On the approximation of L^p functions by trigonometric polynomials

by

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1. A theory of trigonometric, interpolating polynomials suitable for L^p functions was introduced by Marcinkiewicz and Zygmund [4]. A main feature of this was the use of a translation parameter for the interpolation points, a technique which avoided difficulties arising from the fact that the interpolating points were only countable and which made the problem two-dimensional. Thus we write

$$I_{n,u}(x;f) = \frac{2}{2n+1}\sum_{j=0}^{2n} f(u+x_j)D_n(x-u-x_j)\,, \qquad x_j = \frac{2\pi j}{2n+1}\,,$$

where D_n is the Dirichlet kernel. $I_{n,u}(x;f)$ is then a trigonometric polynomial of degree n interpolating the f at the fundamental points of interpolation translated by the parameter u. Their paper included certain fundamental inequalities which were analogues of known results about Fourier series, and in general the analogy with Fourier series was stressed. Immediately, however, differences with respect to convergence were noted in the construction of counterexamples. A positive convergence theorem was later proved by Offord [5] under the assumption of the finiteness of a certain integral. However, no indications were given here concerning the precision of the assumption.

The principal theorem of this paper is a generalization of Offord's result. We assume also the finiteness of an integral and prove the almost everywhere convergence of the sequence $I_{n,u}(x;f)$. Counterexamples are provided to show that very little improvement in the result is possible. The extension to Jackson polynomials is indicated, along with an interpretation of the theorem in terms of fractional integrals. Then a type of continuity called translation continuity, which seems appropriate in this context, is introduced, and certain elementary properties are proved. Under the assumptions of our main theorem, a function is proved to be translation continuous almost everywhere. Finally, a very precise result concerning translation continuity at a point is proved.

We shall be concerned then with the almost everywhere convergence of the sequence $I_{n,u}(x;f)$ for functions f satisfying a condition

$$(\mathrm{C}_{a,p}) \qquad \int_0^{2\pi}\int_0^{2\pi} \frac{|f(x+u)-f(x)|^p}{u^{1+a}}\,du\,dx < \infty .$$

In [5], a hypothesis similar to $(\mathrm{C}_{a,p})$ was used; but the second difference of f replaced the first, and a was taken to be one. However, for the values of a which concern us there is no substantial difference in using the first difference of f rather than the second (cf. [3]). Actually, two different hypotheses were used in [5], each guaranteeing the convergence of $I_{n,u}(x;f)$. However, as we indicate below, the second result is a consequence of the first.

THEOREM 1. (i) *Let f satisfy $(\mathrm{C}_{a,p})$ for some $p>1$ and some $a>a_0 = (\sqrt{5}-1)/2$. For almost every (x,u), $I_{n,u}(x;f)$ converges to $f(x)$.*

(ii) *For every $p>1$ and every $a<1/2$, there exists a function f satisfying $(\mathrm{C}_{a,p})$ such that $I_{n,u}(x;f)$ diverges for almost every (x,u).*

It is a fact of some interest that a_0 is a number of great importance in Diophantine approximation, and that our proof of (ii) relies heavily on this same subject. In broad outline, our proof of (i) follows the pattern established in [5]. Thus f is approximated by f_n, an integral mean of f, such that $I_{n,u}(x;f)$ is close to $I_{n,u}(x;f_n)$ in a certain sense. Then the convergence of $I_{n,u}(x;f_n)$ to $f(x)$ is proved. It is in the latter step that our proof differs substantially from that of [5]. Perhaps the most novel aspect of our proof is the use of a new inequality from the theory of subadditive functions. We begin by presenting this inequality which is known [3]. Since the proof is not long, and since it is a companion to the vital second lemma, we include a proof.

2. We say the positive measurable function φ is *subadditive* on the open interval $(0,A)$, $0<A\leqslant\infty$, if $\varphi(u+v)\leqslant\varphi(u)+\varphi(v)$ where u, v and $u+v$ all belong to $(0,A)$.

LEMMA 1. *Let φ be positive, measurable, and subadditive on $(0,A)$.*

(i) *Let $p\geqslant 1$, and let a be any real number. There exists $C_{a,p}$ depending only on its subscripts such that*

$$\left(\int_0^A \frac{\varphi^p(u)}{u^{1+pa}}\,du\right)^{1/p} \leqslant C_{a,p}\int_0^A \frac{\varphi(u)}{u^{1+a}}\,du .$$

(ii) *If either integral above is finite, then there exists a constant C depending on φ, p, and a but not on u such that $\varphi(u)\leqslant Cu^a$ for u in $(0,A)$.*

If $a\geqslant 0$, and φ is subadditive, then $\varphi(u)/u^a$ is also subadditive. Hence, in the proof it would be enough to restrict attention to the case $a\leqslant 0$. However, this does not account for any simplification in the proof. Let M denote the value of the integral on the right in (i). M may be assumed finite and strictly positive. Let E denote the set of points u such that $\varphi(u)>Mu^a/(\log 4/3)$, and let G be the complement of E. Then

$$(1) \qquad \int_E \frac{1}{u}\,du < \frac{\log(4/3)}{M}\int_0^A \frac{\varphi(u)}{u^{1+a}}\,du = \log(4/3) .$$

