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Incompactness in languages with infinitely long expressions

by

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1. First order predicate logic has the well-known compactness property: If every finite subset of a set Γ of sentences has a model, then the set Γ has a model. This property is not, in general, shared by languages with a greater power of expression. In this paper we consider, corresponding to any given infinite cardinal α, the logic L_α which differs from first order predicate logic in that disjunctions and conjunctions of sequences of formulas of type less than α and quantifications over fewer than α variables are allowed. For many cardinals α we shall show that L_α is incompact; that is, we shall exhibit a set Γ of sentences of L_α which has no model but is such that every subset of Γ of power less than α has a model (1).

For the most part, the set theoretical part of our discussion can be carried out on the basis of Zermelo-Fraenkel set theory. Without some modification, however, some of the set theoretical statements we make would not have their proper meaning in Z-F. For example, we will have occasion to speak of the class of all accessible cardinals. This is not, of course, a set in the sense of Z-F at all. On the other hand, it would do little good to use the Bernays type of set theory (where we have sets and arbitrary classes of sets) since we also have occasion to consider functions defined on classes, etc. Although it could easily be avoided, at one point in Section 3, we use "ordinals" to index the set of all ordinals in the sequence determined by an arbitrary well-order-

(1) Languages with infinitely long expressions were introduced by Tarski in [16]. The author's results on incompactness of L_α for inaccessible α were first stated in the abstract [3]. This paper represents an essential portion of the author's doctoral dissertation presented to the Graduate Division of the University of California, Berkeley, California. Grateful acknowledgement is made to Professor Tarski for his encouragement and assisstance at many stages of this work. It should be pointed out that the Löwenheim-Skolem theorems given in [3] are incorrectly formulated. The correct formulations and references to the earlier work of Carol Karp are given in [5].

ing relation on the ordinals. For this it does not even suffice to further extend Bernays set theory by going a finite number of steps further and considering classes of classes, classes of classes of classes, etc. However, a satisfactory solution is obtained if we extend the theory Z-F with some additional axioms giving the existence of large inaccessible cardinals and then assume that all the cardinals and ordinals (except the "ordinals" mentioned above) which we consider are less than some fixed inaccessible cardinal Ω. Axioms which insure the existence of very large inaccessible cardinals have been formulated, for example, by Tarski and Lévy (see [19] and [10]). Some still stronger axioms can be formulated in terms of some new constructions which are given in Section 3. Since the most important of our results concern inaccessible cardinals, it is desirable that we chose Ω to be a very large inaccessible cardinal in order that our statements concerning inaccessible cardinals are not vacuous.

For the most part, we use standard set theoretical terminology. As is customary, we assume that each ordinal number is the set of all smaller ordinals and that a cardinal number is an initial ordinal. A cardinal a is said to be ***singular*** if, for some ordinal $\beta < a$ there is a β-termed sequence γ such that $\gamma_\eta < a$ for all $\eta < \beta$, and

$$a = \bigcup_{\eta<\beta} \gamma_\eta$$

A cardinal is said to be *regular* if it is not singular. The symbol a^+ is used to denote the *successor* of the cardinal a and a^β denotes cardinal exponentiation, while $+$ and $\cdot$ denote ordinal addition and multiplication, respectively. A cardinal a is said to be *accessible* if it is singular or if $a \leqslant 2^\beta$ for some cardinal $\beta < a$. A cardinal which is either singular or the successor of a smaller cardinal is said to be *strongly accessible*. A cardinal which is not accessible, or not strongly accessible, is said to be *inaccessible*, or *weakly inaccessible*, respectively.

The logic L_a can be described as follows: The signs of L_a include a distinct variables. We will use the symbols w, x, y, z and v_η for $\eta < a$ to denote variables; distinct symbols are assumed to denote distinct variables. $v \upharpoonright \varrho$ will denote the sequence of variables $v_0, \ldots, v_\eta, \ldots$ where η ranges over the ordinals smaller than ϱ. L_a has the equality predicate $=$ and predicate symbols $P_0, \ldots, P_\xi, \ldots$; the number and type of predicate symbols is determined by the similarity type $\mu = \langle \mu_0, \ldots, \mu_\xi, \ldots \rangle$. Thus the atomic formulas of L_a are of the form $x = y$ or of the form $P_\xi v \upharpoonright \varrho$ where the sequence $v \upharpoonright \varrho$ of variables is of type μ_ξ. In case $\mu_0 = 2$, we will write $x < y$ in place of $P_0 xy$. The class of well-formed formulas of L_a is the least class W such that:

(i) *Every atomic formulas is in* W.

(ii) *If* Φ *is in* W, *then its negation* $\neg \Phi$ *is in* W.

(iii) *If* $\varrho < a$ *and* Φ_η *is in* W *for each* $\eta < \varrho$, *then the conjunction* $\bigwedge_{\eta<\varrho} \Phi_\eta$ *is in* W.

(iv) *If* Φ *is in* W *and* $\varrho < a$, *then the universal generalization* $(\forall v \upharpoonright \varrho)\Phi$ *is in* W.

In writing formulas, we will use existential quantifiers $(\exists v \upharpoonright \varrho)$, infinite disjunction $\bigvee_{\eta<\varrho} \Phi_\eta$, and the usual finite sentential connectives $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$; these can be defined in an obvious way in terms of the basic operations for forming well formed formulas. We assume it to be known under what conditions a relational system $\mathfrak{A} = \langle A, R_0, \ldots, R_\xi, \ldots \rangle$ (where A is any non-empty set and each R_ξ is a μ_ξ-ary relation on the elements of A) is regarded as a model of a sentence or a set of sentences of L_a (see [14]).

In this paper, we consider a fundamental problem concerning the language L_a, that of compactness. The compactness property is an important respect in which the languages with infinitely long expressions differ from the language of ordinary logic. Any of the following three conditions express the compactness property of the language L_a:

(I) *If* Γ *is a set of sentences of* L_a *and every subset of* Γ *of power less than* a *has a model, then* Γ *has a model.*

(II) *If a sentence* Φ *of* L_a *is a consequence of a set* Γ *of sentences of* L_a, *then* Φ *is a consequence of some subset of* Γ *of power less than* a.

