties exists, then it is unique. Moreover, h maps $\mathfrak{B}$ onto $\mathfrak{B}'$. The last two statements are direct consequences of (p_3).

If the homomorphism h with the properties mentioned is an isomorphism, then it is called an isomorphism from (11) onto (13), and (11) and (13) are said to be *isomorphic*. Note that then h^{-1} is an isomorphism from (13) onto (11). If one of the isomorphic pseudo-$\mathfrak{m}$-products is an $\mathfrak{m}$-product, so is the other.

For instance, if (11) is a Boolean pseudo-$\mathfrak{m}$-product ($\mathfrak{m}$-product) of (9), and h is an isomorphism from $\mathfrak{B}$ onto another $\mathfrak{m}$-algebra, then $\{\{hi_t\}_{t\in T}, h(\mathfrak{B})\}$ is a Boolean pseudo-$\mathfrak{m}$-product ($\mathfrak{m}$-product) of (9), isomorphic to (11).

It is easy to check that relation (15) is a quasi-ordering in $\mathbf{L}_\mathrm{p}$ and $\mathbf{L}$. Moreover, two pseudo-$\mathfrak{m}$-products ($\mathfrak{m}$-products) (11) and (13) are isomorphic if and only if simultaneously

$$\{\{i'_t\}_{t\in T}, \mathfrak{B}'\} \leqslant \{\{i_t\}_{t\in T}, \mathfrak{B}\} \quad \text{and} \quad \{\{i_t\}_{t\in T}, \mathfrak{B}\} \leqslant \{\{i'_t\}_{t\in T}, \mathfrak{B}'\}.$$

Sometimes it is convenient to identify isomorphic elements in $\mathbf{L}_\mathrm{p}$ (in $\mathbf{L}$). After this identification, $\mathbf{L}$ and $\mathbf{L}_\mathrm{p}$ are partially ordered by $\leqslant$.

Observe that

(xv) *If relation* (15) *holds between pseudo-$\mathfrak{m}$-products* (11) *and* (13), *and* (13) *is an $\mathfrak{m}$-product, then* (11) *is also an $\mathfrak{m}$-product.*

We shall first study pseudo-$\mathfrak{m}$-products. The following theorem assures the existence of pseudo-$\mathfrak{m}$-products of any indexed set (9) of non-degenerate Boolean algebras.

(xvi) *Let* (10) *be the Boolean product of* (9), *and let* $\{i, \mathfrak{B}\}$ *be an* $(\mathfrak{M}, \mathfrak{m})$*-extension of* $\mathfrak{A}^0$. *Then*

(17) $$\{\{ii_t^0\}_{t\in T}, \mathfrak{B}\}$$

is a pseudo-$\mathfrak{m}$-product of (9).

If $\{i', \mathfrak{B}'\}$ *is another* $(\mathfrak{M}, \mathfrak{m})$*-extension of* $\mathfrak{A}^0$, *then*

(18) $$\{\{i' i_t^0\}_{t\in T}, \mathfrak{B}'\} \leqslant \{\{ii_t^0\}_{t\in T}, \mathfrak{B}\} \quad \textit{if and only if} \quad \{i', \mathfrak{B}'\} \leqslant \{i, \mathfrak{B}\}.$$

To prove the first part of (xvi), let us observe that (p_0) follows from (e_1). (p_1) follows from (p_1^0) and (e_2). (p'_2) follows from (p_2^0) and (e_2). (p_3) follows from (p_3^0) and (e_3).

If h is an $\mathfrak{m}$-homomorphism from $\mathfrak{B}$ into $\mathfrak{B}'$ such that $i' = hi$, then $i' i_t^0 = hii_t^0$ for every $t \in T$.

Conversely, if h is an $\mathfrak{m}$-homomorphism from $\mathfrak{B}$ into $\mathfrak{B}'$ such that $i' i_t^0 = hii_t^0$ for every $t \in T$, then $i'(A) = hi(A)$ for every A belonging to the union of all the subalgebras $i_t^0(\mathfrak{A}_t)$. Since the union generates $\mathfrak{A}^0$ (see (p_3^0)), this equality holds for all $A \in \mathfrak{A}^0$, i.e. $i' = hi$.