We assert that for every u in $(0,A)$ there exists v in $(u/3, 2u/3)$ such that v belongs to G and $w=u-v$ belongs to G. If this were not so, say for u_0, then $(u_0/3, 2u_0/3) = E_0\cup E_1$ where $E_0 = E\cap(u_0/3, 2u_0/3)$ and E_1 is the set of points of the form $v=u_0-w$ where w belongs to E_0. Since E_0 and E_1 are reflections of each other through the point $u_0/2$, they have the same measure, $|E_0|$. Thus $|E_0|\geqslant u_0/6$, and

$$\log(4/3) = \int_{u_0/2}^{2u_0/3} \frac{1}{u}\,du \leqslant \int_{E_0}\frac{1}{u}\,du \leqslant \int_E \frac{1}{u}\,du < \log(4/3)$$

by (1). This contradiction proves our assertion. Thus

$$\varphi(u) = \varphi(v+w)\leqslant\varphi(v)+\varphi(w)\leqslant\frac{M}{\log(4/3)}(v^a+w^a) .$$

If $a\geqslant 0$, $v^a+w^a\leqslant 2u^a$. If $a<0$, $v^a+w^a\leqslant 2u^a/3^a$. Hence

$$(2) \qquad \varphi(u)\leqslant C_a M u^a/\log(4/3)$$

and so

$$\frac{\varphi^p(u)}{u^{1+pa}}\leqslant C_{a,p}M^{p-1}\frac{\varphi(u)}{u^{1+a}} .$$

Integration over $(0,A)$ completes the proof of (i). (2) shows that $\varphi(u)\leqslant Cu^a$ if the right side of (i) is finite. If only the left side is finite, then the same proof holds except that M must be replaced by the value of the corresponding integral.

We shall be interested in the case when $\varphi(u)=\varphi_r(u;f)$ where

$$\varphi_r(u;f) = \left(\int_0^{2\pi} |f(x+u)-f(x)|^r\,dx\right)^{1/r}, \qquad r\geqslant 1 ,$$

f is assumed to be periodic, and Minkowski's inequality verifies the subadditivity property. The statement of Lemma 1 in this case is

$$\left(\int_0^{2\pi}\int_0^{2\pi}\frac{|f(x+u)-f(x)|^p}{u^{1+pa}}\,du\,dx\right)^{1/p}\leqslant C_{a,p}\int_0^{2\pi}\frac{du}{u^{1+a}}\left(\int_0^{2\pi}|f(x+u)-f(x)|^p\,dx\right)^{1/p} .$$

The same inequality holds if in each of the above integrals the second symmetric difference of f replaces the first (cf. [3]). It is a consequence of this last fact that the second convergence result of [5] follows from the first, as was mentioned previously.

For direct applications to the proof of our theorem, we need a kind of local version of our first lemma. Let E be an interval $(0, x_E)$ in $(0, 2\pi)$. Let $E+x$ be the translation of E by x, and let $2E = (0, 2x_E)$. We introduce the functions

$$\varphi_E^p(u, x) = \int_{E+x} |f(s+u)-f(s)|^p ds; \qquad g(s) = \int_0^{2\pi} \frac{|f(s+u)-f(s)|^p}{u^{1+\alpha}} du.$$

If f satisfies $(C_{\alpha,p})$, then g is integrable. We cannot expect that for fixed x, $\varphi_E(u, x)$ will be subadditive in u. However, we can prove for it an inequality similar to that of Lemma 1.

LEMMA 2. *Let f satisfy $(C_{\alpha,p})$ with $p \geqslant 1$ and $\alpha \geqslant 0$. For each x in $(0, 2\pi)$ and each u in $(0, x_E)$*

$$\varphi_E^p(u, x) \leqslant \frac{2}{\log 2} u^\alpha \int_{2E+x} g(s)\,ds.$$

Fix x, and let $0 < u$, $v \leqslant x_E$. By Minkowski's inequality,

$$\varphi_E(u+v, x) \leqslant \Big(\int_{E+x+v} |f(s+u)-f(s)|^p ds\Big)^{1/p} + \varphi_E(v, x).$$

Since $0 < v \leqslant x_E$, then $E+v \subset 2E$. Thus

$$\varphi_E(u+v, x) \leqslant \varphi_{2E}(u, x) + \varphi_{2E}(v, x). \tag{3}$$

This inequality is the analogue of the subadditivity property. Now

$$\int_0^{2\pi} \frac{\varphi_{2E}^p(u,x)\,du}{u^{1+\alpha}} = \int_{2E+x} g(s)\,ds.$$

Let M denote the value of this integral, and let S be the set of u such that $\varphi_{2E}(u, x) > M^{1/p} u^{\alpha/p}/(\log 2)^{1/p}$. A slight variation of the argument used in the proof of Lemma 1 shows that for each u in $(0, 2\pi)$, there exists v in $(0, u)$ and $w = u-v$ both in the complement of S. Let $0 < u \leqslant x_E \leqslant 2\pi$. By (3), we may write

$$\varphi_E(u, x) \leqslant \varphi_{2E}(v, x) + \varphi_{2E}(w, x) \leqslant \Big(\frac{M}{\log 2}\Big)^{1/p} (v^{\alpha/p} + w^{\alpha/p}) \leqslant 2\Big(\frac{M}{\log 2}\Big)^{1/p} u^{\alpha/p}.$$

This completes the proof of the lemma.