(III) *If a sentence* Φ *of* L_a *is equivalent to a set* Γ *of sentences of* L_a *(i.e. every model of* Φ *is a model of* Γ *and conversely), then* Φ *is equivalent to some subset of* Γ *of power less than* a.

It is easy to see that these three conditions are equivalent. All three are known to hold for the language L_ω. For example, (II) is an immediate consequence of Gödel's famous completeness theorem. For if a sentence Φ of L_ω is a consequence of a set Γ of sentences of L_ω, then by the completeness theorem there is a proof of Φ using premises from the set Γ; but since any proof is of finite length, only a finite number of sentences of Γ could have been used in the proof and so Φ is a consequence of this finite set of sentences. In the case of those languages L_a for which we have an incompactness result, there can be no notion of proof which is adequate for deductions from arbitrary sets of premises unless we allow proofs to be of length a or greater. Carol Karp has set up formal systems of proof for L_a (see [7]) which for the case of a inaccessible give all logically valid sentences of L_a as theorems. It follows from the results of Section 3 that these formal systems are not adequate for deductions from premises except possibly for very exceptional inaccessible a.

Condition (I) accounts for the name "compactness property". For if we replace each sentence Φ of Γ by the class of all models of Φ, then (I) says that if the intersection of each family of fewer than a classes of Γ is non-empty, then the intersection of all the classes of Γ is non-empty. In the case of L_ω this is just the finite intersection property for elementary classes and shows that the usual topology on the space of all models (i.e. the topology in which the closed sets are arbitrary intersections of elementary classes) is compact.

Condition (III) is related to the method by which the language L_α was constructed. In the case of L_ω it says that there is no sentence which is equivalent to an infinite conjunction of sentences except in the trivial case where the sentence is already equivalent to some finite conjunction of the same sentences. In the case of L_α we have intentionally introduced conjunctions of power less than α. The fact that L_α is incompact shows that we have in the process introduced at least some non-trivial conjunctions of power α or greater.

In order to be more specific in the formulation of theorems, we will say that the cardinal α is *β-incompact* if there exists a set Γ of sentences of L_α such that (i) the power of Γ is β, (ii) Γ has no model, and (iii) every subset of Γ of power less than α has a model. Following Tarski [18], α will be said to be *incompact* if it is β-incompact for some $\beta \geqslant \alpha$ and *strongly incompact* if it is α-incompact.

Certain incompactness results have been known for some time to Tarski. He first obtained a number of theorems concerning α-complete ideals in α-complete fields of sets, their extendability to α-complete prime ideals, relationships between α-distributivity of a Boolean algebra and its isomorphism to a field of sets, etc. These theorems applied exclusively to accessible cardinals. Later he realized that the properties studied by him were very closely related and some of them actually equivalent to the incompactness of the cardinals involved. In this way the incompactness of all accessible cardinals was established; the result is stated implicitly in Scott-Tarski [12], p. 170. For inaccessible cardinals (greater than ω) the compactness problem as well as these related set-theoretic and Boolean-algebraic problems remained entirely open. In Erdös-Tarski [1] it was even speculated that these problems might involve some fundamental difficulties and that their solution might require some essentially new set-theoretical axioms.

In Section 2 and 3, the compactness problem is studied using different, purely metamathematical methods. These methods are closely related to the method used by Tarski in [17] to construct a formal system which was consistent but not ω-consistent. Using these methods, the incompactness theorem is reestablished for all accessible cardinals. Actually we obtain some improvement in that we show that all accessible cardinals of a very comprehensive class are not only incompact, but strongly incompact. This class comprehends, in particular, all strongly accessible cardinals and therefore, under the assumption of the generalized continuum hypothesis, coincides with the class of all accessible cardinals. (Without this assumption, the problem whether all accessible cardinals are strongly incompact is still open.)

What is more important, these metamathematical methods allow the incompactness results to be extended to a very comprehensive class of inaccessible cardinals as well. Unfortunately, these results do not lead to the conclusion that all non-denumerable inaccessible cardinals (and hence all non-denumerable cardinals) are incompact. For this reason it seems important to characterize "constructively" large classes of cardinals to which these methods apply. The ideas of Mahlo (see [11]) can be used to obtain very comprehensive classes of cardinals defined constructively in terms of the class of accessible cardinals. Actually, we shall outline here a construction analogous to that of Mahlo but which yields a class of cardinals which is even larger (unless all non-denumerable cardinals are already obtained by Mahlo's constructions). We will show that each cardinal in this class in incompact. We will also indicate how to obtain even larger classes of inaccessible cardinals. However, it turns out that for each class defined in such a constructive way, it appears almost certain on the basis of "naive" set theory (or what is sometimes called "Cantor's absolute") that not all cardinals belong to the class. And, although we can show incompactness for each member of the class, we can also show incompactness for the first non-denumerable cardinal not in the class. It appears therefore that the results obtained by these methods almost by their very nature cannot be presented in an exhaustive way.

It should be added that Tarski, combining these incompactness results with his earlier observations concerning the relation between the incompactness problems and some problems in set theory and the theory of Boolean algebras, was able to extend to the class of all incompact cardinals various results which were originally established only for accessible cardinals; e.g. the theorem to the effect that every α-complete prime ideal in the field of all subsets of a set of power α is principal. The methods by which these results are obtained are outlined in [18]. In [9] Keisler and Tarski give a purely mathematical method of establishing these results and also give a thorough discussion of various mathematical consequences. In [4] the author shows that the results extend to a problem concerning simply ordered sets and a problem concerning graphs.

Section 2 contains incompactness results for accessible cardinals. Although the proof of Theorem 1 is a direct translation of an algebraic proof due to Tarski (see [15]), the result obtained in this way is slightly stronger than that obtained indirectly using the Boolean algebra which Tarski constructs. Theorems 2 and 3 show that every strongly accessible cardinal is strongly incompact. In Theorem 6 of Section 3, the methods developed for inaccessible cardinals are used to extend strong incompactness results to a class of weakly inaccessible cardinals. Thus, in the absence of the generalized continuum hypothesis, there may be certain accessible cardinals which are not strongly accessible and which are not shown to be strongly incompact by Theorem 1; in this case, a comprehensive class of such cardinals are shown to be strongly incompact in Theorem 6. The remainder of Section 3 is devoted to constructing a large class of inaccessible cardinals, establishing some of its properties (Theorem 4) and proving the main result (Theorem 5) that every cardinal in this class is strongly incompact.