This proves equivalence (18).

(xvii) *If* (11) *is a pseudo-$\mathfrak{m}$-product of* (9), *then there is an* $(\mathfrak{M}, \mathfrak{m})$*-extension* $(i, \mathfrak{B})$ *of* $\mathfrak{A}^0$ *such that* (11) *is identical with* (17).

Let $\mathfrak{A}_0$ be the smallest subalgebra (of $\mathfrak{B}$) which contains all the subalgebras $i_t(\mathfrak{A}_t)$. It is easy to see that $\{\{i_t\}_{t\in T}, \mathfrak{A}_0\}$ is also a Boolean product of (9). Thus there is an isomorphism i from $\mathfrak{A}^0$ onto $\mathfrak{A}_0$ such that $ii_t^0 = i_t$ for every $t \in T$. The $(\mathfrak{M}, \mathfrak{m})$-extension $(i, \mathfrak{B})$ of $\mathfrak{A}^0$ has the required properties.

By (xvi) and (xvii) the examination of pseudo-$\mathfrak{m}$-products can be reduced to the examination of $(\mathfrak{M}, \mathfrak{m})$-extension of Boolean products. To obtain an analogous statement for $\mathfrak{m}$-products we must first prove the following lemma on Boolean algebras.

(xviii) *For* $i = 1, 2$, *let* $\mathbf{S}_i$ *be a dense subset of non-zero elements of a Boolean algebra* $\mathfrak{A}_i$ *such that* $\mathbf{S}_i$ *generates* $\mathfrak{A}_i$. *Every mapping* f *from* $\mathbf{S}_1$ *onto* $\mathbf{S}_2$ *such that*

(19) $$A_1 \subset A_2 \quad \textit{if and only if} \quad f(A_1) \subset f(A_2) \qquad (A_1, A_2 \in \mathbf{S}_1)$$

can be extended uniquely to an isomorphism from $\mathfrak{A}_1$ *onto* $\mathfrak{A}_2$.

(19) implies that

$$A_1 \cap A_2 = 0 \quad \text{if and only if} \quad f(A_1) \cap f(A_2) = 0 \qquad (A_1, A_2 \in \mathbf{S}_1).$$

To prove (xviii) we shall first show that f satisfies condition (4) from theorem 12.2 in [9], i.e. that for any elements $A_1, \ldots, A_m, B_1, \ldots, B_m \in \mathbf{S}_1$,

(20) $$A_1 \cap \ldots \cap A_m \subset B_1 \cup \ldots \cup B_n \quad \text{implies} \quad f(A_1) \cap \ldots \cap f(A_m) \subset f(B_1) \cup \ldots \cup f(B_n).$$

Suppose that the inclusion on the right-hand side does not hold. Then there exists an element $C \in \mathbf{S}_1$ such that $f(C) \subset f(A_j)$ for $j = 1, \ldots, m$ and $f(C) \cap f(B_j) = 0$ for $j = 1, \ldots, n$. Consequently, $C \subset A_j$ for $j = 1, \ldots, m$ and $C \cap B_j = 0$ for $j = 1, \ldots, n$, i.e. the inclusion on the left side of (20) does not hold.

By [9], theorem 12.2, f can be extended to a homomorphism h from $\mathfrak{A}_1$ into $\mathfrak{A}_2$. Since $\mathbf{S}_2$ generates $\mathfrak{A}_2$, h maps $\mathfrak{A}_1$ onto $\mathfrak{A}_2$. Since h transforms a dense set of non-zero elements in $\mathfrak{A}_1$ onto a dense set of non-zero elements in $\mathfrak{A}_2$, h is an isomorphism.

By a $(*)$*-product* of an indexed set (9) of non-degenerate Boolean algebras we shall understand any pair

(21) $$\{\{i_t^*\}_{t\in T}, \mathfrak{A}^*\}$$

such that

(p_1^*) for every $t \in T$, i_t^* is an isomorphism from $\mathfrak{A}_t$ into the Boolean algebra $\mathfrak{A}^*$;

(p_2^*) the subalgebras $i_t^*(\mathfrak{A}_t)$ $(t \in T)$ are $\mathfrak{m}$-independent, i.e. for any set $T' \subset T$, $\overline{\overline{T'}} \leqslant \mathfrak{m}$, and any non-zero elements $A_t \in i_t^*(\mathfrak{A}_t)$, the meet

$$\bigcap_{t \in T'} A_t \tag{22}$$

exists and is not equal to 0;

(p_3^*) the set of all elements (22) is dense in $\mathfrak{A}^*$ and generates $\mathfrak{A}^*$.