As mentioned earlier, the first step in the proof of the theorem is the approximation of f by an integral mean. Thus let

$$f_n(x) = \frac{1}{2\Delta_n} \int_{x-\Delta_n}^{x+\Delta_n} f(t)\,dt, \qquad \Delta_n = \frac{\pi}{(2n+1)^\delta}; \qquad \varphi_n(x) = f_n(x) - f(x).$$

LEMMA 3. *Let f satisfy $(C_{\alpha,p})$ with $p \geqslant 1$ and $\alpha > 0$. Let δ in the definition of Δ_n satisfy $1/\alpha \leqslant \delta$, and let $\beta = \alpha\delta - 1$. Then*

$$\int_0^{2\pi} \sum_{n \geqslant 1} n^\beta |\varphi_n(x)|^p dx \leqslant C \int_0^{2\pi}\int_0^{2\pi} \frac{|f(x+u)-f(x)|^p}{u^{1+\alpha}} du\,dx.$$

C on the right side of this inequality denotes a constant depending in this case on the parameters α, β, δ, and p. Throughout the remainder of the paper, we shall let C denote a constant which may be different in different contexts and usually without specifying its dependence on particular parameters. Lemma 3 is only a slightly more general result than Lemma 2 of [5], and its proof is very close to the proof of the latter. We therefore omit it. The next lemma is a consequence of our Lemma 3 and the following inequality from [4]:

$$\int_0^{2\pi}\int_0^{2\pi} |I_{n,u}(x; f)|^p du\,dx \leqslant C_p \int_0^{2\pi} |f(x)|^p dx, \qquad p > 1.$$

LEMMA 4. *Let f satisfy $(C_{\alpha,p})$, $p > 1$, $\alpha > 0$, and let δ and β be as above. Then for almost every (x, u)*

$$\sum_{n \geqslant 1} n^\beta |I_{n,u}(x; f_n) - I_{n,u}(x; f)|^p < \infty.$$

3. In view of this last result, $I_{n,u}(x, f_n) - I_{n,u}(x; f) = o(1)$ for almost every (x, u). Thus our theorem will be proved if we can show that $I_{n,u}(x; f_n) - f(x) = o(1)$ for almost every (x, u) with an appropriate choice of δ. We choose δ so that $1/\alpha < \delta < 1+\alpha$. This choice is possible if $\alpha > \alpha_0$. It may be assumed that $\alpha < 1$ since decreasing α does not invalidate the $(C_{\alpha,p})$ condition, and we do this now for technical reasons. The $(C_{\alpha,p})$ condition implies that the functions

$$g(x) = \int_0^{2\pi} |f(x+u)-f(x)|^p du/u^{1+\alpha}, \qquad h(x) = \int_0^{2\pi} |f(x-u)-f(x)|^p du/u^{1+\alpha}$$

are integrable. Let x be a point such that $g(x)$ and $h(x)$ exist and such that their integrals have finite derivatives at x. For notational simplicity only we let $x = 0$ be such a point and assume further that $f(0) = 0$. The expression for $I_{n,u}(0; f_n)$ is periodic in u of period $2\pi/(2n+1)$ so that we may let $|u| \leqslant \pi/(2n+1)$. Let ω_n be a step function having positive jumps of $2\pi/(2n+1)$ at the points $2\pi m/(2n+1)$, $m = 0, \pm 1, \ldots$ Then we may write $I_{n,u}$ as a Stieltjes integral over any interval of length 2π. Thus (cf. [6])

$$I_{n,u}(0; f_n) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_n(u+t)D_n(u+t)\,d\omega_n(t)$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f_n\Big(u+t+\frac{2\pi}{2n+1}\Big)D_n\Big(u+t+\frac{2\pi}{2n+1}\Big)\,d\omega_n(t).$$

Taking the mean of these two expressions for $I_{n,u}(0; f_n)$ gives

$$I_{n,u}(0; f_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(f_n(u+t) - f_n\Big(u+t+\frac{2\pi}{2n+1}\Big)\Big)D_n(u+t)\,d\omega_n(t) +$$
$$+\frac{1}{2\pi}\int_{-\pi}^{\pi} f_n\Big(u+t+\frac{2\pi}{2n+1}\Big)\Big(D_n(u+t)+D_n\Big(u+t+\frac{2\pi}{2n+1}\Big)\Big)\,d\omega_n(t).$$

We denote the first integral on the right by $I^*_{n,u}(0; f_n)$, the second by $I'_{n,u}(0; f_n)$ and show that each is $o(1)$. The interval of integration for $I'_{n,u}(0; f_n)$ is divided into three parts: the first with $|t| \leqslant 5\pi/(2n+1)$, the second with t in $(5\pi/(2n+1), \pi)$, and the third with t in $(-\pi, -5\pi/(2n+1))$. Denote the corresponding integrals by D, E_1, and E_2, respectively. D consists of five terms of the form

$$\frac{1}{2n+1}f_n\Big(u+\frac{2\pi(j+1)}{2n+1}\Big)\Big(D_n\Big(u+\frac{2\pi j}{2n+1}\Big)+D_n\Big(u+\frac{2\pi(j+1)}{2n+1}\Big)\Big), \qquad |j| \leqslant 2.$$

Since $|D_n(t)| \leqslant n+1$ for all t, and since $|u| \leqslant \pi/(2n+1)$, it is enough to show that $f_n(v)$ is small if $|v| \leqslant 7\pi/(2n+1)$. By Holder's inequality

$$|f_n(v)|^p \leqslant \frac{1}{2\Delta_n}\int_{v-\Delta_n}^{v+\Delta_n}|f(t)|^p\,dt.$$

The variable of integration satisfies $|t| \leqslant 8\pi/(2n+1)$ so that $1/2\Delta_n \leqslant (2n+1)^\delta \leqslant C/|t|^{1+\alpha}$ since $\delta < 1+\alpha$. Recalling that $f(0) = 0$, we see that $|D|^p$ is majorized by $g(0)+h(0)$ times a term which is $o(1)$ as n increases.

