Algebraic topology methods in the theory of compact fields in Banach spaces

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Let E^m be an m-dimensional Banach space ($m = \infty$ means that the space is of infinite dimension) and let X be a closed and bounded subset of E . A continuous mapping $f\colon X \to E^m$ will be called a *compact field* provided that $f(x) = x - F(x)$ and $\overline{F(X)}$ is a compact subset of E^m. If $m < \infty$ then the notions of continuous mapping and compact field coincide; if $m = \infty$ then these notions are essentially different.

Several well-known theorems on the continuous mappings of subsets of Euclidean spaces are, in general, false for the continuous mappings of subsets of infinite dimensional Banach spaces, but remain true if the continuous mappings are replaced by compact fields. We would like to mention here such examples as Brouwer's Fixed Point Theorem, Borsuk's Antipodensatz or Jordan's Separation Theorem.

Thus the theory of compact fields in infinite dimensional Banach spaces corresponds, in some way, to the theory of continuous mappings in Euclidean spaces. The reader who wishes to get acquainted with this theory is referred to the expository paper of Granas [7]. The present paper is an attempt to introduce algebraic topology methods into the theory of compact fields in Banach spaces.

Let $E^m = R_0^n + E^{m-n}$ be a direct decomposition where R_0^n is of finite dimension n. Let us put

$$P^{m-n} = E^{m-n} \setminus \{0\}, \qquad Q^{m-n} = E^{m-n} \setminus Q_-,$$

where $Q_- \subset E^{m-n}$ is a half-line with the beginning at 0. Let (X, A) be a pair such that X is a closed and compact subset of E^m. If $m < \infty$, then $[X, A; P^{m-n}, Q^{m-n}]$, i.e. the set of all homotopy classes of continuous mappings of the pair (X, A) into the pair (P^{m-n}, Q^{m-n}), can be identified with $[X, A; S^{m-n-1}, p]$, i.e. the set of all homotopy classes of all continuous mappings of (X, A) into (S^{m-n-1}, p), where S^{m-n-1} is an $(m-n-1)$-dimensional sphere and $p \in S^{m-n-1}$. Thus, under suitable conditions on (X, A), a group operation, analogous with the group operation in $(m-n-1)$-th cohomotopy group, can be defined in $[X, A; P^{m-n}, Q^{m-n}]$.

In the case of $m=\infty$ we shall consider the category $\mathcal{E}$ whose objects (X, A) are pairs of closed and bounded subsets of E^∞ and whose mappings are compact fields. Two compact fields $f, g\colon (X, A)\to(Y, B)$ are said to be *homotopic* if there exists a homotopy $h(x, t) = x - H(x, t)$ connecting f and g and such that $\overline{H(X\times I)}$ is a compact subset of E^∞. Thus the set of all compact fields defined on (X, A) and with the values in (Y, B) is divided into disjoint homotopy classes; we shall denote this set by $[X, A; Y, B]$.

In this paper we give the definition of group operation in $[X, A; P^{\infty-n}, Q^{\infty-n}]$; this group will be denoted by $\pi^{\infty-n}(X, A)$. The definition of group operation in $\pi^{\infty-n}(X, A)$ is a generalization to the case of infinitely dimensional Banach spaces of the group operation in cohomotopy groups. The operation is commutative and thus $\pi^{\infty-n}(X, A)$ is an abelian group. Thus, for each pair (X, A) from $\mathcal{E}$ and each non-negative integer n there is defined an abelian group $\pi^{\infty-n}(X, A)$; moreover, if $f\colon (X, A)\to(Y, B)$ is a compact field, then there are defined a homomorphism

$$f^*\colon \pi^{\infty-n}(Y, B)\to\pi^{\infty-n}(X, A)$$

and a homomorphism

$$\delta\colon \pi^{\infty-n}(A)\to\pi^{\infty-n+1}(X, A)\,.$$

Thus $\{\pi^{\infty-n}, f^*, \delta\}$ is a contravariant δ-functor defined on $\mathcal{E}$ with values in the category of abelian groups and homomorphisms. The main result of this paper is that $\{\pi^{\infty-n}, f^*, \delta\}$ fulfil the Eilenberg-Steenrod axioms for cohomology theory. We shall now list the Eilenberg-Steenrod axioms, they appear here as properties of the groups $\pi^{\infty-n}$ and homomorphisms f^* and δ.

1. If $i\colon X\to X$ is the identity compact field, then i^* is the identity homomorphism (Theorem 5.1).

2. If $f\colon (X, A)\to(Y, B)$, $g\colon (Y, B)\to(Z, C)$ and $f, g\in\mathcal{E}$, then $(gf)^* = f^*g^*$ (Theorem 5.4).

3. If $f\colon (X, A)\to(Y, B)$, $f_0\colon A\to B$, $f, f_0\in\mathcal{E}$ and f is an extension of f_0, then $f_0^*\delta = \delta f^*$ (Theorem 5.16).

4. The sequence

$$\pi^{\infty-n}(X, A)\xrightarrow{j^*}\pi^{\infty-n}(X)\xrightarrow{i^*}\pi^{\infty-n}(A)\xrightarrow{\delta}\pi^{\infty-n+1}(X, A)\xrightarrow{j^*}\ldots$$

is exact (Theorem 5.14).

5. If $f, g\colon (X, A)\to(Y, B)$ are homotopic compact fields, then $f^* = g^*$ (Theorem 5.3).

6. If U is an open subset of E^∞ and $i\colon (X\setminus U, A\setminus U)\to(X, A)$ is the inclusion, then

$$i^*\colon \pi^{\infty-n}(X, A)\to\pi^{\infty-n}(X\setminus U, A\setminus U)$$

is an isomorphism (Theorem 5.17).

7. If X is one-point subset, then

$$\pi^{\infty-n}(X)\approx 0 \quad \text{for} \quad n = 0, 1, \ldots$$

(obvious from the definition of $\pi^{\infty-n}(X)$).

§ 1 contains the conventions of notation and the sketch definition of cohomotopy groups and their fundamental properties. In § 2 we give some lemmas on cohomotopy groups and the homomorphism Δ, which plays an important role in the whole theory. § 3 is devoted to compact fields in infinite dimensional Banach spaces. The definition of group operation is given in § 4. § 5 contains the definitions of the homomorphisms f^* and δ and the proofs of the Eilenberg-Steenrod axioms. In § 6 we are concerned with the duality between π^∞-groups and S-homotopy groups. The main result of this section is Theorem 6.12, which says that $\pi^{\infty-n}(X)$ and $\Sigma_n(E^\infty\setminus X)$ are isomorphic. The last section, (§ 7), contains some applications and a list of some known π^∞-groups of the infinite dimensional spheres.

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§1. Preliminaries

DEFINITION 1.1. Denote by $\mathfrak{X}$ the metric space whose points are infinite sequences of real numbers $x = (x_1, x_2, \ldots, x_n, \ldots)$ such that $x_n = 0$ for all but a finite number of n, the metric being

$$\varrho(x, x') = \sqrt{\sum_i (x_i - x_i')^2}\,.$$

We distinguish the following subsets of $\mathfrak{X}$:

$$\begin{aligned}
R^n &= \{x\in\mathfrak{X};\ x_i = 0 \text{ for } i > n\} \quad (n\text{-dimensional Euclidean space}),\\
S^n &= \{x\in R^{n+1}\colon\ \textstyle\sum_i x_i^2 = 1\} \quad (n\text{-dimensional sphere}),\\
E_+^n &= \{x\in S^n;\ x_{n+1}\geqslant 0\},\\
E_-^n &= \{x\in S^n;\ x_{n+1}\leqslant 0\},\\
p &= (1, 0, 0, \ldots),\\
\overline{p} &= (-1, 0, 0, \ldots),\\
S^{-1} &= \Lambda \text{ (the empty set)},\\
S_1^{n+1} &= \{x\in S^{n+2};\ x_{n+2} = 0\},\\
E_{++}^{n+2} &= \{x\in S^{n+2};\ x_{n+2}\geqslant 0,\ x_{n+3}\geqslant 0\},\\
E_{+-}^{n+2} &= \{x\in S^{n+2};\ x_{n+2}\geqslant 0,\ x_{n+3}\leqslant 0\},\\
E_{-+}^{n+2} &= \{x\in S^{n+2};\ x_{n+2}\leqslant 0,\ x_{n+3}\geqslant 0\},\\
E_{--}^{n+2} &= \{x\in S^{n+2};\ x_{n+2}\leqslant 0,\ x_{n+3}\leqslant 0\}.
\end{aligned}$$

In the sequel, *space* will always denote a metric space. By a pair (X, A) we shall understand a space X together with a closed subset A. If X is a space, we shall identify X with the pair (X, A). A triad (X, X_1, X_2) will denote a space X and two closed subspaces X_1 and X_2 of X such that $X_1 \cup X_2 = X$. We shall say that $(X, A; X_1, A_1; X_2, A_2)$ is a *relative triad* if (X, X_1, X_2) is a triad, (X, A) is a pair and $A_i = A \cap X_i$ for $i = 1, 2$. For two spaces X, Y we shall denote by $X \times Y$ their cartesian product. By $S^n \vee S^n$ we shall denote the subset of $S^n \times S^n$ defined by the rule

$$S^n \vee S^n = (S^n \times \{p\}) \cup (\{p\} \times S^n) .$$

We shall say that $f\colon (X, A) \to (Y, B)$ is a *mapping of the pair* (X, A) *into the pair* (Y, B) if f is a continuous mapping of X into Y such that $f(A) \subset B$. If $f, g\colon (X, A) \to (Y, B)$ are two mappings, then $f \times g$ will denote the mapping defined by the rule

$$(f \times g)(x) = \big(f(x), g(x)\big) \in Y \times Y .$$

We shall denote by $\omega\colon S^n \vee S^n \to S^n$ the mapping defined by the rule

$$\omega(x, p) = \omega(p, x) = x .$$

Let $f\colon X \to Y$ be a mapping and let A be a closed subset of X. The mapping f defines a unique mapping $g\colon A \to Y$ such that $g(x) = f(x)$ for each $x \in A$. This mapping g will be called the *restriction* of f to A and will be denoted by $g = f|_A$; f will be called an *extension* of g over X.

In the sequel we shall denote by I the closed interval $\langle 0, 1\rangle$ with the usual topology.