2. Incompactness of accessible cardinals.

THEOREM 1. *Suppose that $\beta, \gamma < \alpha$ and $\alpha \leqslant \beta^\gamma$. Then α is β^γ-incompact.*

Proof. We form a set of sentences of L_α using sentential symbols (zero-placed predicates) $P_{\xi,\eta}$ where $\xi < \gamma$ and $\eta < \beta$ [2]. Let Γ be the set consisting of the sentence

(1) $$\bigwedge_{\xi<\gamma} \bigvee_{\eta<\beta} P_{\xi,\eta}$$

and all sentences of the form

(2) $$\neg \bigwedge_{\xi<\gamma} P_{\xi,f(\xi)},$$

where f is a function mapping γ into β. If (1) holds in a model $\mathfrak{A}$, then for each $\xi < \gamma$ there is an ordinal $\eta < \beta$ such that $P_{\xi,\eta}$ holds in $\mathfrak{A}$. Thus taking for each $\xi < \gamma$ $f(\xi)$ to be least such η, we define a function f such that $P_{\xi,f(\xi)}$ holds in $\mathfrak{A}$ for each $\xi < \gamma$. This contradicts sentence (2) and shows that Γ has no model. This can also be seen by noting that Γ contradicts the (γ, β)-distributive law (see [13]). On the other hand, any subset Γ' of Γ of power less than α (in fact, any proper subset of Γ) has a model since we can find a function f such that (2) is not in Γ' and construct a model in which $P_{\xi,f(\xi)}$ holds but $P_{\xi,\eta}$ fails for all $\eta \neq f(\xi)$. Since the set Γ has power β^γ, we conclude that α is β^γ-incompact.

The proofs of the following theorems (2, 3, and 5) were originally carried out using set theoretical models involving the membership relation ϵ (see the proof given in [18]). Tarski has pointed out that the

(2) It is possible to replace this infinite list of predicates by a single binary predicate. The same holds in later proofs where we use several predicates. As a matter of fact, the results of [2] concerning isomorphism of ineerpreted languages with various lists of predicates were first obtained for the infinite languages. In L_α, the requirements on two lists of predicates to insure isomorphism of the corresponding languages is that each list contain fewer than α predicates each having fewer than α places and that there be at least one binary (or ν-ary where $\nu \geqslant 2$) predicate or else two unary operations on each list.

proofs can equally well make use of well-ordered systems resulting in a somewhat simpler formulation of some of the proofs. In the following, we will write $x < y$ in place of P_0xy and will make use of the sentences

(3) $$(\forall xy)[x < y \vee y < x \vee x = y]$$

and

(4) $$\neg(\exists v \upharpoonright \omega)[\bigwedge_{\xi<\omega} v_{\xi+1} < v_\xi]$$

to characterize a well-ordering relation. Sentence (3) says that the relation is connected and (4) says that there is no infinite decreasing sequence of elements. Thus in any model $\mathfrak{A} = \langle A, R_0, \ldots\rangle$ which satisfies (3) and (4), the relation R_0 must well order the set A (it is easy to check, for example, that a relation satisfying (3) and (4) is transitive). Hence the system $\mathfrak{A}$ must be isomorphic to a system $\langle \varrho, <, \ldots\rangle$ where ϱ is an ordinal and $<$ is the less than relation restricted to ordinals less than ϱ. Thus in the proofs below, when we are considering models satisfying a set of sentences which includes (3) and (4), we will restrict our attention to models $\langle \varrho, <, \ldots\rangle$. Actually, the ability to formulate sentence (4) in L_α is crucial to the method employed here. Indeed, if we add enough individual constants to the language L_α, we can show incompactness for many cardinals (all those less than the first inaccessible to which Theorem 5 does not apply) using only sentences of ordinary logic together with the single infinite sentence (4).

In the following proofs, it will be convenient to have a formula Φ_η which defines the ordinal η. Let $\Phi_\eta(x)$ be the formula

$$(\exists v \upharpoonright \eta)\{[\bigwedge_{\zeta<\eta} \bigwedge_{\xi<\zeta} v_\xi < v_\zeta] \wedge (\forall z)[z < x \leftrightarrow \bigvee_{\xi<\eta} z = v_\xi]\}$$

$\Phi_\eta(y)$ is defined similarly with the free variable x replaced by y. It is clear that $\Phi_\eta(x)$ is a formula of the language L_α whenever $\eta < \alpha$. It is easily seen that in a model $\langle \varrho, <, \ldots\rangle$, an ordinal ν satisfies the formula $\Phi_\eta(x)$ just in case $\nu = \eta$.

THEOREM 2. *If α is singular, then α is strongly incompact.*

Proof. Assuming α singular, we can write

$$\alpha = \bigcup_{\xi<\beta} \gamma_\xi,$$

where $\beta < \alpha$ and $\gamma_\xi < \alpha$ for each $\xi < \beta$. We form a set of sentences of L_α making use of the binary predicate $<$ and a unary predicate P_1. Γ consists of the sentences (3) and (4), the sentence

(5) $$\bigvee_{\xi<\beta} (\exists v \upharpoonright \gamma_\xi)(\forall x)[P_1 x \rightarrow \bigvee_{\eta<\gamma_\xi} x = v_\eta]$$

and, for each $\eta < \alpha$, the sentence

(6) $$(\exists x)[\Phi_\eta(x) \wedge P_1 x].$$

Γ has no model since by sentences (3) and (4), any model of Γ is isomorphic to a system $\langle \varrho, <, R_1\rangle$ and sentence (5) says that, for some $\xi < \beta$, there are at most γ_ξ (and hence fewer than α) ordinals in R_1 whereas by (6), each ordinal $\eta < \alpha$ is in R_1. However, it is easy to see that any subset Γ' of Γ of power less than α has a model since for some $\xi < \beta$, there will be fewer than γ_ξ sentences of form (6) in Γ'. Thus α is strongly incompact.