It easily follows from (p_2^*) that if A_t, A_t' are non-zero elements of $i_t^*(\mathfrak{A}_t)$, $t \in T'$, $T' \subset T$, $\overline{\overline{T'}} \leqslant \mathfrak{m}$, then

$$\bigcap_{t \in T'} A_t \subset \bigcap_{t \in T'} A_t' \quad \text{if and only if} \quad A_t \subset A_t' \quad \text{for every} \quad t \in T'.$$

Hence it follows, on account of (xviii), that all $(*)$-products of (9) are isomorphic in the following sense: if (21) and $\{\{i_t'\}_{t \in T}, \mathfrak{A}'\}$ are $(*)$-products of (9), then there is an isomorphism h from $\mathfrak{A}^*$ onto $\mathfrak{A}'$ such that $i_t' = hi_t^*$ for every $t \in T$ (in other words, the isomorphism h is a common extension of all the isomorphisms $i_t'(i_t^*)^{-1}$). On the other hand, if (21) is a $(*)$-product of (9) and h is an isomorphism from $\mathfrak{A}^*$ onto another Boolean algebra, then $\{\{hi_t\}_{t \in T}, h(\mathfrak{A}^*)\}$ is also a $(*)$-product of (9). Thus the $(*)$-product of (9) is determined by (9) uniquely up to isomorphism.

The $(*)$-product of (9) always exists. For let X_t be the Stone space of $\mathfrak{A}_t$, let s_t be the Stone isomorphism of $\mathfrak{A}_t$ onto the field of all both open and closed subsets of X_t, and let $X = P_{t \in T} X_t$ be the product of all the spaces X_t. For every set $A \subset X_t$, let A^* be the set of all points in X whose t^{th} coordinate is in A. Let $\mathfrak{A}^*$ be the smallest field of sets containing all the intersections

$$\bigcap_{t \in T'} s_t(A_t)^*$$

where $T' \subset T$, $T' \leqslant \mathfrak{m}$, $A_t \in \mathfrak{A}_t$. The mapping

$$i_t^*(A) = s_t(A)^* \quad \text{for} \quad A \in \mathfrak{A}_t \tag{23}$$

is an isomorphism from $\mathfrak{A}_t$ into $\mathfrak{A}^*$. It is easy to check that $\{\{i_t^*\}_{t \in T}, \mathfrak{A}^*\}$ is a $(*)$-product of (9).

It follows directly from the above set-theoretical representation of the $(*)$-product that

(xix) *In any $(*)$-product* (21) *of* (9), *all the isomorphisms i_t^* from $\mathfrak{A}_t$ into $\mathfrak{A}^*$ are complete.*

In other words, every isomorphism i_t^* transforms infinite joins and meets in $\mathfrak{A}_t$ onto corresponding infinite joins and meets in $\mathfrak{A}^*$.

In what follows we shall consider a fixed $(*)$-product (21) of (9). By $\mathfrak{M}^*$ we shall denote the class composed of:

all the sets $i_t^*(S)$ where t any element of T, and S is any subset of $\mathfrak{A}_t$ such that $\overline{\overline{S}} \leqslant \mathfrak{m}$ and the meet $\bigcap_{A \in S} A$ exists in $\mathfrak{A}_t$;

all the sets of elements A_t with $t \in T'$, where $0 \neq A_t \in i_t^*(\mathfrak{A}_t)$ for $t \in T'$, $T' \subset T$ and $\overline{\overline{T'}} \leqslant \mathfrak{m}$.

By (xix) and (p_2^*) the class $\mathfrak{M}^*$ of all such sets satisfies conditions (1) and (2). Consequently the notion of $(\mathfrak{M}^*, \mathfrak{m})$-extensions of $\mathfrak{A}^*$ is well defined.