Two mappings $f, g\colon (X, A) \to (Y, B)$ are said to be *homotopic* (notation $f \sim g$) if there exists a mapping $h\colon (X \times I, A \times I) \to (Y, B)$ such that $h_0 = f$, $h_1 = g$, where $h_t(x) = h(x, t)$. In this case h is called a *homotopy* connecting f and g. The relation $f \sim g$ is an equivalence relation in the set of mappings from (X, A) to (Y, B). As a consequence, mappings are divided into disjoint equivalence classes, called the *homotopy classes* of these mappings. We shall denote by $[X, A; Y, B]$ (or $[X, Y]$ if $A = A$) the totality of these homotopy classes and by $[f]$ the homotopy class of f, that is to say, the homotopy class which contains the mapping f.

All groups under considerations are assumed to be abelian.

Let us suppose that (X, A) is a pair such that X is a compact space and $\dim X \leqslant 2n-2$ ($\dim X$ denotes the topological dimension of the space X). It has been shown by Borsuk that in this case it is possible to introduce, in a natural way, a group structure in $[X, A; S^n, p]$. This group is denoted by $\pi^n(X, A)$ and called the *n-th cohomotopy group* of the pair (X, A) (Borsuk [1], Spanier [11]).

We recall here briefly the definition and properties of $\pi^n(X, A)$. Let α, β be any two elements in $\pi^n(X, A)$. Choose representative mappings $f_\alpha, f_\beta\colon (X, A) \to (S^n, p)$ for α, β respectively. It has been proved by Borsuk that, if $\dim X \leqslant 2n-2$, then $f_\alpha \times f_\beta\colon (X, A) \to (S^n \times S^n, \{p\} \times \{p\})$ is homotopic with some $g\colon (X, A) \to (S^n \times S^n, \{p\} \times \{p\})$ with $g(X) \subset S^n \vee S^n$. Every such g is called a *normalization* of f; it is easy to see that g depends only on the ordered pair α, β. The element $[\omega g]$ is called the *sum* of α and β and denoted by $\alpha + \beta$. It has been proved that $\pi^n(X, A)$ forms an abelian group with this addition.

Suppose that (X, A) and (Y, B) are two pairs such that X and Y are compact and $\dim X, \dim Y \leqslant 2n-2$. Let $f\colon (X, A) \to (Y, B)$ be an arbitrary mapping and let $\alpha \in \pi^n(Y, B)$. If $f_\alpha\colon (Y, B) \to (S^n, p)$ is a representative of α, then $[f_\alpha f]$ depends only on α and $f^*(\alpha) = [f_\alpha f]$ defines a transformation $f^*\colon \pi^n(Y, B) \to \pi^n(X, A)$, which is called *induced* by f. It has been proved that f^* is a homomorphism.

Let α be an arbitrary element of $\pi^n(A)$. Choose a representative $f_\alpha\colon A \to S^n$. Since E_+^{n+1} is contractible, f_α has an extension $\bar{f}_\alpha\colon (X, A) \to (E_+^{n+1}, S^n)$. Let $h\colon (E_+^{n+1} \times I, S^n \times I) \to (S^{n+1}, E_-^{n+1})$ be a homotopy such that h_0 is an inclusion mapping and h_1 maps S^n onto p and $E_+^{n+1} \setminus S^n$ onto $S^{n+1} \setminus \{p\}$ homeomorphically. It is easily seen that such a homotopy exists and that $[h_1 f]$ depends only on α. Thus $\delta(\alpha) = [h_1 f]$ defines a transformation

$$\delta\colon \pi^n(A) \to \pi^{n+1}(X, A) .$$

It is proved that δ is a homomorphism called the *coboundary homomorphism* of the pair (X, A).

It is known that the cohomotopy groups with the induced homomorphism and coboundary operator defined above fulfil the Eilenberg-Steenrod axioms for cohomology theory (Spanier [11]).

§ 2. Finite-dimensional lemmas

Definition 2.1. Let $\alpha \in \pi^n(X, A)$ and $\beta \in \pi^n(S^n, p)$, where (X, A) is a pair such that X is a compact space and $\dim X \leqslant 2n-2$. If α and β are represented by the mappings $f_\alpha\colon (X, A) \to (S^n, p)$, $f_\beta\colon (S^n, p) \to (S^n, p)$, respectively, then $[f_\beta f_\alpha]$ depends only on the elements α, β. Let us put

$$\beta \circ \alpha = [f_\beta f_\alpha] \in \pi^n(X, A) .$$

Lemma 2.2. *The operation $\beta \circ \alpha$ is bilinear, i.e. if $\alpha, \alpha_1, \alpha_2 \in \pi^n(X, A)$, $\beta, \beta_1, \beta_2 \in \pi^n(S^n, p)$, then*

$$(\beta_1 + \beta_2) \circ \alpha = \beta_1 \circ \alpha + \beta_2 \circ \alpha , \qquad \beta \circ (\alpha_1 + \alpha_2) = \beta \circ \alpha_1 + \beta \circ \alpha_2 .$$

Proof. Denote by $f_\alpha, f_{\alpha_1}, f_{\alpha_2}\colon (X, A) \to (S^n, p)$ $f_\beta, f_{\beta_1}, f_{\beta_2}\colon (S^n, p) \to (S^n, p)$ the representatives of $\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2$, respectively. Since

$f_a^*\colon \pi^n(S^n, p) \to \pi^n(X, A)$ is a homomorphism and $\beta \circ a = [f_\beta f_a] = f_a^*(\beta)$, we have

$$(\beta_1 + \beta_2) \circ a = f_a^*(\beta_1 + \beta_2) = f_a^*(\beta_1) + f_a^*(\beta_2) = \beta_1 \circ a + \beta_2 \circ a .$$

To prove the second formula, we may suppose, without loss of generality, that $(f_{a_1} \times f_{a_2})(X) \subset S^n \vee S^n$. Thus $a_1 + a_2 = [\omega(f_{a_1} \times f_{a_2})]$ and, since $(f_\beta f_{a_1} \times f_\beta f_{a_2})(X) \in S^n \vee S^n$, we have $\beta \circ a_1 + \beta \circ a_2 = [\omega(f_\beta f_{a_1} \times f_\beta f_{a_2})]$. Let us observe that if $f_{a_1}(x) = p$ and $f_{a_2}(x) = y$, then $f_\beta\, \omega(f_{a_1} \times f_{a_2})(x) = f_\beta(y)$, whence $(f_\beta f_{a_1} \times f_\beta f_{a_2})(x) = f_\beta\, \omega(f_{a_1} \times f_{a_2})(x)$. The same is true if $f_{a_2}(x) = p$, whence $\omega\big((f_\beta f_{a_1}) \times (f_\beta f_{a_2})\big) = f_\beta\, \omega(f_{a_1} \times f_{a_2})$ and it means that $\beta \circ (a_1 + a_2) = \beta \circ a_1 + \beta \circ a_2$.

Let (X, X_1, X_2) be a triad such that X is compact and $\dim X \leqslant 2n-2$. Let us put $X_0 = X_1 \cap X_2$; following Eilenberg and Steenrod (see [3], I. 15), we define the coboundary homomorphism of the triad (X, X_1, X_2) putting $\Delta = j^*(k^*)^{-1}\delta$, $\Delta\colon \pi^n(X_0) \to \pi^{n+1}(X)$, where $\delta\colon \pi^n(X_0) \to \pi^{n+1}(X_1, X_0)$ is the coboundary homomorphism of the pair (X_1, X_0), and $k\colon (X_1, X_0) \to (X, X_2)$, $j\colon X \to (X, X_2)$ are the inclusions, k^* being the excision isomorphism.

Let $(X, A; X_1, A_1; X_2, A_2)$ be a relative triad such that X is compact and $\dim X \leqslant 2n-2$. Let us put $X_0 = X_1 \cap X_2$, $A_0 = A_1 \cap A_2$. Let us denote by Y the quotient space obtained by the identification of A to a single point $y_0 \in Y$. Let $\eta\colon (X, A) \to (Y, y_0)$ be the projection and let $Y_i = \eta(X_i)$ for $i = 1, 2$, $Y_0 = Y_1 \cap Y_2$. Thus $(Y, y_0; Y_1, y_0; Y_2, y_0)$ is a relative triad, and the mapping $\eta_0\colon (X_0, A_0) \to (Y_0, y_0)$ defined by the rule $\eta_0(x) = \eta(x)$ for $x \in X_0$ is the projection. Obviously, $\eta_0^*\colon \pi^n(Y_0, y_0) \to \pi^n(X_0, A_0)$ and $\eta^*\colon \pi^{n+1}(Y, y_0) \to \pi^{n+1}(X, A)$ are isomorphisms. Let $i_0\colon Y_0 \to (Y_0, y_0)$ and $i\colon Y \to (Y, y_0)$ be natural inclusions, then $i_0^*\colon \pi^n(Y, y_0) \to \pi^n(Y_0)$ and $i^*\colon \pi^{n+1}(Y, y_0) \to \pi^{n+1}(Y)$ are isomorphisms.

DEFINITION 2.3. Let us put

$$\Delta = \eta^*(i^*)^{-1}\Delta i_0^*(\eta_0^*)^{-1};$$

the homomorphism $\Delta\colon \pi^n(X_0, A_0) \to \pi^{n+1}(X, A)$ will be called the *coboundary homomorphism of the relative triad* $(X, A; X_1, A_1; X_2, A_2)$.

LEMMA 2.4. *Let $(X, A; X_1, A_1; X_2, A_2)$ be a relative triad such that X is compact and $\dim X \leqslant 2n-2$. Let us put $X_0 = X_1 \cap X_2$, $A_0 = A \cap X_0 = A_1 \cap A_2$, and let $f\colon (X_0, A_0) \to (S^n, p)$ be an arbitrary mapping. If $f'\colon (X, A) \to (S^{n+1}, p)$ is an extension of f such that $f'(X_1) \subset E_+^{n+1}$, $f(X_2) \subset E_-^{n+1}$ (such an extension exists since E_+^{n+1} and E_-^{n+1} are contractible), then*

$$\Delta([f]) = [f'] .$$

Proof. Let $h\colon (S^{n+1} \times I) \to S^{n+1}$ be a homotopy such that h_0 is the identity mapping and h_1 maps E_-^{n+1} onto p and $E_+^{n+1} \setminus S^n$ onto $S^{n+1} \setminus \{p\}$ homeomorphically. Let us put $f_1 = h_1 f$. It follows immediately from the definition of Δ that $\Delta([f]) = [f_1]$; on the other hand, $[f'] = [f_1]$.