THEOREM 3. *If α is a non-limit cardinal, then α is strongly incompact.*

Proof. Suppose that α is the successor of the cardinal β. Let Γ be the set of sentences of L_α consisting of sentences (3) and (4), the sentence

$$(\exists v \restriction \beta)(\forall x)\big[\textstyle\bigvee_{\xi<\beta} x = v_\xi\big] \tag{7}$$

and, for each ordinal $\eta < \alpha$, the sentence

$$(\exists x)[\Phi_\eta(x)]\,. \tag{8}$$

Suppose that $\langle \varrho, <\rangle$ is a model of Γ. From (8) we see that $\eta < \varrho$ for each $\eta < \alpha$. But (7) says that ϱ has power at most β. Therefore the set Γ has no model. Consider now any subset Γ' of Γ of power less than α. Let ϱ be the least ordinal which is larger than each ordinal η for which the sentence of form (8) is in Γ'. Since there are at most β such ordinals, we see that $\varrho < \alpha$ and that $\langle \varrho, <\rangle$ is a model of Γ'. Thus we have shown that α is strongly incompact.

3. Incompactness of inaccessible cardinals. We wish now to extend the class of accessible cardinals to a much more comprehensive class of incompact cardinals. A set C of ordinals is said to be *relatively closed* if whenever a subset of C has an upper bound in C, its union is in C. That is, if the union of C is added to C, then the resulting set is closed in the order topology on the ordinals. Given an arbitrary class $\boldsymbol{X}$ of ordinals, we let $\boldsymbol{M}(\boldsymbol{X})$ be the class of all ordinals ξ such that, for some relatively closed $C \subseteq \boldsymbol{X}$, $\xi = \bigcup C$. This process of forming the class $\boldsymbol{M}(\boldsymbol{X})$ from a class $\boldsymbol{X}$ was first employed by Mahlo (see [11]); if $\boldsymbol{X}$ is the class of all strongly accessible cardinals, then $\boldsymbol{M}(\boldsymbol{X})$ is the class of all cardinals which are not ϱ_0 numbers in Mahlo's sense. The present formulation in terms of relatively closed sets was suggested by Tarski. Let $\boldsymbol{AC}$ be the class of all accessible cardinals. We will be concerned with iterated applications of the operation $\boldsymbol{M}$ to the class $\boldsymbol{AC}$.

To illustrate the effect of the operation $\boldsymbol{M}$, let $\theta_0, \ldots, \theta_\xi, \ldots$ be the sequence of all inaccessible cardinals. First we see that $\theta_\eta \in \boldsymbol{M}(\boldsymbol{AC})$ unless possibly there are θ_η inaccessible cardinals smaller than θ_η; that is, unless $\eta = \theta_\eta$. For suppose $\eta < \theta_\eta$. Since θ_η is regular, $\bigcup_{\xi<\eta}\theta_\xi < \theta_\eta$ and taking C to be the set of all cardinals strictly between $\bigcup_{\xi<\eta}\theta_\xi$ and θ_η, we see that C is a relatively closed set of accessible cardinals whose union is θ_η. Thus $\theta_\eta \in \boldsymbol{M}(\boldsymbol{AC})$. But even many of those cardinals θ_η for which $\eta = \theta_\eta$

are in $\boldsymbol{M}(\boldsymbol{AC})$. Let $\theta_0', \ldots, \theta_\eta', \ldots$ be the sequence of all such fixed points of the original sequence. Again we can show that $\theta_\eta' \in \boldsymbol{M}(\boldsymbol{AC})$ unless possibly $\eta = \theta_\eta'$. For if $\eta < \theta_\eta'$, we take C to be the relative closure of the set of cardinals which are immediate successors of cardinals of the form θ_ξ which lie between $\bigcup_{\xi<\eta}\theta_\xi'$ and θ_ξ'. If is then easy to see that C is a relatively closed subset of $\boldsymbol{AC}$ whose union is θ_η'. This process can be carried still further by taking fixed points of this new sequence, etc. For stronger results along this line, see Mahlo [11] and Lévy [10]. From such results it can be seen that the cardinals not in $\boldsymbol{M}(\boldsymbol{AC})$ must be very rare indeed. However, we do not stop here.

Mahlo also considered iterations of the operation $\boldsymbol{M}$ to obtain still more comprehensive classes of cardinals. We define $\boldsymbol{M}^\eta(\boldsymbol{X})$ recursively by the conditions: (i) $\boldsymbol{M}^0(\boldsymbol{X}) = \boldsymbol{X}$, (ii) $\boldsymbol{M}^{\eta+1}(\boldsymbol{X}) = \boldsymbol{M}\big(\boldsymbol{M}^\eta(\boldsymbol{X})\big)$, and (iii), for η a limit ordinal, $\boldsymbol{M}^\eta(\boldsymbol{X}) = \bigcup_{\xi<\eta}\boldsymbol{M}^\xi(\boldsymbol{X})$. $\boldsymbol{M}^\eta(\boldsymbol{AC})$ is the class of cardinals which are not hyperinaccessible of type η as defined in Lévy [10]. An immediate consequence of Theorem 5 below will be that if α is inaccessible and $\alpha \in \boldsymbol{M}^\alpha(\boldsymbol{AC})$, then α is strongly incompact. Thus we will have shown that all cardinals $\alpha > \omega$ are incompact unless possibly α is hyperinaccessible of type α. However, Theorem 5 will be formulated in such a way as to give strong incompactness for a comprehensive class even of those cardinals which are hyperinaccessible of their own type.

Suppose now that α is inaccessible and $\alpha \in \boldsymbol{M}^\alpha(\boldsymbol{AC})$. It follows from the definition of $\boldsymbol{M}^\eta$ that if ζ is the first ordinal such that $\alpha \in \boldsymbol{M}^\zeta(\boldsymbol{AC})$, then ζ is not a limit ordinal and $\zeta < \alpha$. Taking η to be the predecessor of the ordinal ζ, we have $\alpha \notin \boldsymbol{M}^\eta(\boldsymbol{AC})$ but $\alpha \in \boldsymbol{M}^{\eta+1}(\boldsymbol{AC})$ and hence there exists a relatively closed subset C of $\boldsymbol{M}^\eta(\boldsymbol{AC})$ such that $\alpha = \bigcup C$. This ordinal η and set C give us a description of α in terms of smaller ordinals in much the same way that α was described in terms of the sequence $\gamma_0, \ldots, \gamma_\xi, \ldots$ in the proof of Theorem 2. Such a description permits us to show that α is strongly incompact. (The notion of describability and its relationship to incompactness is given explicitly in [6].) In carrying out this incompactness proof in a natural way, it is seen that the $<$ relation plays a dual rôle. On one hand it is used in the definition of inaccessible cardinal and relatively closed set. On the other hand it is used to index the degrees of hyperinaccessibility. The generalization of Mahlo's construction which will be presented here consists in allowing distinct relations to play these two separate rôles. Suppose then that $\boldsymbol{R}$ is any well-ordering relation on the ordinals. Let $\theta_0, \ldots, \theta_\eta, \ldots$ be the enumeration of the ordinals in the order determined by $\boldsymbol{R}$. We define $\boldsymbol{M}^{(\boldsymbol{R},\eta)}$ recursively as follows:

(i) $\boldsymbol{M}^{(\boldsymbol{R},0)}(\boldsymbol{X}) = \boldsymbol{X}$.