To estimate E_1, we use the easily verifiable fact that $\int_0^{2\pi}|D_n(t)|\,d\omega_n(u+t) \leqslant C\log n$, for any u. An application of Hölder's inequality shows that

$$|E_1|^p \leqslant C(\log n)^{p-1}\int_{5\pi/(2n+1)}^{\pi}\Big|f_n\Big(u+t+\frac{2\pi}{2n+1}\Big)\Big|^p\Big|D_n(u+t)+$$
$$+D_n\Big(u+t+\frac{2\pi}{2n+1}\Big)\Big|\,d\omega_n(t).$$

For t in the given interval and for $|u| \leqslant \pi/(2n+1)$,

$$\Big|D_n(u+t)+D_n\Big(u+t+\frac{2\pi}{2n+1}\Big)\Big| \leqslant \frac{C}{(2n+1)(u+t)^2} \leqslant \frac{4C}{(2n+1)t^2}.$$

We substitute this into the above integral, introduce the function

$$\Psi_n(t) = \int_{5\pi/(2n+1)}^{t}\Big|f_n\Big(u+v+\frac{2\pi}{2n+1}\Big)\Big|^p\,d\omega_n(v)$$

and integrate by parts to obtain

$$|E_1|^p \leqslant C\frac{(\log n)^{p-1}}{2n+1}\cdot\frac{\Psi_n(\pi)}{\pi^2} + C\frac{(\log n)^{p-1}}{2n+1}\int_{5\pi/(2n+1)}^{\pi}\frac{\Psi_n(t)}{u^3}\,dt.$$

Let $\mathcal{I}(v)$ denote the interval $(v-\Delta_n, v+\Delta_n)$. By Hölder's inequality,

$$\Psi_n(t) \leqslant C(2n+1)^\delta\int_{5\pi/(2n+1)}^{t}d\omega_n(v)\int_{\mathcal{I}(u+v+2\pi/(2n+1))}|f(s)|^p\,ds$$
$$\leqslant C(2n+1)^{\delta-1}\int_0^{2t}|f(s)|^p\,ds.$$

The justification for the second inequality is that the intervals $\mathcal{I}\big(u+v+ +2\pi/(2n+1)\big)$ corresponding to the jumps of ω_n are disjoint and contained in $(0, 2t)$. Thus

$$|E_1| \leqslant C(\log n)^{p-1}(2n+1)^{\delta-2}\int_0^{2\pi}|f(s)|^p\,ds +$$
$$+C(\log n)^{p-1}(2n+1)^{\delta-2}\int_{5\pi/(2n+1)}^{\pi}\frac{dt}{t^3}\int_0^{2t}|f(s)|^p\,ds.$$

The first term on the right is $o(1)$ since $\alpha < 1$ and so $\delta < 2$. For the second we write $t^{-3} = t^{-2+\alpha}t^{-1-\alpha} \leqslant Ct^{-2+\alpha}s^{-1-\alpha}$ if s belongs to the interval $(0, 2t)$. Thus the second term above does not exceed

$$C(\log n)^{p-1}(2n+1)^{\delta-2}\int_{5\pi/(2n+1)}^{\pi}\frac{dt}{t^{2-\alpha}}\int_0^{2t}\frac{|f(s)|^p}{s^{1+\alpha}}\,ds.$$

Recalling that $f(0) = 0$, we see that the inner integral does not exceed $g(0)$. An integration then shows that $E_1 = o(1)$ since $\delta < 1+\alpha$. A similar argument holds for E_2 so that $|I'_{n,u}(0; f_n)| = o(1)$.

To show that $I^*_{n,u}(0; f_n) = o(1)$, we again divide the interval of integration into three parts: the first with $|t| < 3\pi/(2n+1)$, the second with t in $(3\pi/(2n+1), \pi)$, and the third with t in $(-\pi, -3\pi/(2n+1))$. Designate the three integrals by A, B_1 and B_2, respectively. For the same reasons that D above is small, $A = o(1)$. B_1 may be treated as E_1 above. Thus since $|D_n(u+1)| \leqslant C/(u+t)$,

$$|B_1| \leqslant C(\log n)^{p-1}\frac{\Phi_n(\pi)}{\pi} + C(\log n)^{p-1}\int\limits_{3\pi/(2n+1)}^{\pi}\frac{\Phi_n(t)}{(u+t)^2}\,dt\,, \tag{4}$$

where

$$\Phi_n(t) = \int\limits_{3\pi/(2n+1)}^{t}\left|f_n(u+v) - f_n\left(u+v+\frac{2\pi}{2n+1}\right)\right|^p d\omega_n(v)\,.$$