The following lemma is obvious from the definition:

LEMMA 2.5. *Let $(X, A; X_1, A_1; X_2, A_2)$ and $(Y, B; Y_1, B_1; Y_2, B_2)$ be two relative triads such that X and Y are compact and $\dim X, \dim Y \leqslant 2n-2$. If $f\colon (X, A) \to (Y, B)$, $f_0\colon (X_0, A_0) \to (Y_0, B_0)$ are mappings such that $f(X_i) \subset Y_i$ for $i = 1, 2$ and f is an extension of f_0, then the following diagram is commutative:*

$$\begin{array}{ccc}
\pi^n(Y_0, B_0) & \xrightarrow{f_0^*} & \pi^n(X_0, A_0) \\
\downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} \\
\pi^{n+1}(Y, B) & \xrightarrow{f^*} & \pi^{n+1}(X, A)
\end{array}$$

LEMMA 2.6. *If $(X, A; X_1, A_1; X_2, A_2)$ is a relative triad such that X is compact and $\dim X \leqslant 2n-2$, then the following diagram is anticommutative (i.e. $\delta\Delta = -\Delta\delta$):*

$$\begin{array}{ccc}
\pi^n(A_0) & \xrightarrow{\delta} & \pi^{n+1}(X_0, A_0) \\
\downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} \\
\pi^{n+1}(A) & \xrightarrow{\delta} & \pi^{n+2}(X, A)
\end{array}$$

Proof. Let a be an arbitrary element of $\pi^n(A_0)$ with a representative $f_a\colon A_0 \to S^n$. Let $f'\colon X_0 \to E_+^{n+1}$, $f''\colon A \to S^{n+1}$ be two extensions of f_a, with $f''(A_1) \subset E_+^{n+1}$, $f''(A_2) \subset E_-^{n+1}$. Define a mapping $l\colon S^{n+2} \to S^{n+2}$ by the rule

$$l(x_1, x_2, \ldots, x_{n+2}, x_{n+3}) = (x_1, x_2, \ldots, x_{n+3}, x_{n+2});$$

obviously $l(S^{n+1}) = S_1^{n+1}$. Let $g\colon X_0 \cup A \to S^{n+2}$ be a mapping defined by the rule

$$g(x) = \begin{cases} f''(x) & \text{for } x \in A, \\ lf'(x) & \text{for } x \in X_0. \end{cases}$$

Let $g'\colon X \to E_+^{n+2}$ be an extension of g, with $g'(X_1) \subset E_{++}^{n+2}$, $g'(X_2) \subset E_{+-}^{n+2}$. Let $h(x, t) = h_t(x)$ be a homotopy $h\colon (E_+^{n+2} \times I, S^{n+1} \times I) \to (S^{n+2}, E_-^{n+2})$ such that

(i) h_0 is the inclusion mapping,

(ii) $h_1\colon (E_+^{n+2}, S^{n+1}) \to (S^{n+2}, p)$ and h_1 maps $E_+^{n+2} \setminus S^{n+1}$ onto $S^{n+2} \setminus \{p\}$ homeomorphically,

(iii) if $h_t(x_1, \ldots, x_{n+2}, x_{n+3}) = (y_1, \ldots, y_{n+2}, y_{n+3})$, then $x_{n+2} \geqslant 0$ $(x_{n+1} \leqslant 0)$ implies $y_{n+2} \geqslant 0$ $(y_{n+2} \leqslant 0)$.

Such a homotopy exists since S^{n+2} is a suspension of S^{n+1}. It is easy to see that

$$[h_1 g'] = \delta\Delta(a) \quad \text{and} \quad [lh_1 g'] = \Delta\delta(a) ;$$

on the other hand, it is known (see for example Hu [8], p. 212) that $[lh_1 g] = -[h_1 g]$ in $\pi^{n+2}(X, A)$, which completes the proof.

§ 3. Compact fields

In the sequel, the symbol E^∞ will be reserved to denote an arbitrary but fixed infinite dimensional Banach space. We will suppose that there are given two sequences of linear subspaces of E^∞, $\{R_0^n\}$, $\{E^{\infty-n}\}$ and the subspace Q, with the following properties:

(i) R_0^n is n-dimensional,

(ii) $R_0^n \subset R_0^{n+1}$,

(iii) $E^{\infty-n} \subset E^{\infty-n+1}$,

(iv) for each $x \in E^\infty$ and $n = 0, 1, 2, \ldots$ there exists a unique decomposition $x = x_1 + x_2$, $x_1 \in R_0^n$, $x_2 \in E^{\infty-n}$,

(v) Q is one-dimensional and $Q \subset \bigcap_{n=1}^{\infty} E^{\infty-n}$.

It follows from (iv) that there exist linear projections of E^∞ onto R_0^n and $E^{\infty-n}$; we will denote these projections by $p_1^n\colon E^\infty \to R_0^n$, $p_2^n\colon E^\infty \to E^{\infty-n}$, respectively. Thus, for each $x \in E^\infty$ and $n = 0, 1, \ldots$, there exists a unique decomposition $x = p_1^n(x) + p_2^n(x)$.

The subspace $E^{\infty-n-1}$ disconnects $E^{\infty-n}$ onto two (closed) half-subspaces. Let us choose one of them and denote it by $E_+^{\infty-n}$, denoting the other by $E_-^{\infty-n}$. We will suppose that there is distiguished a point $q \in Q$ such that $\|q\| = 1$. Let us put

$$Q_- = \{x \in Q,\ x = tq,\ t < 0\},$$
$$P^{\infty-n} = E^{\infty-n} \setminus \{0\}, \qquad Q^{\infty-n} = E^{\infty-n} \setminus Q_-.$$

Let X be an arbitrary space. We shall say that a mapping $F\colon X \to E^\infty$ is *compact* if $\overline{F(X)}$ is a compact subset of E^∞. The set of all compact mappings defined on X will be denoted by $C(X)$. If L is a linear subspace of E^∞, then $C(X, L)$ will denote the set of all compact mappings defined on X, with values in L.

We shall say that (X, A) is a *closed* (*bounded*) *pair* in E^∞ if X is a closed (bounded) subset of E^∞.

Let (X, A) and (Y, B) be two pairs in E^∞; a mapping $f\colon (X, A) \to (Y, B)$ is said to be a *compact field* defined on (X, A) with values in (Y, B) if the mapping $F\colon X \to E^\infty$, defined by the rule $F(x) = x - f(x)$, is compact.

We shall use the following notation:

$D_L(X, A; Y, B)$—the set of all compact fields defined on (X, A) with values in (Y, B), with $F(x) = x - f(x) \in L$ for $x \in X$,

$DH_L(X, A; Y, B)$—the set of all mappings $h\colon (X \times I, A \times I) \to (Y, B)$, $h(x, t) = x - H(x, t)$, with $H(x, t) \in L$ for $x \in X$, $t \in I$, H being compact.

In the sequel we shall use the following abbreviations:

$$D_L(X, Y) = D_L(X, A; Y, B),$$
$$DH_L(X, Y) = DH_L(X, A; Y, B),$$
$$D_L^n(X, A) = D_L(X, A; P^{\infty-n}, Q^{\infty-n}),$$
$$DH_L^n(X, A) = DH_L(X, A; P^{\infty-n}, Q^{\infty-n}),$$
$$D(X, A; Y, B) = D_{E^\infty}(X, A; Y, B),$$
$$DH(X, A; Y, B) = DH_{E^\infty}(X, A; Y, B),$$
$$D^n(X, A) = D_{E^\infty}^n(X, A),$$
$$DH_{E^\infty}^n(X, A) = DH^n(X, A).$$

Two compact fields $f, g \in D_L(X, A; Y, B)$ are said to be *L-homotopic* (notation $f \underset{L}{\approx} g$) if there exists a homotopy $h \in DH_L(X, A; Y, B)$ connecting f and g. The relation $\underset{L}{\approx}$ is an equivalence relation in $D_L(X, A; Y; B)$. As a consequence the compact fields in $D_L(X, A; Y, B)$ are divided into disjoint equivalence classes, which will be called *L-homotopy classes*. We shall denote by $[X, A; Y, B]_L$ (or $[X, Y]_L$ if $A = A$) the totality of these classes and denote by $[f]_L$ the L-homotopy class of f, that is to say, the L-homotopy class which contains f. In the case of $L = E^\infty$, we shall write $[X, A; Y, B]$ and $[f]$ instead of $[X, A; Y, B]_{E^\infty}$ and $[f]_{E^\infty}$. If f and g are two E^∞-homotopic compact fields, we shall say that f and g are *homotopic* (notation $f \approx g$). Observe that in this terminology it may happen that f and g are homotopic as mappings and not homotopic as compact fields.

The following lemma, first proved by Schauder, will be used later:

LEMMA 3.1. *If $F\colon X \to E^\infty$ is a compact mapping then for every $\varepsilon > 0$ there exist a finite dimensional subspace $L \subset E^\infty$ and a compact mapping $F_\varepsilon\colon X \to L$ such that*

$$\|F(x) - F_\varepsilon(x)\| < \varepsilon \quad \text{for} \quad x \in X.$$

For the proof see [7], Theorem II. 5.

LEMMA 3.2. *If X is a closed subset of E^∞ and $h \in DH(X, E^\infty)$, then $Y = h(X \times I)$ is a closed subset of E^∞.*

This is a simple generalization of a known lemma, see [7], Prop. III. 5.

Proof. Let $h(x, t) = x - H(x, t)$. Let $\{y_n\}$ be a sequence of points of Y, i.e. $y_n = h(x_n, t_n) = x_n - H(x_n, t_n)$, and suppose that $\lim_{n\to\infty} y_n = y_0$. Since H is compact, we may assume, without loss of generality, that the sequence $H(x_n, t_n)$ is convergent to a point y^*, i.e. $\lim_{n\to\infty} H(x_n, t_n) = y^*$. Similarly we may assume that $\lim_{n\to\infty} t_n = t_0$. We have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \big(y_n + H(x_n, t_n)\big) = y_0 + y^*$$

and hence, by the continuity of h, $\lim_{n\to\infty} h(x_n, t_n) = h(y_0+y, t_0)$, i.e. $y_0 = h(y_0+y^*, t_0) \in h(X\times I)$.