(ii) $\boldsymbol{M}^{(\boldsymbol{R},\eta+1)}(\boldsymbol{X}) = \boldsymbol{M}^{(\boldsymbol{R},\eta)}(\boldsymbol{X}) \cup \Big(\boldsymbol{M}\big(\boldsymbol{M}^{(\boldsymbol{R},\eta)}(\boldsymbol{X})\big) - (\theta_\eta + 1)\Big)$.

(iii) *If η is a limit ordinal, then*

$$M^{(R,\eta)}(X) = \bigcup_{\xi<\eta} M^{(R,\xi)}(X)\,.$$

Let $M^R(X)$ be the union over all η of the sets $M^{(R,\eta)}(X)$. The set $\theta_\eta+1$ is subtracted in (ii) to insure that the cardinals added at the $(\eta+1)$st step of the iteration are all larger than θ_η; thus the condition $\alpha \in M^R(X)$ depends only on the relation R restricted to ordinals less than α. Due to this fact, if $\alpha \in M^R(X)$, then we obtain a description of α in terms of the relation R restricted to ordinals less than α, some ordinal θ_η smaller than α, and a relatively closed subset of α.

Taking R to be the ordinary $<$ relation on the ordinals, we have $\theta_\eta = \eta$ and by induction on η we easily show that $M^{(<,\eta)}(AC)$ is the set of cardinals α such that either $\alpha \leqslant \eta$ and $\alpha \in M^\alpha(AC)$ or $\alpha > \eta$ and $\alpha \in M^\eta(X)$. Hence $\alpha \in M^<(AC)$ just in case $\alpha \in M^\alpha(X)$, that is, just in case α is not hyper-inaccessible of its own type. Thus we have formed a comprehensive class $M^<(AC)$ of cardinals. Using this class as a starting point, we can form further classes $M\big(M^<(AC)\big)$, $M^<\big(M^<(AC)\big)$, $M\Big(M^<\big(M^<(AC)\big)\Big)$, etc. Assuming that none of these classes contains all cardinals, it is easily seen that each one is larger than the preceeding. Furthermore, given two well orderings R and S on the ordinals, we can define $(M^S)^{(R,\eta)}$ and $(M^S)^R$ by replacing the function M by M^S in the definitions of $M^{(R,\eta)}$ and M^R. Thus, the operation $(M^<)^<$ is the result of iterating the operation $M^<$. It can be seen that the class $(M^<)^<(AC)$ is more comprehensive than any of the classes mentioned above. However, in the next theorem we establish certain closure principles which show that each of these classes is already of the form $M^R(AC)$ for suitable R.

THEOREM 4. *Suppose that S and T are well-ordering relations on the ordinals. Then there exist well orderings R and R' such that, for any set X of cardinals,*

(i) $M^R(X) = M^S\big(M^T(X)\big)$,

(ii) $M^{R'}(X) = (M^S)^T(X)$.

Proof. Any ordinal θ can be written uniquely in the form $\omega\cdot\nu+n$ for some ordinal ν and integer $n \in \omega$. We call $\omega\cdot\nu+2n$ the even ordinal corresponding to θ and $\omega\cdot\nu+2n+1$ the odd ordinal corresponding to θ. Let $\theta_0, \ldots, \theta_\eta, \ldots$ and $\theta'_0, \ldots, \theta'_\eta, \ldots$ be enumerations of the ordinals in the orders determined by S and T, respectively. (We may assume that T is such that this second sequence has no last element.) We then construct a new sequence $\theta''_0, \ldots, \theta''_\eta, \ldots$ consisting of the even ordinals corresponding to $\theta'_0, \ldots$ followed by the odd ordinals corresponding to $\theta_0, \ldots$ Let R be the relation on the ordinals determined by this new sequence. We then show by induction on η that, for every set X of cardinals,

$$M^{(R,\eta)}(X) = M^{(T,\eta)}(X)$$

when θ''_η is even and

$$M^{(R,\eta)}(X) = M^{(S,\zeta)}\big(M^T(X)\big)$$

when θ''_η is the odd ordinal corresponding to θ_ζ. Thus (i) follows.

To show (ii) we chose distinct ordinals $\theta''_{\eta,\zeta}$ corresponding to each pair θ_η and θ'_ζ of ordinals in the original sequences in such a way that the power of $\theta''_{\eta,\zeta}$ is the maximum of the powers of θ_η and θ'_ζ (except that we only require $\theta''_{\eta,\zeta}$ to be finite in case θ_η and θ'_ζ are both finite). Then we say that $\theta''_{\eta,\zeta}$ is in the relation R' to $\theta''_{\eta',\zeta'}$ just in case $\zeta<\zeta'$ or $\zeta=\zeta'$ and $\eta<\eta'$. By an inductive argument we then show that if $\theta''_{\eta,\zeta}$ is the ξth ordinal in the sequence determined by R', then

$$M^{(R',\xi)}(X) = (M^S)^{(T,\zeta)}(X) \cup \Big(M^{(S,\eta)}\big((M^S)^{(T,\zeta)}(X)\big) - (\theta'_\zeta+1)\Big)\,.$$

Taking the union over all ξ, we obtain the desired conclusion (ii).