Now we come back to study $\mathfrak{m}$-products of (9). The following theorem ensures the existence of $\mathfrak{m}$-products of any indexed set (9) of non-degenerate Boolean algebras.

(xx) *Let* (21) *be the $(*)$-product of* (9), *and let $\{i, \mathfrak{B}\}$ be an $(\mathfrak{M}^*, \mathfrak{m})$-extension of $\mathfrak{A}^*$. Then*

$$\{\{ii_t^*\}_{t \in T}, \mathfrak{B}\} \tag{24}$$

is an $\mathfrak{m}$-product of (9).

If $\{i', \mathfrak{B}'\}$ is another $(\mathfrak{M}^*, \mathfrak{m})$-extension of $\mathfrak{A}^*$, then

$$\{\{i'i_t^*\}_{t \in T}, \mathfrak{B}'\} \leqslant \{\{ii_t^*\}_{t \in T}, \mathfrak{B}\} \quad \textit{if and only if} \quad \{i', \mathfrak{B}'\} \leqslant \{i, \mathfrak{B}\}. \tag{25}$$

The proof is similar to that of (xvi).

To prove the first part, let us observe that (p_0) follows from (e_1). (p_1) follows from (p_1^*) and (e_2). (p_2) follows from (p_2^*) and (e_2). (p_3) follows from (p_3^*) and (e_3).

If h is an $\mathfrak{m}$-homomorphism from $\mathfrak{B}$ into $\mathfrak{B}'$ such that $i' = hi$, then $i'i_t^* = hii_t^*$ for every $t \in T$.

Conversely, if h is an $\mathfrak{m}$-homomorphismf rom $\mathfrak{B}$ into $\mathfrak{B}'$ such that $i'i_t^* = hii_t^*$ for every $t \in T$, then $i'(A) = h(i(A))$ for every A belonging to the union of all the subalgebras $i_t^*(\mathfrak{A}_t)$. Consequently also $i'(A) = h(i(A))$ for all elements A of the form (22). Since those elements generate $\mathfrak{A}^*$, this equality holds for all $A \in \mathfrak{A}^*$, i.e. $i' = hi$.

This proves equivalence (25).

(xxi) *If* (11) *is an $\mathfrak{m}$-product of* (9), *then there is an $(\mathfrak{M}^*, \mathfrak{m})$-extension $(i, \mathfrak{B})$ of $\mathfrak{A}^*$ such that* (11) *is identical with* (24).

Let $\mathfrak{A}_*$ be the smallest subalgebra (of $\mathfrak{B}$) which contains all the elements $\bigcap_{t \in T} A_t$ where $A_t \in i_t(\mathfrak{A}_t)$, $T' \subset T$, $\overline{\overline{T'}} \leqslant \mathfrak{m}$. It is easy to see that $\{\{i_t\}_{t \in T}, \mathfrak{A}_*\}$ is a $(*)$-product of (9). Thus there is an isomorphism i from $\mathfrak{A}^*$ onto $\mathfrak{A}_*$ such that $ii_t^* = i_t$ for every $t \in T$. The $(\mathfrak{M}^*, \mathfrak{m})$-extension $(i, \mathfrak{B})$ of $\mathfrak{A}^*$ has the required properties.

By (xx) and (xxi) the examination of $\mathfrak{m}$-products can be reduced to the examination of $(\mathfrak{M}^*, \mathfrak{m})$-extensions of $(*)$-products.

By (vi), (xvi) and (xvii) the set $\mathbf{L}_p$ has the greatest element. Namely, if $(i, \mathfrak{B})$ is a maximal $(\mathfrak{M}, \mathfrak{m})$-extension of the Boolean product (10) of (9), then (17) is the greatest element of $\mathbf{L}_p$. Assuming in (xv) as (13) any $\mathfrak{m}$-product and as (11) the greatest element of $\mathbf{L}_p$, we infer that the greatest element of $\mathbf{L}_p$ is also the greatest element of $\mathbf{L}$. The greatest element (17) defined above and all its isomorphs will be called *maximal* $\mathfrak{m}$*-products*.