Let f be a compact field, putting $h(x, t) = f(x)$ we obtain the following corollary:

Corollary 3.3. *If X is a closed subset of E^∞ and $f \in D(X, E^\infty)$, then $Y = f(X)$ is a closed subset of E^∞.*

Definition 3.4. Let $f \in D_L^n(X, A)$; we shall denote by f_L the restriction of f onto $X\cap L$:

$$f_L\colon (X\cap L, A\cap L)\to(L\cap P^{\infty-n}, L\cap Q^{\infty-n}).$$

Lemma 3.5. *Let L be a finite dimensional linear subspace of E^∞, with $R_0^n+Q\subset L$, and let (X, A) be a closed and bounded pair in E^∞. If $f, g\in D_L^n(X, A)$ and $f_L\sim g_L$, then $f \underset{L}{\approx} g$.*

Proof. Let $h\colon \big((X\cap L)\times I, (A\cap L)\times I\big)\to(L\cap P^{\infty-n}, L\cap Q^{\infty-n})$ be a homotopy connecting f_L and g_L. Let H be a mapping, defined on $Y = (X\times\{0\})\cup(X\times\{1\})\cup\big((X\cap L)\times I\big)$ by the following rules:

$$\begin{aligned}
H(x, 0) &= x-f(x) &&\text{for}\quad x\in X,\ t=0,\\
H(x, t) &= x-h(x, t) &&\text{for}\quad x\in X\cap L,\ t\in I,\\
H(x, 1) &= x-g(x) &&\text{for}\quad x\in X,\ t=1.
\end{aligned}$$

It is easy to see that H is continuous and $H\colon Y\to L$. By the Dugundji Extension Theorem (Dugundji [2]) H can be extended to a mapping $\bar H \in C(X\times I, L)$. Let us put

$$h'(x, t) = x-p_1^n(x)-p_2^n\bar H(x, t).$$

It is clear that $h'\in DH(X, E^\infty)$. On the other hand, $h'(x, t) = x-\bar H(x,t)-p_1^n\big(x-\bar H(x, t)\big) = p_2^n\big(x-\bar H(x,t)\big)$, so $h'(x,t)\in E^{\infty-n}$. Since $f(x)$, $g(x)\in E^{\infty-n}$, we have $h'(x, 0) = p_2^n\big(f(x)\big) = f(x)$, $h'(x, 1) = p_2^n\big(g(x)\big) = g(x)$. Since $\bar H(x, t)\in L$ and $R_0^n\subset L$, we have $p_2^n\bar H(x, t) = \bar H(x, t)-p_1^n\bar H(x, t)\in L$. If $x\in X\cap L$, then $x-\bar H(x, t) = h'(x, t)\in E^{\infty-n}$; so $h'(x, t) = h(x, t)$; if $x\in X-L$, then $p_1^n(x), p_2^n\bar H(x, t)\in L$; so $h'(x, t)\notin L$. From that, since $Q_-\subset L$, we conclude that $h\in DH_L^n(X, A)$.

Lemma 3.6. *Let L be a finite dimensional linear subspace of E^∞, with $R_0^n+Q\subset L$. If (X, A) is a closed and bounded pair in E^∞, then for an arbitrary mapping $\varphi\colon (X\cap L, A\cap L)\to(L\cap P^{\infty-n}, L\cap Q^{\infty-n})$ there exists an $f\in D_L^n(X, A)$ such that $f_L = \varphi$.*

Proof. Let $\bar F\in C(X, L)$ be an arbitrary extension of F, $F(x) = x-\varphi(x)$. Let us put $f(x) = p_2^n\big(x-\bar F(x)\big)$. Using calculations quite similar to those in the proof of the preceding lemma, it can be shown that $f\in D_L^n(X, A)$ and $f_L = \varphi$.

Lemma 3.7. *If X is a closed subset of E^∞ and $f\in D(X, E^\infty)$, then for every $\varepsilon>0$ there exist a finite dimensional subspace $L\subset E^\infty$ and $f_1\in D_L(X, E^\infty)$ such that*

$$\|f(x)-f_1(x)\|<\varepsilon \quad \textit{for}\quad x\in X.$$

Proof. Let $f(x) = x-F(x)$; by Lemma 3.1 there exist a finite dimensional subspace $L\subset E$ and a compact mapping $F_\varepsilon\colon X\to L$ such that $\|F(x)-F_\varepsilon(x)\|<\varepsilon$ for $x\in X$. Let us put $f_1(x) = x-F_\varepsilon(x)$. Obviously, f_1 has the desidered properties.

Definition 3.8. For an arbitrary subset $X\subset E^\infty$ and $\varepsilon>0$, let us put

$$X^{(\varepsilon)} = \{x\in E^\infty,\ \varrho(x, X)\leqslant\varepsilon\}.$$

Lemma 3.9. *If (X, A) and (Y, B) are two closed and bounded pairs in E^∞ and $f\in D(X, A;\ Y, B)$, then for every $\varepsilon>0$ there exist a finite dimensional subspace $L\subset E^\infty$ and $f_1\in D_L(X, A;\ Y^{(\varepsilon)}, B^{(\varepsilon)})$ such that $\|f(x)-f_1(x)\|<\varepsilon$ and $f_1\approx if$, where $i\colon (Y, B)\to(Y^{(\varepsilon)}, B^{(\varepsilon)})$ is the inclusion mapping.*

Proof. By Lemma 3.7 there exist a finite dimensional subspace L and $f_1\in D_L(X, E^\infty)$ such that $\|f(x)-f_1(x)\|<\varepsilon$. Obviously, $f_1\in D_L(X, A;\ Y^{(\varepsilon)}, B^{(\varepsilon)})$. Let us put

$$h(x, t) = t\cdot f(x)+(1-t)f_1(x);$$

evidently, $\|h(x, t)-f(x)\|<\varepsilon$ for $x\in X$, $t\in I$, and so $h(x, t)\in Y^{(\varepsilon)}$ for $x\in X$, $t\in I$ and $h(x, t)\in B^{(\varepsilon)}$ for $x\in A$, $t\in I$.

Lemma 3.10. *If (X, A) is a closed and bounded pair in E^∞ and $f\in D^n(X, A)$, then there exist a finite dimensional subspace $L\subset E^\infty$ and $g\in D_L^n(X, A)$ such that $f\approx g$.*

Proof. Let $Y = f(X)$, $B = f(A)$. By Corollary 3.3, Y and A are closed subsets of E^∞; thus (Y, B) is a closed and bounded pair in $E^{\infty-n}$. There exists an $\varepsilon>0$ such that $\varrho(Y, R_0^n)>\varepsilon$ and $\varrho(B, R_0^n+Q)>\varepsilon$. By Lemma 3.9 there exist a finite dimensional subspace $L\subset E^\infty$ and $f_0\in D_L(X, A;\ Y^{(\varepsilon)}, B^{(\varepsilon)})$ such that $\|f(x)-f_0(x)\|<\varepsilon$. Let us put $g(x) = p_2^n f_0(x)$, $h_1(x, t) = t\cdot f(x)+(1-t)f_0(x)$ and $h(x, t) = p_2^n\big(t\cdot f(x)+(1-t)f_0(x)\big) = x-p_1^n(x)-p_2^n\big(t\cdot F(x)+(1-t)F_0(x)\big)$. Thus $h(x, 0) = g(x)$, $h(x, 1) = f(x)$, $h(x, t)\in E^{\infty-n}$ for $x\in X$, $t\in I$. Obviously, $h_1(x, t)\in Y^{(\varepsilon)}$ for $x\in X$, $t\in I$ and $h_1(x, t)\in B^{(\varepsilon)}$ for $x\in A$, $t\in I$; thus $h_1(x, t)\notin R_0^n$ for $x\in X$ and $h_1(x, t)\notin R^n+Q_-$ for $x\in A$, whence $h(x, t)\in P^{\infty-n}$ for $x\in X$ and $h(x, t)\in Q^{\infty-n}$ for $x\in A$. Thus $h\in DH^n(X, A)$ and the proof is completed.

Lemma 3.11. *If (X, A) is a closed and bounded pair in E^∞, $f, g\in D^n(X, A)$ and $f\approx g$, then there exist a finite dimensional subspace $L\subset E^\infty$ and $f_1, g_1\in D_L^n(X, A)$ such that $f_1\approx f$, $g_1\approx g$ and $f_1\underset{L}{\approx} g_1$.*

Proof. Let $h(x,t) = x - H(x,t)$ be a homotopy connecting f and g. By Lemma 3.2, $Y = h(X\times I)$ and $B = h(A\times I)$ are closed subsets of E^∞. Obviously, $Y\subset P^{\infty-n}$, $B\subset Q^{\infty-n}$ and there exists an $\varepsilon>0$ such that $\varrho(Y, R_0^n) > \varepsilon$ and $\varrho(B, R_0^n+Q) > \varepsilon$; thus $Y^{(\varepsilon)}\cap R_0^n = \Lambda$ and $B^{(\varepsilon)}\cap(R_0^n+Q) = \Lambda$. Let us put $Z = \overline{H(X\times I)}$. If $F\colon Z\to Z$ is the identity mapping, by Lemma 3.1 there exist a finite dimensional subspace $L\subset E^\infty$ and a compact mapping $F_\varepsilon\colon Z\to L$ such that $\|F(x)-F_\varepsilon(x)\| = \|x-F_\varepsilon(x)\| < \varepsilon$ for $x\in Z$. Let us put $h'(x,t) = x - F_\varepsilon H(x,t)$, $h''(x,t) = p_2^n\big(x-F_\varepsilon H(x,t)\big) = x - p_1^n x - p_2^n F_\varepsilon H(x,t)$; evidently, $p_1^n(x)+p_2^nF_\varepsilon H(x,t)\in L$ for $x\in X$, $t\in I$. Since

$$\|h''(x,t)-h'(x,t)\| = \|H(x,t)-F_\varepsilon H(x,t)\| < \varepsilon,$$

we have $h''(x,t)\notin R_0^n$ for $x\in X$, $t\in I$ and $h''(x,t)\notin R_0^n+Q_-$ for $x\in A$, $t\in I$ and thus $h''\in DH_L^n(X,A)$.