Theorem 4 (i) and (ii) correspond to the addition and multiplication of order types, respectively. Similar results corresponding to exponentiation and higher order operations on order types are possible. For example, the definition which corresponds to exponentiation is as follows:

$$M^{(R^{(S,0)})}(X) = M(X)\,,$$

$$M^{(R^{(S,\eta+1)})}(X) = M^{(R^{(S,\eta)})}(X) \cup \big((M^{(R^{(S,\eta)})})^R(X) - (\theta_\eta+1)\big)\,,$$

and, for η a limit ordinal,

$$M^{(R^{(S,\eta)})}(X) = \bigcup_{\xi<\eta} M^{(R^{(S,\xi)})}(X)\,.$$

As in Theorem 4 we then have that $M^{(R^S)}$ (defined as the union over all η) is of the form M^T for suitable T.

Before proceeding to the main theorem we note that instead of starting with the class AC of all accessible cardinals, we could equally well start with the class AO of all accessible cardinals together with all ordinals which are not cardinals. In fact, the class $M^R(AO)$ contains just the cardinals in $M^R(AC)$ together with the ordinals which are not cardinals. This follows from the fact that $M(AO \cup X) = M(AC \cup X) \cup \cup AO$. For if a cardinal is the union of a relatively closed subset C of $AO \cup X$, then it is the union of the relatively closed set C' of $AC \cup X$ which is obtained from C by replacing each ordinal which is not a cardinal by the next larger cardinal.

THEOREM 5. *Suppose that R is a well-ordering relation on the ordinals and α is an inaccessible cardinal larger than ω. Then $\alpha \in M^R(AC)$ implies that α is strongly incompact.*

Proof. Let $\theta_0, \ldots, \theta_\eta, \ldots$ enumerate the ordinals in the sequence determined by R. Let η be the least ordinal such that $\alpha \in M^{(R,\eta)}(AC)$.

Looking at the definition of $M^{(R,\eta)}(AC)$ we see from (i) that $\eta \neq 0$, since α is inaccessible and from (iii) that η is not a limit ordinal since for all $\xi < \eta$, $\alpha \notin M^{(R,\xi)}(AC)$. Hence (ii) must apply and so $\eta = \zeta+1$ and since $\alpha \notin M^{(R,\zeta)}(AC)$, we must have $\alpha \in M\big(M^{(R,\zeta)}(AC)\big)$ and $\alpha \notin \theta_\zeta+1$. Hence $\theta_\zeta < \alpha$ and for some relatively closed set $C \subseteq M^{(R,\zeta)}(AC)$ we have $\alpha = \bigcup C$. We will now use the relation R (restricted to α), the ordinal θ_ζ, and the set C to construct a set Γ of sentences of L_α such that Γ has no model but every subset of Γ of power less than α has a model. In forming Γ, we make use of four binary predicates $<$, P_1, P_2, and P_3, two ternary predicates P_4 and P_5, and three unary predicates P_6, P_7, and P_8. (The use of the two ternary predicates P_4 and P_5 to express accessibility is due to Keisler, see [8].) The meaning of these predicates and the sentences of Γ will be made clear in the proof which follows. Γ consists of the sentences (3) and (4) of Section 2, the sentences

$$(\forall wxyz)[P_4wxy \wedge P_4wxz \to y = z], \tag{9}$$

$$(\forall w)(\exists z)[z < w \wedge (\forall xy)[P_4wxy \to x < z \wedge y < w]], \tag{10}$$

$$(\forall wz)[P_6w \wedge z < w \to (\exists xy)[P_4wxy \wedge z < y]], \tag{11}$$

$$(\forall wxy)[x < w \wedge y < w \wedge (\forall z)[P_5wxz \leftrightarrow P_5wyz] \to x = y], \tag{12}$$

$$(\forall w)\big[P_7w \to (\exists z)\big[z < w \wedge (\forall xy)[P_5wxy \to y < z]\big]\big], \tag{13}$$

$$(\forall wx)\big[P_2wx \to P_6w \vee P_7w \vee (\exists y)\big[P_1yx \wedge y < w \wedge (\forall z)[P_3wz \to P_2zy]\big]\big], \tag{14}$$

$$(\forall wx)[x < w \to (\exists y)[x \leqslant y \wedge y < w \wedge P_3wy]], \tag{15}$$

$$(\forall wx)\big[(\forall z)[z < x \to (\exists y)[z < y \wedge y < x \wedge P_3wy]] \wedge x < w \to P_3wx\big], \tag{16}$$

$$(\exists x)(\forall y)[y < x \vee y = x], \tag{17}$$

$$(\forall x)\big[(\forall z)[z < x \to (\exists y)[z < y \wedge y < x \wedge P_8y]] \to P_8x\big], \tag{18}$$

$$(\exists x)\big[\Phi_{\theta_\zeta}(x) \wedge (\forall y)[P_8y \to P_2yx]\big], \tag{19}$$

and, for all ordinals ξ and η smaller than α and such that $\langle \xi, \eta\rangle \in R$, the sentence

$$(\exists xy)[\Phi_\xi(x) \wedge \Phi_\eta(y) \wedge \neg P_1xy] \tag{20}$$

and, finally, for every ordinal $\eta \in C$, the sentence

$$(\exists x)[\Phi_\eta(x) \wedge P_8x]\,. \tag{21}$$

By (3) and (4), any model of Γ is isomorphic to a model of the form

$$\mathfrak{A} = \langle \varrho, <, R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8\rangle,$$

where R_1, R_2, and R_3 are binary and R_4 and R_5 are ternary relations on ϱ and R_6, R_7, and R_8 are subsets of ϱ. We proceed to establish the following:

(a) *If* $\varkappa \in R_6$ *or* $\varkappa \in R_7$, *then* $\varkappa \in AO$. Suppose $\varkappa \in R_6$. Let $f_\varkappa$ be the set of ordered pairs $\langle \xi, \eta\rangle$ such that $\langle \varkappa, \xi, \eta\rangle \in R_4$. By (9), $f_\varkappa$ is a function and by (10), there is an ordinal $\lambda < \varkappa$ such that the domain of $f_\varkappa$ is a subset of λ and the range of $f_\varkappa$ is a subset of $\varkappa$. Sentence (11) then says that $\varkappa$ is the union of the range of $f_\varkappa$. Hence $\varkappa$ must be either a singular cardinal or else an ordinal which is not a cardinal. Thus $\varkappa \in AO$. Suppose now that $\varkappa \in R_7$. For each $\eta < \varkappa$, let $F_\varkappa(\eta)$ be the set of all ordinals ξ such that $\langle \varkappa, \eta, \xi\rangle \in R_5$. By (12), $F_\varkappa$ is one to one and by (13), there is an ordinal $\lambda < \varkappa$ such that each element of the range of $F_\varkappa$ is a subset of λ. Thus we conclude that the power of $\varkappa$ is less than or equal to 2^λ and hence that $\varkappa \in AO$.