(xxii) *In order that a pseudo-*$\mathfrak{m}$*-product* (11) *be a maximal* $\mathfrak{m}$*-product it is necessary and sufficient that*

(p) *for any* $\mathfrak{m}$*-homomorphism* h_t *from* $\mathfrak{A}_t$ *into any Boolean* $\mathfrak{m}$*-algebra* $\mathfrak{C}$ *there exist an* $\mathfrak{m}$*-homomorphism* h *from* $\mathfrak{B}$ *into* $\mathfrak{C}$ *such that* $h_t = hi_t$ *for every* $t \in T$.

Property (p) is equivalent to the following one:

(p′) *for any* $\mathfrak{m}$*-homomorphisms* h'_t *from* $i_t(\mathfrak{A}_t)$ *into a Boolean* $\mathfrak{m}$*-algebra* $\mathfrak{C}$ *there exists a homomorphism* h *from* $\mathfrak{B}$ *into* $\mathfrak{C}$ *which is a common extension of all* h'_t, $t \in T$.

To obtain the equivalence it is sufficient to assume $h'_t = h_t i_t^{-1}$ or $h_t = h'_t i_t$.

To prove the necessity of (p), it suffices to show that the $\mathfrak{m}$-product (17) where $(i, \mathfrak{B})$ is a maximal $(\mathfrak{M}, \mathfrak{m})$-extension of (10) has property (p). Indeed, by [9], theorem 13.3, there exist a homomorphism h_0 from $\mathfrak{A}^0$ into $\mathfrak{C}$ such that $h_t = h_0 i_t$ for every $t \in T$, i.e. h_0 is a common extension of all the homomorphisms $h_t i_t^{-1}$. Consequently h_0 is also an $\mathfrak{M}$-homomorphism. By (v) there exists an $\mathfrak{m}$-homomorphism h from $\mathfrak{B}$ into $\mathfrak{C}$ such that $h_0(B) = h\big(i(B)\big)$ for every $B \in \mathfrak{A}^0$. Assuming $B = i_t(A)$, where A is any element of $\mathfrak{A}_t$, we obtain $h_0\big(i_t(A)\big) = h\Big(i\big(i_t(A)\big)\Big)$. This proves that $h_t = h_0 i_t = hii_t$ for every $t \in T$.

On the other hand, it is easy to see that all pseudo-$\mathfrak{m}$-products with property (p) are isomorphic. Since the maximal $\mathfrak{m}$-product has property (p), every pseudo-$\mathfrak{m}$-product with property (p) is isomorphic to a maximal $\mathfrak{m}$-product, i.e. it is also a maximal $\mathfrak{m}$-product.

Consider now the pseudo-$\mathfrak{m}$-product (17) where $\{i, \mathfrak{B}\}$ is a minimal $\mathfrak{m}$-extension of (10). By (xi) and (xvii) the pseudo-$\mathfrak{m}$-product just defined is a minimal element in $\mathbf{L}_p$. This pseudo-$\mathfrak{m}$-product and all its isomorphs will be called *minimal pseudo-*$\mathfrak{m}$*-products* of (9).

Consider the $\mathfrak{m}$-product (24) where $\{i, \mathfrak{B}\}$ is a minimal $(\mathfrak{M}^*, \mathfrak{m})$-extension of (21). By (xx) and (xxi), the $\mathfrak{m}$-product just defined is a minimal element in $\mathbf{L}$. This $\mathfrak{m}$-product and all its isomorphs are called *minimal* $\mathfrak{m}$*-products* of (9).

(xxiii) *A pseudo-*$\mathfrak{m}$*-product* (11) *(an* $\mathfrak{m}$ *product* (11)*) of* (9) *is a minimal pseudo-*$\mathfrak{m}$*-product (is a minimal* $\mathfrak{m}$*-product) if and only if the class of all the elements* $\bigcap_{t \in T'} A_t$ *where* $A_t \in i(\mathfrak{A}_t)$ *for* $t \in T'$ *and where* T' *is any finite subset of* T *(where* T' *is any subset of* T *of a power* $\leqslant \mathfrak{m}$*) is dense in* $\mathfrak{B}$.

The easy proof is left to the reader.

It is not known whether the minimal pseudo-$\mathfrak{m}$-product is the least element in $\mathbf{L}_p$. Neither it is known whether the minimal $\mathfrak{m}$-product is the smallest element of $\mathbf{L}$. The answer to the second problem is affirmative if all $\mathfrak{A}_t$ are $\mathfrak{m}$-distributive. To prove this, let us assume the following definition.