THEOREM 3.12. *If (X,A) is a closed and bounded pair in E_∞, then for each compact field $f\in D^n(X,A)$ there exist a finite dimensional subspace $L\subset E^\infty$ and a compact field $g\in D_L^n(X,A)$ such that $f\approx g$. Two compact fields $f,g\in D^n(X,A)$ are homotopic if and only if there exist a finite dimensional subspace $L\subset E^\infty$ and $f_1, g_1\in D_L^n(X,A)$ such that $f_1 \underset{L}{\approx} g_1\cdot f\approx f_1$ and $g\approx g_1$.*

The theorem follows immediately from Lemmas 3.10 and 3.11.

THEOREM 3.13. *If (X,A) is a closed and bounded pair in E^∞, $f_0, g_0\in D^n(A)$, $f_0\approx g_0$ and if there exists an $f\in D^n(X)$ which is an extension of f_0, then there exists a $g\in D^n(X)$ which is an extension of g_0 such that $f\approx g$.*

The above theorem is a special case of the Homotopy Extension Theorem proved for compact fields in [6].

§ 4. Definition of $\pi^{\infty-n}(X,A)$

In this section (X,A) denotes a closed and bounded pair in E^∞.

DEFINITION 4.1. Let L be a linear subspace of E^∞. We shall say that a compact field $f\in D_L^n(X,A)$ is *L-normal* if

(i) $f(x) = q$ for $x\in A\cap L$,

(ii) $\|f(x)\| = 1$ for $x\in X\cap L$.

The subset of $D_L^n(X,A)$ consisting of all L-normal compact fields will be denoted by $ND_L^n(X,A)$.

We shall say that a homotopy $h\in DH_L^n(X,A)$ is *L-normal* if

(iii) $h(x,t) = q$ for $x\in A\cap L$, $t\in I$,

(iv) $\|h(x,t)\| = 1$ for $x\in X\cap L$, $t\in I$.

The subset of $DH_L^n(X,A)$ consisting of all L-normal homotopies will be denoted by $NDH_L^n(X,A)$. If $A = \Lambda$, then we shall write $ND_L^n(X)$ and $NDH_L^n(X)$ instead of $ND_L^n(X,\Lambda)$ and $NDH_L^n(X,\Lambda)$.

DEFINITION 4.2. We shall say that a linear subspace $L\subset E^\infty$ is *n-admissible* if L is finite dimensional, $R_0^n+Q\subset L$ and $\dim L\cap E^{\infty-n}\geqslant n+4$. We shall denote by $\mathfrak{L}^n$ the set of all n-admissible subspaces of E^∞ partially ordered by inclusion.

If L is an n-admissible subspace, we shall denote by S_L^m the subset of L defined by the rule

$$S_L^m = \{x\in L\cap E^{\infty-n},\ \|x\| = 1\},$$

where $m+1 = \dim(L\cap E^{\infty-n})$; thus $\dim L = n+m+1$ and S_L^m is homeomorphic with the m-dimensional sphere.

LEMMA 4.3. *If L is an n-admissible subspace of E^∞, then*

(i) *for each $f\in D_L^n(X,A)$ there exists an $f_1\in ND_L^n(X,A)$ such that $f\underset{L}{\approx} f_1$,*

(ii) *two compact fields $f,g\in ND_L^n(X,A)$ are L-homotopic if and only if there exists an L-normal homotopy connecting f and g.*

Proof. Let $\varphi\colon (L\cap P^{\infty-n})\to S_L^m$ be a mapping defined by the rule $\varphi(x) = x/\|x\|$. For an arbitrary $\varepsilon>0$, let us put $D_\varepsilon = \{x\in S_L^m,\ \|x+q\|<\varepsilon\}$. Let $\eta^{(\varepsilon)}\colon (S_L^m\times I)\to S_L^m$ be a homotopy such that $\eta_0^{(\varepsilon)}$ is the identity mapping and $\eta^{(\varepsilon)}\colon (S_L^m, S_L^m\setminus D_\varepsilon)\to(S_L^m, q)$ maps D_ε onto $S_L^m\setminus\{q\}$ homeomorphically. If $f\in D_L^n(X,A)$, then for $x\in A\cap L$, $f_L(x)\notin Q_-$, and hence $\varphi f_L(x)\neq -q$. Since $f_L(A\cap L)$ is a closed subset of S_L^m, there exists an $\varepsilon>0$ such that $\varphi f_L(A\cap L)\in S_L^m\setminus D_\varepsilon$. By Lemma 3.6 there exists an $f_1\in D_L^n(X,A)$ which is an extension of $f_L\colon (X\cap L, A\cap L)\to(S_L^m, q)$. Obviously, $f_1\in ND_L^n(X,A)$ and, by Lemma 3.5, $f_1\underset{L}{\approx} f$. Thus the first part is proved. The proof of the second part is analogous.

Let L be an n-admissible subspace and α,β be any elements of $[X,A;P^{\infty-n},Q^{\infty-n}]_L$. By Lemma 4.3 there exist representatives $f_\alpha, f_\beta\in ND_L^n(X,A)$ of α and β, respectively. If $\dim L = n+m+1$, then S_L^m is homeomorphic to S^m; let $\xi\colon (S_L^m, q)\to(S^m, p)$ be an arbitrary homeomorphism. Thus

$$\xi(f_\alpha)_L,\ \xi(f_\beta)_L\colon (X\cap L, A\cap L)\to(S^m, p).$$

Since $\dim(X\cap L)\leqslant m+n+1\leqslant 2m-2$, the group $\pi^m(X\cap L, A\cap L)$ is defined. Let us put $\delta = [\xi(f_\alpha)_L]$, $\gamma = [\xi(f_\beta)_L]$, $\delta,\gamma\in\pi^m(X\cap L, A\cap L)$.

LEMMA 4.4. *Let $\alpha,\beta\in[X,A;P^{\infty-n},Q^{\infty-n}]_L$ and let $\xi\colon (S_L^m,q)\to(S^m,p)$ be an arbitrary homeomorphism; if $\varphi\colon (X\cap L, A\cap L)\to(S^m,p)$ is a representative of $\alpha+\beta$, and if $g\in D_L^n(X,A)$ is an extension of $\xi^{-1}\varphi$, then $[g]_L$ depends only on α and β.*

Proof. It is clear that $[(f_\alpha)_L]$ and $[(f_\beta)_L]$ are independent of the choice of the representatives f_α and f_β. By Lemma 3.5, $[g]_L$ does not depend on the choice of the extension of $\xi^{-1}\varphi$. We are going to show that $[\xi^{-1}\varphi]$

does not depend on ξ. Let $\xi_1, \xi_2\colon (S^m_L, q)\to(S^m, p)$ be any two homeomorphisms. Let us put $\xi_0 = \xi_1\xi_2^{-1}\colon (S^m, p)\to(S^m, p)$,

$$\alpha_i = [\xi_i(f_\alpha)_L], \qquad \beta_i = [\xi_i(f_\beta)_L], \qquad i = 1, 2,$$

$\alpha_i, \beta_i \in \pi^m(X\cap L, A\cap L)$. Let $f_{\alpha_1}, f_{\alpha_2}, f_{\beta_1}, f_{\beta_2}, f_{\alpha_1+\beta_1}, f_{\alpha_2+\beta_2}$ be representatives of $\alpha_1, \alpha_1, \beta_1, \beta_2, \alpha_1+\beta_1, \alpha_2+\beta_2$, respectively. By Lemma 2.2

$$[\xi_1(f_\alpha)_L]+[\xi_1(f_\beta)_L] = [\xi_0\xi_2(f_\alpha)_L]+[\xi_0\xi_2(f_\beta)_L] = [\xi_0]\circ\big([\xi_2(f_\alpha)_L]+[\xi_2(f_\beta)_L]\big).$$

Thus $f_{\alpha_1+\beta_1}\sim\xi_0 f_{\alpha_2+\beta_2}$. Since ξ_1^{-1} is a homeomorphism, we have

$$\xi_1^{-1}f_{\alpha_1+\beta_1}\sim\xi_1^{-1}\xi_0 f_{\alpha_2+\beta_2} = \xi_2^{-1}f_{\alpha_2+\beta_2}.$$

DEFINITION 4.5. Let (X, A) be a closed and bounded pair in E^∞ and $\alpha, \beta\in[X, A; P^{\infty-n}, Q^{\infty-n}]_L$, where L is an n-admissible subspace of E^∞. Let us put

$$\alpha+\beta = [g],$$

where $g\in D^n_L(X, A)$ is an arbitrary extension of the mapping $\xi^{-1}f_{\alpha_1+\beta_1}$.

This definition is correct, since, by Lemma 4.4, $[g]_L$ depends only on α and β.

DEFINITION 4.6. Let $\xi\colon (S^m_L, q)\to(S^m, p)$ be an arbitrary homeomorphism. For an arbitrary $f\in ND^n_L(X, A)$, let us put

$$\Phi_\xi([f]_L) = [\xi f_L].$$

The correctness of this definition follows immediately from the previous lemmas.

The following lemma is an immediate consequence of the definition of Φ_ξ:

LEMMA 4.7. *The transformation*

$$\Phi_\xi\colon [X, A; P^{\infty-n}, Q^{\infty-n}]_L\to\pi^m(X\cap L, A\cap L)$$

is one-to-one; moreover,

$$\Phi_\xi(\alpha+\beta) = \Phi_\xi(\alpha)+\Phi_\xi(\beta).$$

COROLLARY 4.8. *$[X, A; P^{\infty-n}, Q^{\infty-n}]_L$ is an abelian group, isomorphic to the group $\pi^m(X\cap L, A\cap L)$. If $\xi_1, \xi_2\colon (S^m_L, q)\to(S^m, p)$ are homotopic, then $\Phi_{\xi_1} = \Phi_{\xi_2}$; if not, $\Phi_{\xi_1} = -\Phi_{\xi_2}$.*

In the sequel we shall denote the group $[X, A; P^{\infty-n}, Q^{\infty-n}]_L$ by $\pi^{\infty-n}_L(X, A)$.

Let L and M be two n-admissible subspace of E^∞ and let $L\subset M$. If $f, g\in D^n_L(X, A)$, then $f, g\in D^n_M(X, A)$; if $f\underset{L}{\approx}g$, then $f\underset{M}{\approx}g$.

For an arbitrary $f\in D^n_L(X, A)$, let us put

$$\sigma^M_L([f]_L) = [f]_M;$$

thus $\sigma^M_L\colon \pi^{\infty-n}_L(X, A)\to\pi^{\infty-n}_M(X, A)$.