To estimate Φ_n, we write

$$\left|f_n(u+v) - f_n\left(u+v+\frac{2\pi}{2n+1}\right)\right|^p \leqslant Cn^{\delta}\int\limits_{\mathcal{I}(u+v)}\left|f(s) - f\left(s+\frac{2\pi}{2n+1}\right)\right|^p ds$$

and so

$$\begin{aligned}\Phi_n(t) &\leqslant Cn^{\delta}\int\limits_{3\pi/(2n+1)}^{t} d\omega_n(v)\int\limits_{\mathcal{I}(u+v)}\left|f(s) - f\left(s+\frac{2\pi}{2n+1}\right)\right|^p ds \\ &\leqslant Cn^{\delta-1}\int\limits_{0}^{t+u+\Delta_n}\left|f(s) - f\left(s+\frac{2\pi}{2n+1}\right)\right|^p ds\,.\end{aligned} \tag{5}$$

Since $t+u+\Delta_n \leqslant t+2\pi/(2n+1)$, we consider

$$\varphi_E^p\left(\frac{2\pi}{2n+1}, 0\right) = \int\limits_{0}^{t+2\pi/(2n+1)}\left|f(s) - f\left(s+\frac{2\pi}{2n+1}\right)\right|^p ds\,,$$

where E is the set $(0, t+2\pi/(2n+1))$. An application of Lemma 2 gives

$$\varphi_E^p\left(\frac{2\pi}{2n+1}, 0\right) \leqslant \frac{2^p}{\log 2}\left(\frac{2\pi}{2n+1}\right)^{a}\int\limits_{0}^{2t+4\pi/(2n+1)} g(s)\,ds\,.$$

We note that $2t+4\pi/(2n+1) \leqslant 4t$ for the range of t values we consider, and that the integral of g has, by assumption, a finite derivative at 0. Thus $\int_0^{4t} g(s)\,ds = O(t)$. Substitution of these inequalities into (5) shows that

$$\Phi_n(t) = (2n+1)^{\delta-1-a}O(t)\,.$$

It follows from (4) that $|B_1|^p$ does not exceed a constant multiple of

$$(\log 2n+1)^{p-1}(2n+1)^{\delta-1-a}\left[1 + \int\limits_{3\pi/(2n+1)}^{\pi}\frac{t}{(u+t)^2}\,dt\right].$$

The integral on the right is $O(\log n)$. Since $\delta-1-a<0$, $B_1 = o(1)$. By an analogous argument involving the differentiability of h at 0, we may show that $B_2 = o(1)$, and the proof of (i) is complete.

4. The function to be constructed for the proof of (ii) is of the form $\sum_{i\geqslant 1} f_{m(i)}(x)$, where $f_m(x)$ is a step function equal to the positive real A_m if $|x-2\pi j/m| \leqslant 4\pi A_m^{1/2}/m^2$ for some integer j and equal to 0 otherwise. The numbers A_m increase slowly to ∞ with m and can be chosen in many ways. To be definite, we shall say $A_m = (\log m)^{1/2}$. It is known [1] that for (x, u) in a set E_m, $\sup_{n\leqslant m}|I_{n,u}(x; f_m)| > CA_m^{1/2}$. The set E_m is a subset of the square in the xu-plane of side 2π, and E_m has measure $4\pi^2-\varepsilon_m$ where $\varepsilon_m = o(1)$. Thus, if the sequence $m(i)$ increases rapidly enough, the sequence $I_{n,u}(x; f)$ diverges for almost every (x, u).

It now remains only to describe the sequence $m(i)$ so that f satisfies the condition $(\mathrm{C}_{a,p})$. Let J_m be the set for which $f_m(x) \neq 0$. Since $|J_m|$, the measure of J_m, equals $8\pi A_m^{1/2}/m$, we may demand that $\sum_{i\geqslant 1}|J_{m(i)}| < \infty$. This implies that $f(x)$ is at least defined for almost all x. Let K_i be the subset of $J_{m(i)}$ such that $f_{m(j)} = 0$ if $j > i$. The K_i's are mutually disjoint, and their union is, except for a set of measure 0, the set where $f(x) \neq 0$. Since A_m increases and $A_m^p|J_m|$ decreases, we may further specify that

$$\left|\sum_{k\leqslant j} A_{m(k)}\right|^p \leqslant 2A_{m(k)}^p\,; \qquad \sum_{k\geqslant j} A_{m(k)}^p|J_{m(k)}| \leqslant 2A_{m(j)}^p|J_{m(j)}|\,. \tag{6}$$

If $f_{m(j)}(x+u) - f_{m(j)}(x) \neq 0$, then $x+u$ belongs to $J_{m(j)}$ and x does not, or the opposite is true. Thus, if $0 \leqslant u \leqslant \pi/m(j)$, then $f_{m(j)}(x+u) - f_{m(j)}(x) = 0$ except for a set of x measure not exceeding $2um(j)$. Hence

$$\int\limits_{0}^{2\pi}|f_{m(j)}(x+u) - f_{m(j)}(x)|^p dx \leqslant 2um(j)\,A_{m(j)}^p\,. \tag{7}$$