(b) *If* $\varkappa \leqslant \alpha$ *and* $\theta_\eta < \alpha$ *and* $\langle \varkappa, \theta_\eta\rangle \in R_2$, *then* $\varkappa \in M^{(R,\eta)}(AO)$. We proceed by induction on η. Assume therefore that, for all $\xi < \eta$, if $\lambda \leqslant \alpha$ and $\theta_\xi < \alpha$ and $\langle \lambda, \theta_\xi\rangle \in R_2$, then $\lambda \in M^{(R,\xi)}(AO)$. Suppose that $\varkappa \leqslant \alpha$, $\theta_\eta < \alpha$, and $\langle \varkappa, \theta_\eta\rangle \in R_2$. Let $D_\varkappa$ be the set of all ordinals $\lambda < \varkappa$ such that $\langle \varkappa, \lambda\rangle \in R_3$. Now if $\varkappa \in R_6$ or R_7, then by (a) $\varkappa \in AO \subseteq M^{(R,\eta)}(AO)$ and so we are done. But (14) says that either $\varkappa$ is in R_6 or R_7 or for some ordinal θ we have $\langle \theta, \theta_\eta\rangle \in R_1$, $\theta < \varkappa$, and for all $\lambda \in D_\varkappa$, $\langle \lambda, \theta\rangle \in R_2$. Thus we assume the latter. Now if $\langle \theta, \theta_\eta\rangle \notin R$ then (since both θ and θ_η are less than α) by (20) we would have $\langle \theta, \theta_\eta\rangle \notin R_1$. Hence $\langle \theta, \theta_\eta\rangle \in R$ and so θ is θ_ξ for some $\xi < \eta$. Thus $\langle \lambda, \theta_\xi\rangle \in R_2$ for all $\lambda \in D_\varkappa$. Hence we conclude by the hypothesis of induction that $D_\varkappa \subseteq M^{(R,\xi)}(AO)$. By (15), the union of $D_\varkappa$ is $\varkappa$ and by (16), $D_\varkappa$ is relatively closed. Therefore $\varkappa \in M\big(M^{(R,\xi)}(AO)\big)$. Since $\varkappa \notin \theta_\xi+1$, we conclude that $\varkappa \in M^{(R,\xi+1)}(AO) \subseteq M^{(R,\eta)}(AO)$.

(c) *Γ has no model.* By (21), $C \subseteq R_8 \subseteq \varrho$. Since $\alpha = \bigcup C$, we conclude that $\alpha \leqslant \varrho$. But (17) says that there is a largest ordinal in $\mathfrak{A}$; that is, that ϱ is a non-limit ordinal. Since α is a limit ordinal, we conclude that $\alpha < \varrho$. By (18), R_8 is closed and so $\alpha \in R_8$ and by (19), we conclude that $\langle \alpha, \theta_\zeta\rangle \in R_2$. Applying (b) it follows that $\alpha \in M^{(R,\zeta)}(AO) = M^{(R,\zeta)}(AC)$ contrary to our choice of ζ. Therefore the assumption that $\mathfrak{A}$ is a model of Γ leads to a contradiction.

Suppose now that Γ' is a subset of Γ of power less than α. Let ϱ be the union of all ordinals η such that the formula $\Phi_\eta(x)$ or $\Phi_\eta(y)$ appears in a sentence (of form (19), (20), or (21)) of Γ'. Since α is not singular, $\varrho < \alpha$. We construct a model

$$\mathfrak{A} = \langle \varrho+1, <, R_1, \ldots, R_8\rangle$$

of Γ' as follows: R_1 is the relation R restricted to $\varrho+1$. R_2 is the set of all pairs $\langle \varkappa, \theta_\eta\rangle$ where $\varkappa, \theta_\eta \leqslant \varrho$ and $\varkappa \in M^{(R,\eta)}(AC)$. For each $\varkappa \leqslant \varrho$, if for some η, $\varkappa \in M^{(R,\eta+1)}(AC) - M^{(R,\eta)}(AC)$, then we choose a relatively closed set $D_\varkappa \subseteq M^{(R,\eta)}(AC)$ such that $\varkappa = \bigcup D_\varkappa$; otherwise, let $D_\varkappa = \varkappa$.

R_3 is then the set of all pairs $\langle \varkappa, \lambda \rangle$ where $\lambda \in D_\varkappa$. For each ordinal $\varkappa \leqslant \varrho$, if $\varkappa$ is a singular cardinal or an ordinal which is not a cardinal, we choose a particular function $f_\varkappa$ with domain some ordinal $\lambda < \varkappa$ and with a range which is included in $\varkappa$ and has its union equal to $\varkappa$; otherwise, let $f_\varkappa$ be empty. R_4 is the set of triples $\langle \varkappa, \xi, \eta \rangle$ where $\langle \xi, \eta \rangle \in f_\varkappa$. For each ordinal $\varkappa \leqslant \varrho$, if $\varkappa \leqslant 2^\lambda$ for some $\lambda < \varkappa$, then let $F_\varkappa$ be a function mapping $\varkappa$ one-to-one onto some family of subsets of λ; otherwise let $F_\varkappa$ be defined by $F_\varkappa(\eta) = \{\eta\}$ for each $\eta < \varkappa$. R_5 is then the set of all triples $\langle \varkappa, \eta, \xi \rangle$ such that $\xi \in F_\varkappa(\eta)$. R_6 is the set of all $\varkappa \leqslant \varrho$ such that $\varkappa$ is a singular cardinal or an ordinal which is not a cardinal. R_7 is the set of all ordinals $\varkappa \leqslant \varrho$ such that $\varkappa \leqslant 2^\lambda$ for some $\lambda < \varkappa$. Finally, R_8 is the intersection of C and $\varrho+1$. It is now a routine matter to check that $\mathfrak{A}$ is a model of Γ'.