Let L and M be two admissible subspaces of E^∞ such that $L\subset M$ and $\dim M = \dim L+1 = m+n+2$. In this case L intersects M into two halfspaces, which will be denoted by M_+ and M_-.

LEMMA 4.9. *If $\xi\colon (S^m_L, q)\to(S^m, p)$, $\eta\colon (S^{m+1}_M, q)\to(S^{m+1}, p)$ are two homeomorphisms such that*

$$\begin{aligned} \eta(x) &= \xi(x) && \text{for}\quad x\in S^m_L,\\ \eta(x) &\in E^{m+1}_+ && \text{for}\quad x\in S^{m+1}_M\cap M_+,\\ \eta(x) &\in E^{m+1}_- && \text{for}\quad x\in S^{m+1}_M\cap M_-, \end{aligned}$$

and if Δ is the coboundary homomorphism of the relative triad $(X\cap M, A\cap M; X\cap M_+, A\cap M_+; X\cap M_-, A\cap M_-)$, then the following diagram is commutative:

$$\begin{array}{ccc} \pi^{\infty-n}_L(X, A) & \xrightarrow{\sigma^M_L} & \pi^{\infty-n}_M(X, A)\\ \downarrow{\scriptstyle \Phi_\xi} & & \downarrow{\scriptstyle \Phi_\eta}\\ \pi_m(X\cap L, A\cap L) & \xrightarrow{\Delta} & \pi^{m+1}(X\cap M, A\cap M). \end{array}$$

Proof. Let α be an arbitrary element of $\pi^{\infty-n}_L(X, A)$. By Lemma 4.3, there exists a representative compact field $f\in ND^n_L(X, A)$. Thus $f_L\colon (X\cap L, A\cap L)\to(S^m_L, q)$. Let $\varphi\colon (X\cap M, A\cap M)\to(S^{m+1}_M, q)$ be an extension of f_L such that $\varphi(X\cap M_+)\subset S^{m+1}_M\cap M_+$, $\varphi(X\cap M_-)\subset S^{m+1}_M\cap M_-$ and let $g\in D^n_M(X, A)$ be an arbitrary extension of φ. Obviously, $[\xi g_M] = \sigma^M_L(\alpha)$ and $\Phi_\eta\sigma^M_L(\alpha) = \Delta\Phi_\xi(\alpha)$.

LEMMA 4.10. *If L and M are two n-admissible subspaces of E^∞ such that $L\subset M$, then $\sigma^M_L\colon \pi^{\infty-n}_L(X, A)\to\pi^{\infty-n}_M(X, A)$ is a homomorphism.*

Proof. If $\dim M = 1+\dim L$, then this is a direct consequence of Lemma 4.9. In the general case, there exists a sequence of subspaces $L_0 = L, L_1, L_2, \ldots, L_k = M$, such that $L_i\subset L_{i+1}$ and $\dim L_{i+1} = 1+\dim L_i$. Since

$$\sigma^M_L = \sigma^{L_1}_{L_0}\sigma^{L_2}_{L_1}\ldots\sigma^{L_k}_{L_{k-1}}$$

is a composition of homomorphisms, it is a homomorphism.

Let L be an n-admissible subspace and $f\in D^n_L(X, A)$. Putting $\sigma_L([f]_L) = [f]$ we obtain a transformation

$$\sigma_L\colon \pi^{\infty-n}_L(X, A)\to[X, A; P^{\infty-n}, Q^{\infty-n}].$$

We shall denote by $\pi^{\infty-n}_\infty(X, A)$ the direct limit of the system $\{\pi^{\infty-n}_L(X, A), \sigma^M_L\}$ indexed by $\mathfrak{L}_n$ — the set of all n-admissible subspaces partially ordered by inclusion (for the definitions, see Eilenberg-Steenrod [3], Ch. VIII).

If L and M are two n-admissible subspaces with $L \subset M$, then $\sigma_M \sigma_L^M = \sigma_L$; thus there is defined a limit transformation

$$\sigma\colon \pi^{\infty-n}(X, A) \to [X, A; P^{\infty-n}, Q^{\infty-n}].$$

By Theorem 3.12, σ is one-to-one.

Definition 4.11. Let α, β be any two elements of $[X, A; P^{\infty-n}, Q^{\infty-n}]$. Let

$$\alpha + \beta = \sigma\big(\sigma^{-1}(\alpha) + \sigma^{-1}(\beta)\big).$$

Thus $[X, A; P^{\infty-n}, Q^{\infty-n}]$ is a group, which will be denoted by $\pi^{\infty-n}(X, A)$;

$$\sigma\colon \pi_\infty^{\infty-n}(X, A) \to \pi^{\infty-n}(X, A)$$

is an isomorphism.

§ 5. The algebraic properties of $\pi^{\infty-n}(X, A)$

Let (X, A) and (Y, B) be two closed and bounded pairs in E^∞ and let $f\colon (X, A) \to (Y, B)$ be a compact field. For any compact field $g \in D^n(Y, B)$ $gf \in D^n(X, A)$; if $g_1, g_2 \in D^n(Y, B)$ are homotopic compact fields, then $g_1 f$ and $g_2 f$ are homotopic. The assignment $f^*([g]) = [gf]$ defines the induced transformation

$$f^*\colon \pi^{\infty-n}(Y, B) \to \pi^{\infty-n}(X, A).$$

If L, M are two n-admissible subspaces such that $L \subset M$, $g \in D_L^n(Y, B)$ and $f \in D_M^n(X, A; Y, B)$, then $gf \in D_M^n(X, A)$; the assignment $[g]_M \to [gf]_M$ defines the induced transformation

$$f^*\colon \pi_M^{\infty-n}(Y, B) \to \pi_M^{\infty-n}(X, A).$$

The following theorem is obvious by the definition of f^*:

Theorem 5.1. *If (X, A) is a closed and bounded pair in E^∞ and $f\colon (X, A) \to (X, A)$ is the identity compact field, then, for each n,*

$$f^*\colon \pi^{\infty-n}(X, A) \to \pi^{\infty-n}(X, A)$$

is the identity isomorphism.

Lemma 5.2. *Let (X, A) and (Y, B) be two compact and bounded pairs in E^∞. If L is an n-admissible subspace of E^∞ and $f \in D_L(X, A; Y, B)$, then the induced transformation*

$$f^*\colon \pi^{\infty-n}(Y, B) \to \pi^{\infty-n}(X, A)$$

is a homomorphism.

Proof. Let α, β be any two elements of $\pi^{\infty-n}(Y, B)$. By Lemma 4.3 there exists an n-admissible subspace M such that it is possible to choose representatives $f_\alpha, f_\beta \in ND_M^n(Y, B)$. We may suppose, without loss of generality, that $L \subset M$. Thus, $f_\alpha f, f_\beta f \in ND_M^n(X, A)$. Let $\xi\colon S_M^m \to S^m$ be an arbitrary homeomorphism. We have the following commutative diagram:

$$\begin{array}{ccc}
\pi_M^{\infty-n}(Y, B) & \xrightarrow{f^*} & \pi_M^{\infty-n}(X, A) \\
\downarrow{\scriptstyle \Phi_\xi} & & \downarrow{\scriptstyle \Phi_\xi} \\
\pi^m(Y \cap M, A \cap M) & \xrightarrow{f_M^*} & \pi^m(X \cap M, A \cap M).
\end{array}$$

Since f_M is a homomorphism,

$$f^*\colon \pi_M^{\infty-n}(Y, B) \to \pi_M^{\infty-n}(X, A)$$

is a homomorphism. Let us put $\alpha_1 = [f_\alpha]_M$, $\beta_1 = [f_\beta]_M$. Thus, $f(\alpha+\beta) = \sigma_M f^*(\alpha_1 + \beta_1) = \sigma_M f^*(\alpha_1) + \sigma_M f^*(\beta_1) = f^*(\alpha) + f^*(\beta)$.

By the definition of the induced transformation the following theorems are obvious:

Theorem 5.3. *If (X, A) and (Y, B) are two closed and bounded pairs in E^∞, $f, g \in D(X, A; Y, B)$ and $f \approx g$, then $f^* = g^*$.*

Theorem 5.4. *If (X, A), (Y, B), (Z, C) are closed and bounded pairs in E^∞, $f \in D(X, A; Y, B)$, $g \in D(Y, B; Z, C)$, then $gf \in D(X, A; Z, C)$ and $(gf)^* = f^* g^*$.*

Lemma 5.5. *If (X, A) is a closed and bounded pair in E^∞ and L is an n-admissible subspace of E^∞, then for each $f \in D_L^n(X, A)$ there exist an $\varepsilon > 0$ and $f_1 \in D_L^n(X^{(\varepsilon)}, A^{(\varepsilon)})$ such that f_1 is an extension of f.*

Proof. Let $f(x) = x - F(x)$ and let $f_1(x) = x - F_1(x)$, $f_1 \in D_L^n(X^{(1)}, E^{\infty-n})$ be an arbitrary extension of f. We are going to prove that there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

(i) if $\varrho(x, X) \leqslant \varepsilon_1$, then $f_1(x) \neq 0$,

(ii) if $\varrho(x, A) \leqslant \varepsilon_2$, then $f_1(x) \notin Q_-$.

Suppose, on the contrary, that there is no such ε_1. Hence for every natural m there exists an $x_m \in E^\infty$ such that $\varrho(x_m, X) \leqslant 1/m$ and $f_1(x_m) = x_m - F_1(x_m) = 0$. Thus, $x_m \in L \cap X$ and we may suppose, without loss of generality, that x_m is convergent. Let $x_0 = \lim_{n\to\infty} x_m$. From this $x_0 \in X$ and by the continuity of f_1, $f_1(x_0) = f(x_0) = 0$, which is a contradiction. The proof of (ii) is quite analogous.

Corollary 5.6. *If (X, A) is a closed and bounded pair in E^∞, then for each $\alpha \in \pi^{\infty-n}(X, A)$ there exist an $\varepsilon > 0$ and $\alpha_1 \in \pi^{\infty-n}(X^{(\varepsilon)}, A^{(\varepsilon)})$ such that $\alpha = i^*(\alpha_1)$, where*

$$i\colon (X, A) \to (X^{(\varepsilon)}, A^{(\varepsilon)})$$

is the inclusion mapping.