Let $b_i = \pi/m(i)^{\beta}$ where $1/(1-a) < \beta < 1/a$. This choice is possible if $a < 1/2$. We write $F_i(x) = \sum_{j\leqslant i} f_{m(j)}(x)$, $G_i(x) = f(x) - F_i(x)$. By Hölder's inequality,

$$|F_i(x+u) - F_i(x)|^p \leqslant i^{p-1}\sum_{j=1}^{i}|f_{m(j)}(x+u) - f_{m(j)}(x)|^p$$

and thus from (7) if $0 \leqslant u \leqslant \pi/m(i)$,

$$\int\limits_{0}^{2p}|F_i(x+u) - F_i(x)|^p dx \leqslant 2i^{p-1}\sum_{j=1}^{i} A_{m(j)}^p m(j) \leqslant 2i^p m(i)\,uA_{m(i)}^p\,. \tag{8}$$

Also from (6)

$$(9)\qquad \int_0^{2\pi} |G_i(x+u)-G_i(x)|^p dx \leqslant 2^p \int_0^{2\pi} |G_i(x)|^p dx \leqslant 2^p \sum_{k>i} A^p_{m(k)} |K_k| \leqslant 2^{p+1} A^p_{m(i+1)} |J_{m(i+1)}|.$$

Now

$$\int_{b_{i+1}}^{b_i} \frac{du}{u^{1+a}} \int_0^{2\pi} |f(x+u)-f(x)|^p dx \leqslant 2^{p+1} \int_{b_{i+1}}^{b_i} \frac{du}{u^{1+a}} \int_0^{2\pi} |F_i(x+u)-F_i(x)|^p dx + 2^{p+1} \int_{b_{i+1}}^{b_i} \frac{du}{u^{1+a}} \int_0^{2\pi} |G_i(x+u)-G_i(x)|^p dx.$$

Denote the integral on the left by a_i. In view of (8) and (9),

$$a_i \leqslant (2i)^p m(i) A^p_{m(i)} \int_{b_{i+1}}^{b_i} \frac{du}{u^a} + 2^{2p} A^p_{m(i+1)} |J_{m(i+1)}| \int_{b_{i+1}}^{b_i} \frac{du}{u^{1+a}} \leqslant (2i)^p m(i) A^p_{m(i)} \frac{b_i^{1-a}}{1-a} + 2^{2p+3}\pi \frac{A^{p+1/2}_{m(i+1)}}{m(i+1)} \cdot \frac{b_{i+1}^{-a}}{a}.$$

Now $m(i) b_i^{1-a} \leqslant C m^{\mu}(i)$ and $b_{i+1}^{-a}/m(i+1) \leqslant C m^{\nu}(i+1)$ where $\mu = 1+\beta(a-1)$ and $\nu = a\beta-1$. Since both exponents are negative because of restrictions on β, the sequence $m(i)$ may now easily be chosen so that $\sum_{i\geqslant 1} a_i$ converges. This establishes the fact that f satisfies $(C_{a,p})$ and so completes the proof of (ii).

Theorem 1 holds also for Jackson polynomials. In fact, we may be slightly more precise in part (i) by allowing $a = a_0$. Since the proofs of both (i) and (ii) are much simpler in this case, details are not necessary. The involved argument centering around the function $\Phi_n(t)$ may be omitted entirely. Paper [1] contains a proof that our construction holds for Jackson polynomials.

5. The next theorem is an interpretation of Theorem 1 in terms of fractional integrals and generalizes Theorem 5 of [5].

THEOREM 2. (i) *Let $p>1$, and let $a > a_0 = (\sqrt{5}-1)/2$. Let f be the fractional integral of order a/p of a function g of class L^p. For almost every (x,u), $I_{n,u}(x;f)$ converges to $f(x)$.*

(ii) *For every $p>1$ and a, $0<a<1/2$, there exists a function f which is the fractional integral of order a/p of a function g of class L^p such that $I_{n,u}(x;f)$ diverges for almost every (x,u).*

The theorem is a rather easy consequence of Theorem 1 and the following lemma, the first part of which, at least, is elementary.

LEMMA 5. (i) *Let $p>1$ and $0<a<p$. If f is the fractional integral of order a/p of a function g of class L^p, then $\int_0^{2\pi} |f(x+u)-f(x)|^p dx = O(u^a)$.*

(ii) *Let $0<\beta<p$. If f satisfies $(C_{\beta,p})$, then for every a, $0<a<\beta$, f is the fractional integral of order a/p of a function g of class L^p.*

Both parts of the lemma can be made more precise in special cases, but it contains all the informations we need. Let $\sum c_n e^{inx}$ be the Fourier series of f. It has been implicitly assumed that $c_0 = 0$. For notational convenience, assume further that $c_n = 0$ if $n \leqslant 0$. The Fourier series of $f(x+u)-f(x-u)$ is then $2i\sum_{n\geqslant 1} c_n (\sin nu) e^{inx}$. u is temporarily fixed, and we consider separately the sums corresponding to $n \leqslant 1/u$ and to $n > 1/u$. The first sum can be thought of as the transform of the sum $\sum_{1\leqslant n\leqslant 1/u} (nu)^{a/p} c_n e^{inx}$ after application of the multipliers $(nu)^{-a/p}(\sin nu)$. Since these multipliers are bounded by one and essentially monotone, we have

$$\int_0^{2\pi} \Big|\sum_{n=1}^{1/u} c_n (\sin nu) e^{inx}\Big|^p dx \leqslant C u^a \int_0^{2\pi} \Big|\sum_{n=1}^{1/u} n^{a/p} c_n e^{inx}\Big|^p dx.$$