A Boolean algebra B is said to have the *strong* $\mathfrak{m}$*-extension property* if, for every Boolean $\mathfrak{m}$-algebra $\mathfrak{B}'$ and any set S $\mathfrak{m}$-generating $\mathfrak{B}'$, every mapping f from S into $\mathfrak{B}$ such that

$$\bigcap_{A \in S_0} \varepsilon(A) \cdot A = 0 \text{ in } \mathfrak{B}', \; S_0 \subset S, \; \overline{\overline{S}}_0 \leqslant \mathfrak{m}$$
$$\text{imply} \quad \bigcap_{A \in S_0} \varepsilon(A) \cdot h(A) = 0 \text{ in } \mathfrak{B}$$

can be extended to an $\mathfrak{m}$-homomorphism from $\mathfrak{B}'$ into $\mathfrak{B}$.

Here $\varepsilon(A)$ is any function with values $+1$ or -1, and $+1 \cdot A$ denotes the element A, and $-1 \cdot A$ denotes the complement of A.

It follows from [9], theorem 24.6 and theorem 34.1, that every $\mathfrak{m}$-distributive Boolean $\mathfrak{m}$-algebra has the strong $\mathfrak{m}$-extension property. Of course, the strong extension property implies the weak $\mathfrak{m}$-extension property examined on p. 104. It is easy to show that the properties are not equivalent.

(xxiv) *If* (11) *is an* $\mathfrak{m}$*-product of* (9) *and* $\mathfrak{B}$ *has the strong* $\mathfrak{m}$*-extension property, then* (11) *is the minimal* $\mathfrak{m}$*-extension of* (9) *and it is the smallest element of* $\mathbf{L}$.

In fact, let $\big\{\{i'_t\}_{t \in T}, \mathfrak{B}'\big\}$ be any element in $\mathbf{L}$. Since $\mathfrak{B}$ has the weak $\mathfrak{m}$-extension property, all the isomorphisms $i_t i'^{-1}$ can be extended to an $\mathfrak{m}$-homomorphism h from $\mathfrak{B}'$ into $\mathfrak{B}$ (take as S the union of all the algebras $i'_t(A_t)$!). This proves that (11) is smaller than any element in $\mathbf{L}$. Consequently it is the smallest element in $\mathbf{L}$ and therefore it is a minimal extension of (9).

(xxv) *If all the algebras* $\mathfrak{A}_t$ *are distributive, then the minimal* $\mathfrak{m}$*-product of* (9) *is the smallest element of* $\mathbf{L}$.

Christensen and Pierce [1] proved that if all the algebras $\mathfrak{A}_t$ are distributive, then the minimal $\mathfrak{m}$-product of (9) is also $\mathfrak{m}$-distributive (for another proof of this theorem, see Sikorski and Traczyk [10]). Thus (xxv) follows directly from (xxiv).

Let $\mathbf{L}_r$ and $\mathbf{L}_{pr}$ denote respectively the classes of all $\mathfrak{m}$-products and all pseudo-$\mathfrak{m}$-products (11) of (9), such that the algebra $\mathfrak{B}$ is $\mathfrak{m}$-representable. By definition, $\mathbf{L}_r$ is a subclass of $\mathbf{L}_{pr}$. If one of the algebras $\mathfrak{A}_t$ in (9) is not $\mathfrak{m}$-representable, then $\mathbf{L}_{pr}$ (and consequently $\mathbf{L}_r$ too) is empty. For if the algebra $\mathfrak{B}$ is $\mathfrak{m}$-representable, so is $i_t(\mathfrak{A}_t)$ for every $t \in T$ since it is an $\mathfrak{m}$-regular subalgebra of $\mathfrak{B}$. By isomorphism, $\mathfrak{A}_t$ is also $\mathfrak{m}$-representable.