We return now to those weakly inaccessible cardinals for which Theorem 1 gives incompactness but not strong incompactness. Let $\boldsymbol{BC}$ be the class of all cardinals α such that either α is strongly accessible or $\alpha = \beta^\gamma$ for some cardinals $\beta, \gamma < \alpha$. By the theorems of Section 2, every cardinal in $\boldsymbol{BC}$ is strongly incompact. In the next theorem, we show that this class can be extended to a much wider class of inaccessible cardinals. (The author originally obtained this result using the class of strongly accessible cardinals as a starting point; Tarski suggested starting with this wider class.)

THEOREM 6. *Suppose that $\boldsymbol{R}$ is a well-ordering relation on the ordinals and α is a nondenumerable cardinal in $\boldsymbol{M}^{\boldsymbol{R}}(\boldsymbol{BC})$. Then α is strongly incompact.*

Proof. If $\alpha \in \boldsymbol{BC}$, we have strong incompactness by Theorem 1, 2, or 3. In case $\alpha \notin \boldsymbol{BC}$, the proof follows closely the proof of Theorem 5. Relations R_5 and R_7 are replaced by relations which express the fact that either the cardinal $\varkappa$ is the successor of a smaller cardinal λ (i.e. one-to-one functions are introduced which map each ordinal smaller than $\varkappa$ into λ) or that the ordinals less than $\varkappa$ are in one-to-one correspondence with functions mapping γ into β where $\beta, \gamma < \varkappa$. In connection with the latter, we must include, for each $\beta, \gamma < \alpha$, an infinite sentence which says that every function mapping γ into β appears in such a correspondence.

Let $\boldsymbol{M}^*(\boldsymbol{X})$ be the union over all well-ordering relations $\boldsymbol{R}$ of the sets $\boldsymbol{M}^{\boldsymbol{R}}(\boldsymbol{X})$. By the preceeding theorems, every non-denumerable cardinal in $\boldsymbol{M}^*(\boldsymbol{AC})$ is incompact and every non-denumerable cardinal in $\boldsymbol{M}^*(\boldsymbol{BC})$ is strongly incompact. However, it is possible to show that the first non-denumerable cardinal, if any, not in $\boldsymbol{M}^*(\boldsymbol{AC})$ (or not in $\boldsymbol{M}^*(\boldsymbol{BC})$) is strongly incompact. In fact, we can consider various ways of enlarging the class $\boldsymbol{M}^*(\boldsymbol{AC})$ and show, for example, that all the non-denumerable cardinals in $\boldsymbol{M}(\boldsymbol{M}^*(\boldsymbol{AC}))$ or $\boldsymbol{M}^*(\boldsymbol{M}^*(\boldsymbol{AC}))$ are incompact. We could also define various classes $(\boldsymbol{M}^*)^{(\boldsymbol{R},\eta)}(\boldsymbol{AC})$, $(\boldsymbol{M}^*)^{\boldsymbol{R}}(\boldsymbol{AC})$, and $(\boldsymbol{M}^*)^*(\boldsymbol{AC})$ formed by iterating the $\boldsymbol{M}^*$ operation or even the classes $\boldsymbol{M}^{(*(\boldsymbol{R},\eta))}(\boldsymbol{AC})$, $\boldsymbol{M}^{(*\boldsymbol{R})}(\boldsymbol{AC})$, and $\boldsymbol{M}^{(**)}(\boldsymbol{AC})$ formed by iterating the * operation itself. Unless one of these classes contains all cardinals, then each class we have mentioned is much more comprehensive than the earlier ones. Furthermore, we can show that the non-denumerable cardinals in each of these classes are incompact (in fact, strongly incompact except possibly for certain accessible cardinals). These facts can be proved in a manner quite analogous to the proof of Theorem 5. In the same way that the relation R_3 gave us an indexed family of relatively closed sets, we make use, for example, of indexed families of well-ordering relations.

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On a problem of Erdös and Tarski

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In the paper Erdös-Tarski [1], the following properties of a cardinal number α are formulated:

$\boldsymbol{P}_1$: *There exists a simply ordered set of power α which has no well ordered or inversely well ordered subset of power α.*

$\boldsymbol{P}_2$: *There exists a division of the complete graph on a set of power α into two subgraphs neither of which includes a complete graph on a subset of power α.*

$\boldsymbol{P}_3$: *In the set algebra of all subsets of a set of power α, every α-complete prime ideal is principal.*

$\boldsymbol{P}_4$: *There is an α-complete and α-distributive Boolean algebra which is not isomorphic to any α-complete set algebra.*

$\boldsymbol{Q}$: *There is a ramification system $\mathcal{R}$ of order α such that for any $\eta < \alpha$ there are fewer than α elements of order η and every well ordered subset of $\mathcal{R}$ has power less than α.*

Here α is assumed to be an infinite cardinal.

It is shown in [1] that $\boldsymbol{P}_1$ implies $\boldsymbol{P}_2$, $\boldsymbol{P}_2$ implies $\boldsymbol{P}_3$, and $\boldsymbol{P}_3$ implies $\boldsymbol{P}_4$. It is also shown that $\boldsymbol{Q}$ implies $\boldsymbol{P}_3$ and that, for every inaccessible cardinal α, $\boldsymbol{P}_2$ implies $\boldsymbol{Q}$. Erdös and Tarski raised the question whether any of these implications holds in the opposite direction as well. It should be mentioned that all the properties $\boldsymbol{P}_1$-$\boldsymbol{P}_4$ hold for all accessible cardinals but fail for the smallest inaccessible cardinal ω. The problem whether $\boldsymbol{Q}$ holds for all accessible cardinals is not completely settled; it is known, however, that under the assumption of the generalized continuum hypothesis, the solution of the problem is affirmative for every accessible cardinal α which is not the immediate successor of a singular cardinal. For proofs and references for these results, see [1].

In Keisler-Tarski [6], the following condition is discussed:

$\boldsymbol{S}$: *There is an α-complete set algebra which is α-generated by α elements and in which every α-complete prime ideal is principal.*

It is known that, for all inaccessible α, $\boldsymbol{S}$ is equivalent to $\boldsymbol{Q}$. Like $\boldsymbol{P}_1$-$\boldsymbol{P}_4$, $\boldsymbol{S}$ is known to hold for all accessible α's and to fail for ω. Hence