For the second sum

$$\int_0^{2\pi} \Big|\sum_{n>1/u} c_n (\sin nu) e^{inx}\Big|^p dx \leqslant C \int_0^{2\pi} \Big|\sum_{n>1/u} c_n e^{inx}\Big|^p dx.$$

The series $\sum_{n>1/u} c_n e^{inx}$ is the transform of $\sum_{n>1/u} (nu)^{a/p} c_n e^{inx}$ after application of the multipliers $(nu)^{-a/p}$ which are bounded by one and decrease to zero. Thus

$$\int_0^{2\pi} \Big|\sum_{n>1/u} c_n e^{inx}\Big|^p dx \leqslant C u^a \int_0^{2\pi} \Big|\sum_{n>1/u} n^{a/p} c_n e^{inx}\Big|^p dx.$$

Since the sequence $n^{a/p} c_n$ is, except for a complex constant, the sequence of Fourier coefficients of g, part (i) of the lemma follows by combining two of the above inequalities and noting that

$$\int_0^{2\pi} |f(x+u)-f(x-u)|^p dx = \int_0^{2\pi} |f(x+2u)-f(x)|^p dx.$$

The hypothesis of Lemma 5 (ii) insures by Lemma 1 that

$$\int_0^{2\pi} |f(x+u)-f(x)|^p dx = O(u^\beta).$$

Thus if $a < \beta$,

$$\int_0^{2\pi} \frac{du}{u^{1+a/p}} \left\{ \int_0^{2\pi} |f(x+u)-f(x)|^p dx \right\}^{1/p} < \infty .$$

This implies that f is the desired fractional integral [3].

To apply the lemma to the proof of part (i) of the theorem, we choose β such that $a_0 < \beta < a$ and note that the conclusion of the lemma insures that f satisfies $(C_{\beta,p})$. Hence Theorem 1 is applicable. For part (ii) of the theorem, we choose β such that $a < \beta < 1/2$, and construct f as before to satisfy $(C_{\beta,p})$ and so that $I_{n,u}(x;f)$ diverges for almost every (x, u).

6. The underlying idea in our construction of examples f for which $I_{n,u}(x;f)$ diverges almost everywhere is that for some n, depending on x and u, the functional value at the interpolating point $u+2\pi j/(2n+1)$ which is closest to x is large (cf. [1]). This suggests the concept of translation continuity which we now introduce. Given two real numbers, x and u, which do not differ by a rational multiple of π, let $r = r(x, u, n)$ be the integer uniquely defined by the relation $|x-u-2\pi r/n| < \pi/n$. We shall say that a measurable, periodic function f is *translation continuous at* x if for almost every u

$$\lim_n f\left(u + \frac{2\pi r}{n}\right) = f(x) .$$

The exceptional set is specifically meant to include those u which differ from x by a rational multiple of π so that there is no ambiguity in the definition of r. There are several reasons for using this type of continuity in connection with interpolating polynomials. Here we consider only certain aspects of it related to our previous work.

The conclusions of Theorem 1 are applicable to the notion of translation continuity, even in a slightly more precise form: i.e. *if f satisfies $(C_{a_0,p})$, then f is translation continuous almost everywhere, and there are functions f satisfying $(C_{a,p})$ for $a < 1/2$ such that f is translation continuous almost nowhere.* The proof follows the lines of that for Theorem 1, but it is much simpler. Thus, in the proof of the positive part, we show by the previous technique that $f(u+2\pi r/n) - f_n(u+2\pi r/n) = o(1)$ and then that $f_n(u+2\pi r/n) - f(x) = o(1)$. An examination of the proof of the latter step at the pertinent spot reveals that what is involved is a kind of unsymmetric differentiation of the integral of f.

There is, as before, a slight gap in the bounds of a used for the positive and the negative parts of the theorem. However, as regards translation continuity at a point, we can be very precise. Let us say that f satisfies the condition $(D_{a,p})$ at the point x if

$$(D_{a,p}) \qquad \int_{x-\pi}^{x+\pi} \frac{|f(u)-f(x)|^p}{|u-x|^{1+a}}\, du < \infty .$$

THEOREM 3. (i) *Let $p \geq 1$, and let f satisfy $(D_{1,p})$ at the point x. Then f is translation continuous at x.*

(ii) *For every $p \geq 1$ and $a < 1$, there is a function f satisfying the condition $(D_{a,p})$ at the point 0 and such that f is not translation continuous at 0.*

Let $g(u) = |f(u)-f(x)|/|x-u|^{2/p}$. The hypothesis implies that g is in class L^p over $(x-\pi, x+\pi)$. It is to be extended periodically outside this interval. As is known [2], for almost every u,

$$\lim_n \max_j \frac{|g(u+2\pi j/n)|^p}{n^2} = 0 .$$

Now choose r as before so that $|x-u-2\pi r/n| < \pi/n$. Then

$$|f(u+2\pi r/n)-f(x)|^p = |x-u-2\pi r/n|^2 |g(u+2\pi r/n)|^p \leq \frac{\pi^2 |g(u+2\pi r/n)|^p}{n^2} .$$

By virtue of the previous formula, the term on the right is $o(1)$.