Suppose now that all the algebras $\mathfrak{A}_t$ in (9) are $\mathfrak{m}$-representable. Then the class $\mathbf{L}_r$ is not empty. To construct an element in $\mathbf{L}_r$, let us denote by X_t the Stone space of $\mathfrak{A}_t$. Let s_t be the Stone isomorphism from $\mathfrak{A}_t$ onto the field of all both open and closed subsets of X_t, let $X = \mathbf{P}_{t\in T} X_t$ be the product of all the spaces X_t with the usual topology, and let $\mathfrak{F}_{\mathfrak{m}}$ denote the smallest $\mathfrak{m}$-field (of subsets of X) containing all both open and closed subsets of X. For every set $A \subset X_t$, let A^* be the set of all points in X whose t^{th} coordinate is in A. Let $\Delta_{\mathfrak{m}}$ be the ideal (in the field $\mathfrak{F}_{\mathfrak{m}}$) generated by all the sets A^* where A is an $\mathfrak{m}$-nowhere dense subset of X_t, $t \in T$. It is easy to check that no set of the form

(26) $\quad \bigcap_{t\in T'} s_t(A_t)^*,\quad$ where $\quad A_t$ is a non-zero element of $\mathfrak{A}_t$

and $T' \subset T$, $\overline{\overline{T'}} \leqslant \mathfrak{m}$, belongs to $\Delta_{\mathfrak{m}}$. Hence it follows that the formula

$$i_t(A) = [s_t(A)^*]\Delta_{\mathfrak{m}} \quad \text{for} \quad A \in \mathfrak{A}_t$$

defines an isomorphism from $\mathfrak{A}_t$ into $\mathfrak{F}_{\mathfrak{m}}/\Delta_{\mathfrak{m}}$, and that the subalgebras $i_t(\mathfrak{A}_t)$ are $\mathfrak{m}$-independent. It is easy to verify that i_t is an $\mathfrak{m}$-isomorphism from $\mathfrak{A}_t$ into $\mathfrak{F}_{\mathfrak{m}}/\Delta_{\mathfrak{m}}$ and that the union of the subalgebras $i_t(\mathfrak{A}_t)$ $\mathfrak{m}$-generates $\mathfrak{F}_{\mathfrak{m}}/\Delta_{\mathfrak{m}}$. Thus

(27) $$\{\{i_t\}_{t\in T},\ \mathfrak{F}_{\mathfrak{m}}/\Delta_{\mathfrak{m}}\}$$

is an $\mathfrak{m}$-product of (9). This $\mathfrak{m}$-product and all its isomorphs are said to be *maximal $\mathfrak{m}$-representable $\mathfrak{m}$-products* of (9).

Now let $\mathbf{I}_r$ ($\mathbf{I}_{pr}$) be the class of all $\mathfrak{m}$-ideals Δ in $\mathfrak{F}_{\mathfrak{m}}$ satisfying the following conditions:

(p_*) $\Delta_{\mathfrak{m}}$ is a subset of Δ;

(p_{**}) no set of the form (26) where $\overline{\overline{T'}} \leqslant \mathfrak{m}$ (where T' is finite) belongs to Δ.

For any ideal Δ in $\mathbf{I}_{pr}$, let

$$i_t^{\Delta}(A) = [s_t(A)^*]_{\Delta} \quad \text{for} \quad A \in \mathfrak{A}_t.$$

As in the case of (27), we verify that

(xxvi) *For every $\Delta \in \mathbf{I}_r$ (for every $\Delta \in \mathbf{I}_{pr}$),*

(28) $$\{\{i_t^{\Delta}\}_{t\in T},\ \mathfrak{F}_{\mathfrak{m}}/\Delta\},$$

is an element of $\mathbf{L}_r$ (of $\mathbf{L}_{pr}$).

The proof of the following theorem is similar to that of theorem 38.2 in [9].

(xxvii) *In order that an element (11) of $\mathbf{L}_{pr}$ be a maximal $\mathfrak{m}$-representable $\mathfrak{m}$-product of (9) it is sufficient and necessary that, for any $\mathfrak{m}$-homomorphisms h_t from $\mathfrak{A}_t$ into any $\mathfrak{m}$-representable $\mathfrak{m}$-algebra $\mathfrak{C}$, there exist an $\mathfrak{m}$-homomorphism h from $\mathfrak{B}$ into $\mathfrak{C}$ such that $h_t = hi_t$ for every $t \in T$.*

In other words, an element (11) of $\mathbf{L}_{pr}$ is a maximal $\mathfrak{m}$-representable $\mathfrak{m}$-product if and only if any homomorphisms h_t' from $i(\mathfrak{A}_t)$ into an $\mathfrak{m}$-representable Boolean $\mathfrak{m}$-algebra $\mathfrak{C}$ can be extended to an $\mathfrak{m}$-homomorphism h from $\mathfrak{B}$ into $\mathfrak{C}$.