Theorem 5.7. *If (X, A) and (Y, B) are two closed and bounded pairs in E^∞ and $f \in D(X, A; Y, B)$, then f^* is a homomorphism.*

Let α, β be any two elements of $\pi^{\infty-n}(Y, B)$. There exist, by Corollary 5.6, an $\varepsilon > 0$ and $\alpha_1, \beta_1 \in \pi^{\infty-n}(Y^{(\varepsilon)}, B^{(\varepsilon)})$ such that $i^*(\alpha_1) = \alpha$, $i^*(\beta_1) = \beta$. By Lemma 3.7 there exists a compact field $f_1\colon X \to E^\infty$ such that $\|f(x) - f_1(x)\| < \varepsilon$ for $x \in X$. Thus, $f_1 \in D(X, A;\, Y^{(\varepsilon)}, B^{(\varepsilon)})$.

Let us regard the following homotopy:

$$h(x, t) = t\cdot f(x) + (1-t)f_1(x) = x - t\cdot F(x) - (1-t)F_1(x)\,.$$

Obviously, $h \in DH(X, A;\, Y^{(\varepsilon)}, B^{(\varepsilon)})$, whence $if \approx f_1$. By Theorems 5.3 and 5.4, $f_1^* = (if)^* = f^* i^*$, whence

$$f^*(\alpha+\beta) = f^*i^*(\alpha_1+\beta_1) = f_1^*(\alpha_1+\beta_1) = f_1^*(\alpha_1) + f_1^*(\beta_1) = f^*(\alpha) + f^*(\beta)\,.$$

Let (X, A) and (Y, B) be two closed and bounded pairs in E^∞. A compact field $f \in D(X, A;\, Y, B)$ will be called a *homotopy equivalence* if there exists a compact field $g \in D(Y, B;\, X, A)$ such that gf and fg are homotopic to the identity compact fields.

THEOREM 5.8. *If (X, A) and (Y, B) are two closed and bounded pairs in E^∞ and if $f \in D(X, A;\, Y, B)$ is a homotopy equivalence then, for each n,*

$$f^*\colon \pi^{\infty-n}(Y, B) \to \pi^{\infty-n}(X, A)$$

is an isomorphism.

Proof. The theorem is an easy consequence of Theorems 5.1, 5.3 and 5.4.

THEOREM 5.9. *If (X, A) and (Y, B) are two closed and bounded pairs in E^∞, $f\colon (X, A) \to (Y, B)$ is a homeomorphism and $f \in D(X, A;\, Y, B)$ then, for each n,*

$$f^*\colon \pi^{\infty-n}(Y, B) \to \pi^{\infty-n}(X, A)$$

is an isomorphism.

Proof. Let $f(x) = x - F(x)$ and let $g = f^{-1}\colon (Y, B) \to (X, A)$. Since $f(g(x)) = g(x) - F(g(x)) = x$, we have $g(x) = x + F(g(x))$ and $g \in D(Y, B;\, X, A)$. Thus, f is a homotopy equivalence and, by Theorem 5.8, f^* is an isomorphism for each n.

Let (X, A) be a closed and bounded pair in E^∞ and let L be an n-admissible subspace of E^∞, $\dim L = n+m+1$. Let $\xi\colon (S_L^m, q) \to (S^m, p)$, $\eta\colon (S_L^{m+1}, q) \to (S^{m+1}, p)$ be any two homeomorphisms such that

(i) $\xi(x) = \eta(x)$ for $x \in S_L^m$,

(ii) $\eta(x) \in E_+^{m+1}$ for $x \in S_L^{m+1} \cap E_+^{\infty-n+1}$,

(iii) $\eta(x) \in E_-^{m+1}$ for $x \in S_L^{m+1} \cap E_+^{\infty-n+1}$.

Consider the diagram

$$\begin{array}{ccc}
\pi_L^{\infty-n}(A) & & \pi_L^{\infty-n+1}(X, A) \\
\downarrow{\scriptstyle \Phi_\xi} & & \downarrow{\scriptstyle \Phi_\eta} \\
\pi^m(A\cap L) & \xrightarrow{\delta} & \pi^{m+1}(X\cap L, A\cap L)\,,
\end{array}$$

in which Φ_ξ, Φ_η are isomorphisms.

DEFINITION 5.10. The homomorphism

$$\delta = (\Phi_\eta)^{-1}\delta\Phi_\xi\colon \pi_L^{\infty-n}(A) \to \pi_L^{\infty-n+1}(X, A)$$

will be called the *coboundary operator* of the pair (X, A).

Observe that $\delta(\alpha)$ does not depend on ξ and η provided conditions (i)-(iii) are fulfilled, since in this case $[\eta]$ is determined by $[\xi]$ and changes the sign together with $[\xi]$.

LEMMA 5.11. *If (X, A) is a closed and bounded pair in E^∞ and if L, M are two n-admissible subspaces of E^∞ such that $L \subset M$, then the following diagram is commutative:*

$$\begin{array}{ccc}
\pi_L^{\infty-n}(A) & \xrightarrow{\delta} & \pi_L^{\infty-n+1}(X, A) \\
\downarrow{\scriptstyle \sigma_L^M} & & \downarrow{\scriptstyle \sigma_L^M} \\
\pi_M^{\infty-n}(A) & \xrightarrow{\delta} & \pi_M^{\infty-n+1}(X, A)\,.
\end{array}$$

Proof. It is sufficient to prove this in the case $1 + \dim L = \dim M$. Denote by M_+ and M_- two halfspaces such that $M_+ \cup M_- = M$, $M_+ \cap M_- = L$. Let $\dim L \cap E^{\infty-n} = m+1$. Let $\xi\colon (S_L^m, q) \to (S^m, p)$ be an arbitrary homeomorphism and let $\eta\colon (S_M^{m+1}, q) \to (S^{m+1}, p)$, $\xi_1\colon (S_L^{m+1}, q) \to (S^{m+1}, p)$ be two extensions of ξ such that ξ, ξ_1 fulfil conditions (i)-(iii) of Definition 5.10 and

$$\begin{aligned}
&\eta(x) \in E_+^{m+1} \quad \text{for} \quad x \in S_M^{m+1} \cap M_+\,,\\
&\eta(x) \in E_-^{m+1} \quad \text{for} \quad x \in S_M^{m+1} \cap M_-\,.
\end{aligned}$$

Let $l\colon S_+^{m+2} \to S^{m+2}$ be the mapping defined by the rule

$$l(x_1, x_2, \ldots, x_{m+2}, x_{m+3}) = (x_1, x_2, \ldots, x_{m+3}, x_{m+2})\,.$$

Let $\varphi\colon (S_M^{m+1} \cup S_L^{m+1}) \to S^{m+2}$ be the mapping defined by the rule

$$\varphi(x) = \begin{cases} \eta(x) & \text{for} \quad x \in S_M^{m+1}, \\ l\xi_1(x) & \text{for} \quad x \in S_L^{m+1}. \end{cases}$$

Let

$$\eta_1\colon (S_M^{m+2}, q) \to (S^{m+1}, p)$$

be an extension of φ such that

$$\begin{aligned}
&\eta_1(x) \in E_{++}^{m+2} \quad \text{for} \quad x \in S_M^{m+2} \cap M_+ \cap E_+^{\infty-n+1}\,,\\
&\eta_1(x) \in E_{+-}^{m+2} \quad \text{for} \quad x \in S_M^{m+2} \cap M_+ \cap E_-^{\infty-n+1}\,,\\
&\eta_1(x) \in E_{-+}^{m+2} \quad \text{for} \quad x \in S_M^{m+2} \cap M_- \cap E_+^{\infty-n+1}\,,\\
&\eta_1(x) \in E_{--}^{m+2} \quad \text{for} \quad x \in S_M^{m+2} \cap M_- \cap E_-^{\infty-n+1}\,.
\end{aligned}$$

Let us regard the following diagram, in which

$$\begin{aligned}
&\Delta\colon \pi^m(A \cap L) \to \pi^{m+1}(A \cap M)\,,\\
&\Delta\colon \pi^{m+1}(X \cap L, A \cap L) \to \pi^{m-2}(X \cap M, A \cap M)
\end{aligned}$$

are coboundary operators of the triad $(A \cap M, A \cap M_+, A \cap M_-)$ and the relative triad $(X \cap M, A \cap M; X \cap M_+, A \cap M_+; X \cap M_-, A \cap M_-)$, respectively:

$$\begin{tikzcd}
\pi_L^{\infty-n}(A) \arrow[rrr, "\delta"] \arrow[dr, "\Phi_\xi"] \arrow[ddd, "\sigma_L^M"'] \arrow[drr, phantom, "\mathrm{II}"] & & & \pi_L^{\infty-n+1}(X, A) \arrow[dl, "\Phi_{\xi_1}"'] \arrow[ddd, "\sigma_L^M"] \\
& \pi^m(A \cap L) \arrow[r, "\delta"] \arrow[d, "\Delta"] \arrow[dr, phantom, "\mathrm{I}"] \arrow[l, phantom, "\mathrm{III}"] & \pi^{m+1}(X \cap L, A \cap L) \arrow[d, "-\Delta"] \arrow[r, phantom, "\mathrm{V}"] & \\
& \pi^{m+1}(A \cap M) \arrow[r, "\delta"] \arrow[dr, phantom, "\mathrm{IV}"] & \pi^{m+2}(X \cap M, A \cap M) & \\
\pi_M^{\infty-n}(A) \arrow[rrr, "\delta"] \arrow[ur, "\Phi_\eta"] & & & \pi_M^{\infty-n+1}(X, A) \arrow[ul, "\Phi_{\eta_1}"']
\end{tikzcd}$$

We assert that in the above diagram all regions are commutative:
I: by Lemma 2.6,
II, IV: by Definition 5.10,
III: by Lemma 4.9,
V: by Lemma 4.9, since $l\eta_1$ is an extension of ξ_1.
Thus, since Φ_ξ, Φ_η, Φ_{ξ_1}, Φ_{η_1} are isomorphisms, the exterior square is commutative.