For the construction of f in the proof of (ii), we first let γ_i be a sequence of reals decreasing to 0 and such that $\sum_{i\geq 1} \gamma_i < \infty$. Let $m(i)$ be a sequence of integers increasing rapidly to ∞. Several conditions will be imposed on the sequence $m(i)$, the first being that $1/\gamma_i < \log m(i)$. Let the periodic function $f_i(x)$ be defined as 1 if $0 \leq |x-1/2m(i)| < 1/\gamma_i m^2(i)$ and as 0 otherwise. The intervals $\big(1/2m(i)-1/\gamma_i m^2(i),\ 1/2m(i)+1/\gamma_i m^2(i)\big)$, $i = 1, 2, \ldots$ may be taken as disjoint and lying in $(0, \pi)$. We note that for u in the ith interval $u \geq 1/4m(i)$. Now let $f(x) = \sum_{i\geq 1} f_i(x)$, an everywhere convergent series such that $f(0) = 0$, and such that

$$\int_{-\pi}^{\pi} \frac{|f(u)|^p}{|u|^{1+a}}\, du \leq 2^{3+2a} \sum_{i=1}^{\infty} \frac{1}{\gamma_i m^{1-a}(i)} .$$

The sum on the right is convergent for proper choice of the $m(i)$ since $1-a > 0$ so that f satisfies the condition $(D_{a,p})$ for any $p \geq 0$ at the point 0. Let E_i be the subset of $(0, 2\pi)$ such that for u in E_i,

$$|u+2\pi j/n - 1/2m(i)| < 1/\gamma_i m^2(i)$$

for some integers j and n with $\gamma_i m(i) < n \leq m(i)$. For u in E_i, $f(u+2\pi j/n) = f_i(u+2\pi j/n) = 1$. But

$$|u+2\pi j/n| \leq |u+2\pi j/n - 1/2m(i)| + 1/2m(i) < \pi/m(i)$$

so that j is the integer r associated with u, 0, and n as in our definition of translation continuity. Thus for u in E_i

$$f(u+2\pi r/n) = f_i(u+2\pi r/n) = 1 . \tag{10}$$

It is known [1] that $|E_i|$, the measure of E_i, exceeds $2\pi - C\gamma_i$ so that $|\liminf E_i| = 2\pi$. Thus for almost every u, the equation (10) occurs infinitely often.

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On Vietoris mapping theorem and its inverse

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0. We shall use Čech cohomology groups with coefficients in an arbitrary abelian group A. Let X, Y, T be compact Hausdorff spaces and let $f\colon X\to Y$, $g_1\colon X\to T$, $g_2\colon Y\to T$ be continuous onto maps such that $g_2 f = g_1$. In this paper we prove (Theorem 1) that if f induces isomorphisms of ith cohomology groups for $i = 0, 1, \dots, n$ and a monomorphism of the $(n+1)$st cohomology groups of fibres $g_2^{-1}(x)$, $g_1^{-1}(x)$, for all $x \in T$, then f induces isomorphisms of ith cohomology groups for $i = 0, 1, \dots, n$ and a monomorphism of the $(n+1)$st cohomology groups of spaces Y and X. The result generalizes the well-known Vietoris-Begle theorem [1].

On the other hand, we show (Theorem 2) that if there exists a totally disconnected subset $T_1 \subset T$ such that the fibres $g_1^{-1}(x)$, $g_2^{-1}(x)$ are $(n+1)$-acyclic for $x \in T - T_1$ and f induces isomorphisms of ith cohomology groups of Y and X for $i = 0, 1, \dots, n$, then f induces isomorphisms of ith cohomology groups of fibres $g_2^{-1}(x)$, $g_1^{-1}(x)$ for all $x \in T$ and $i = 0, 1, \dots, n$. In the last part we give some applications of the theorems connected with an Eilenberg-Kuratowski theorem [2].

1. All topological spaces considered here are Hausdorff. Let T denote a topological compact space. $\mathcal{F}, \mathcal{G}, \mathcal{I}$ will denote sheaves of abelian groups. If $\mathcal{F}$ is a sheaf over T and $x \in T$, then $\mathcal{F}_x$ denotes the stalk of $\mathcal{F}$ over x. For any abelian group A, A^T denotes the constant sheaf over T with stalks A. If $U \subset T$, then $\Gamma(U, \mathcal{F})$ denotes the group of all cross-sections over U into $\mathcal{F}$. If $d \in \Gamma(U, \mathcal{F})$, then $\bar{d}$ denotes the carrier of d, i.e., the subset of U composed of all $x \in U$ such that $d(x) \neq 0$. $\bar{d}$ is a closed subset of U. We shall write $\Gamma(\mathcal{F})$ instead of $\Gamma(T, \mathcal{F})$. We say that $\mathcal{F}$ has its support in $U \subset T$ if $\mathcal{F}_x = 0$ for any $x \in T - U$. $\mathcal{F}|U$ will denote the restriction of $\mathcal{F}$ to U. It is known that, if $\mathcal{F}$ is an injective sheaf and U is a closed subset of T, then $\mathcal{F}|U$ is a soft sheaf. The ith cohomology group of X with coefficients in $\mathcal{F}$ is denoted by $H^i(X, \mathcal{F})$, $i = 0, 1, 2, \dots$

F, I, J will denote positive cochain complexes of sheaves. The mth sheaf of F will be denoted by F_m, $m = 0, 1, \dots$ If we write $F_m \to F_{m+1}$,