(xxviii) *For every element (11) in $\mathbf{L}_r$ (in $\mathbf{L}_{pr}$) there exists and ideal Δ in $\mathbf{I}_r$ (in $\mathbf{I}_{pr}$) such that (11) is isomorphic to (28).*

The proof is similar to that of theorem 38.4 in [9].

It is easy to verify that

(xxix) *For any $\mathfrak{m}$-ideals Δ', Δ'' in $\mathbf{I}_{pr}$,*

$$\{\{i_t^{\Delta'}\}_{t\in T},\ \mathfrak{F}_{\mathfrak{m}}/\Delta'\} \leqslant \{\{i_t^{\Delta''}\}_{t\in T},\ \mathfrak{F}_{\mathfrak{m}}/\Delta''\} \quad \textit{if and only if} \quad \Delta'' \subset \Delta'.$$

Thus $\{\{i_t^{\Delta'}\}_{t\in T}, \mathfrak{F}_{\mathfrak{m}}/\Delta'\}$ is isomorphic to $\{\{i_t^{\Delta''}\}_{t\in T}, \mathfrak{F}_{\mathfrak{m}}/\Delta''\}$ if and only if $\Delta' = \Delta''$. Hence it follows that the set $\mathbf{L}_r$ (the set $\mathbf{L}_{pr}$) partially ordered by $\leqslant$ is ordering-isomorphic to the set $\mathbf{I}_r$ (the set $\mathbf{I}_{pr}$) partially ordered by the converse of the set-theoretical inclusion. The maximal $\mathfrak{m}$-representable $\mathfrak{m}$-extension of (9) is the greatest element of $\mathbf{L}_{pr}$ and of $\mathbf{L}_r$.

Observe, moreover, that

(xxx) *If (11) is in $\mathbf{L}_{pr}$ (in $\mathbf{L}_r$), (13) is in $\mathbf{L}_p$ (in $\mathbf{L}$) and (15) holds, then (13) is also in $\mathbf{L}_{pr}$ (in $\mathbf{L}_r$).*

It is not known whether the minimal pseudo-$\mathfrak{m}$-product (the minimal $\mathfrak{m}$-product) of (9) belongs to $\mathbf{L}_{pr}$ (to $\mathbf{L}_r$). It is surely in $\mathbf{L}_{pr}$ (in $\mathbf{L}_r$) by (xxx) if it is the smallest element of $\mathbf{L}_p$ (of $\mathbf{L}_r$).

In the case where $\mathfrak{m} = \sigma$ we have $\mathbf{L} = \mathbf{L}_r$ and $\mathbf{L}_p = \mathbf{L}_{pr}$ since every Boolean algebra is σ-representable.

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Correction to the paper "Reduced direct products" *

by

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At the suggestion of Professor Tarski, the authors wish to revise their account of Tarski's role in the development of the notion of reduced products. The following more accurate statement should replace lines 7-16 of the first paragraph of the introduction on p. 195:

In [29] Łoś defined the notion of logical measures in algebraic systems and established some basic properties of this notion referring primarily to the case of the two-valued measure. A seemingly quite different construction was carried through and applied by Chang and Morel in [1]. Tarski realized that both the construction of Łoś in the two-valued case and that of Chang and Morel could be subsumed as two special cases under one general notion, that of reduced products. He formulated the definition of this concept for arbitrary algebraic systems and suggested that the notion could be used to give a proof of the compactness theorem in the theory of models by means of a mathematical construction; in particular, using the construction of Chang and Morel, he gave such a proof for the class of the so-called Horn sentences (cf. p. 211) At his suggestion Frayne then extended the definition of reduced products to arbitrary relational systems, and several results (in particular, a proof of the general compactness theorem along the lines of Tarski's suggestions) were subsequently obtained by Frayne and Morel.

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