Lemma 5.12. *If (X, A) is a closed and bounded pair in E^∞, L is an n-admissible subspace of E^∞ and $i\colon A \to X$, $j\colon X \to (X, A)$ are natural inclusions, then the following sequence is exact:*

$$\pi_L^{\infty-n}(X, A) \xrightarrow{j^*} \pi_L^{\infty-n}(X) \xrightarrow{i^*} \pi_L^{\infty-n}(A) \xrightarrow{\delta} \pi_L^{\infty-n+1}(X, A) \to \ldots \to \pi_L^\infty(A).$$

Proof. Let $\dim L = n+m+1$, $S_L^{m+k} = \{x \in L \cap E^{\infty-n+k},\ \|x\| = 1\}$. Let us define, by induction, a sequence of homeomorphisms $\xi_0, \xi_1, \ldots, \xi_n$, $\xi_k\colon (S_L^{m+k}, q) \to (S^{m+k}, p)$ as follows: let ξ_0 be an arbitrary homeomorphism; suppose that ξ_k is defined, and let ξ_{k+1} be such that ξ_k, ξ_{k+1} fulfil conditions (i)-(iii) of Definition 5.10. In the diagram

$$\begin{array}{ccccccc}
\pi_L^{\infty-n+k}(X, A) & \xrightarrow{j^*} & \pi_L^{\infty-n+k}(X) & \xrightarrow{i^*} & \pi_L^{\infty-n+k}(A) & \xrightarrow{\delta} & \pi_L^{\infty-n+k+1}(X, A) \\
\downarrow{\scriptstyle \Phi_{\xi_k}} & & \downarrow{\scriptstyle \Phi_{\xi_k}} & & \downarrow{\scriptstyle \Phi_{\xi_k}} & & \downarrow{\scriptstyle \Phi_{\xi_k}} \\
\pi^{m+k}(X \cap L, A \cap L) & \xrightarrow{j^*} & \pi^{m+k}(X \cap L) & \xrightarrow{i^*} & \pi^{m+k}(A \cap L) & \xrightarrow{\delta} & \pi^{m+k+1}(X \cap L, A \cap L)
\end{array}$$

all squares are commutative and Φ_{ξ_k} are isomorphisms. Thus, since the cohomotopy sequence of the pair $(X \cap L, A \cap L)$ is exact, the lemma is proved.

Definition 5.13. Let (X, A) be a closed and bounded pair in E^∞. We shall denote by

$$\delta\colon \pi_\infty^{\infty-n}(A) \to \pi_\infty^{\infty-n+1}(X, A)$$

the homomorphism which is the limit of the homomorphisms

$$\delta\colon \pi_L^{\infty-n}(A) \to \pi_L^{\infty-n+1}(X, A).$$

We shall denote by

$$\delta\colon \pi^{\infty-n}(A) \to \pi^{\infty-n+1}(X, A)$$

the homomorphism defined by the rule

$$\delta = \sigma\delta_\infty\sigma^{-1}.$$

Theorem 5.14. *If (X, A) is a closed and bounded pair in E^∞ and $i\colon A \to X$, $j\colon X \to (X, A)$ are inclusions, then the sequence*

$$\pi^{\infty-n}(X, A) \xrightarrow{j^*} \pi^{\infty-n}(X) \xrightarrow{i^*} \pi^{\infty-n}(A) \xrightarrow{\delta} \pi^{\infty-n+1}(X, A) \to \ldots \to \pi^\infty(A)$$

is exact.

Proof. Let us regard the commutative ladder

$$\begin{array}{ccccccc}
\pi_\infty^{\infty-n}(X, A) & \xrightarrow{j_\infty^*} & \pi_\infty^{\infty-n}(X) & \xrightarrow{i_\infty^*} & \pi_\infty^{\infty-n}(A) & \xrightarrow{\delta_\infty} & \pi_\infty^{\infty-n+1}(X, A) \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\
\pi^{\infty-n}(X, A) & \xrightarrow{j^*} & \pi^{\infty-n}(X) & \xrightarrow{i^*} & \pi^{\infty-n}(A) & \xrightarrow{\delta} & \pi^{\infty-n+1}(X, A)
\end{array}$$

in which i_∞^* and j_∞^* are limits of the homomorphisms

$$i^*\colon \pi_L^{\infty-n}(X) \to \pi_L^{\infty-n}(A), \qquad j^*\colon \pi_L^{\infty-n}(X, A) \to \pi_L^{\infty-n}(X).$$

By Lemma 5.12 and Theorem VIII. 5.4 of [3] the top row is exact. Thus the bottom row is also exact.

Lemma 5.15. *If (X, A) and (Y, B) are two closed and bounded pairs in E^∞, L is a finite dimensional subspace of E^∞ and $f \in D_L(X, A; Y, B)$, then the following diagram is commutative:*

$$\begin{array}{ccc}
\pi^{\infty-n}(B) & \xrightarrow{\delta} & \pi^{\infty-n+1}(Y, B) \\
\downarrow{\scriptstyle (f|_A)^*} & & \downarrow{\scriptstyle f^*} \\
\pi^{\infty-n}(A) & \xrightarrow{\delta} & \pi^{\infty-n+1}(X, A)
\end{array}$$

Proof. If M is an n-admissible subspace and $L \subset M$, then the diagram

$$\begin{array}{ccc}
\pi_M^{\infty-n}(B) & \xrightarrow{\delta} & \pi_M^{\infty-n+1}(Y, B) \\
\downarrow{\scriptstyle (f|_A)^*} & & \downarrow{\scriptstyle f^*} \\
\pi_M^{\infty-n}(A) & \xrightarrow{\delta} & \pi_M^{\infty-n+1}(X, A)
\end{array}$$

is commutative. The subset of $\mathfrak{L}^n$ consisting of all n-admissible subspaces containing L is confinal in $\mathfrak{L}^n$. Hence, by Theorem VIII. 4.12 of [3], the following diagram is commutative:

$$\begin{array}{ccc}
\pi^{\infty-n}(B) & \xrightarrow{\delta_\infty} & \pi^{\infty-n+1}(Y, B) \\
\downarrow{\scriptstyle (f|_A)^*_\infty} & & \downarrow{\scriptstyle f_\infty^*} \\
\pi^{\infty-n}(A) & \xrightarrow{\delta_\infty} & \pi^{\infty-n+1}(X, A).
\end{array}$$

In this diagram f_∞^* and $(f|_A)_\infty^*$ are limits of the induced homomorphisms f^* and $(f|_A)^*$. Now it is sufficient to apply to the last diagram the isomorphism σ.

THEOREM 5.16. *If (X, A) and (Y, B) are two closed and bounded pairs in E^∞ and $f \in D(X, A; Y, B)$, then the following diagram is commutative:*

$$\begin{tikzcd} \pi^{\infty-n}(B) \arrow[r, "\delta"] \arrow[d, "(f|_A)^*"'] & \pi^{\infty-n+1}(Y, B) \arrow[d, "f^*"] \\ \pi^{\infty-n}(A) \arrow[r, "\delta"] & \pi^{\infty-n+1}(X, A). \end{tikzcd}$$

Proof. Let α be an arbitrary element of $\pi^{\infty-n}(B)$ and let $\beta = \delta(\alpha) \in \pi^{\infty-n+1}(Y, B)$. By Corollary 5.6 there exist an $\varepsilon > 0$, $\alpha_1 \in \pi^{\infty-n}(B^{(\varepsilon)})$ and $\beta_1 \in \pi^{\infty-n+1}(Y^{(\varepsilon)}, B^{(\varepsilon)})$ such that $i_0^*(\alpha_1) = \alpha$, $i^*(\beta_1) = \beta$ where $i_0\colon B \to B^{(\varepsilon)}$, $i\colon (Y, B) \to (Y^{(\varepsilon)}, B^{(\varepsilon)})$ are isomorphisms. By Lemma 3.9 there exist an n-admissible subspace $L \subset E^\infty$ and a compact field $f' \in D_L(X, A; Y, B)$ such that $f' \approx if$ and $f'|_A \approx i_0 f|_A$.

Consider the following diagram:

$$\begin{tikzcd} \pi^{\infty-n}(B) \arrow[rrr, "\delta"] \arrow[ddd, "(f|_A)^*"'] \arrow[dr, phantom, "\mathrm{I}"] & & & \pi^{\infty-n+1}(Y, B) \arrow[ddd, "f^*"] \arrow[dl, phantom, "\mathrm{III}"] \\ & \pi^{\infty-n}(B^{(\varepsilon)}) \arrow[r, "\delta"] \arrow[ul, "i_0^*"'] \arrow[dl, "(f'|_A)^*"] \arrow[r, phantom, shift left=4ex, "\mathrm{II}"] \arrow[r, phantom, shift right=4ex, "\mathrm{IV}"] & \pi^{\infty-n+1}(Y^{(\varepsilon)}, B^{(\varepsilon)}) \arrow[ur, "i^*"] \arrow[dr, "(f')^*"'] & \\ & & & \\ \pi^{\infty-n}(A) \arrow[rrr, "\delta"] & & & \pi^{\infty-n+1}(X, A). \end{tikzcd}$$

Regions I and III are commutative by Theorems 5.3 and 5.4, and regions II and IV by Lemma 5.15. Thus

$$f^*\delta(\alpha) = f^*(\beta) = (f')^*(\beta_1) = \delta(f'|_A)^*(\alpha_1) = \delta(f|_A)^*(\alpha).$$

THEOREM 5.17. *If (X, A) is a closed and bounded pair in E^∞, U is an open subset of E^∞ and $i\colon (X \setminus U, A \setminus U) \to (X, A)$ is the inclusion mapping, then*

$$i^*\colon \pi^{\infty-n}(X, A) \to \pi^{\infty-n}(X \setminus U, A \setminus U)$$

is an isomorphism.

Proof. Let L be an n-admissible subspace of E^∞, $\dim L = n+m+1$. Thus

$$i_L^*\colon \pi^m(X \cap L, A \cap L) \to \pi^m\big((X \setminus U) \cap L, (A \setminus U) \cap L\big)$$

is an isomorphism by Theorem 7.6 of [12]. Thus, by Theorem VIII. 4.13 of [3], in the commutative diagram

$$\begin{tikzcd} \pi_\infty^{\infty-n}(X, A) \arrow[r, "i_\infty^*"] \arrow[d, "\sigma"] & \pi^{\infty-n}(X \setminus U, A \setminus U) \arrow[d, "\sigma"] \\ \pi^{\infty-n}(X, A) \arrow[r, "i^*"] & \pi^{\infty-n}(X \setminus U, A \setminus U) \end{tikzcd}$$

i_∞^* is an isomorphism. Since σ are isomorphisms, i^* is an isomorphism.

So far we have defined, for every closed and bounded pair (X, A), the sequence of groups $\pi^{\infty-n}(X, A)$ together with the homomorphisms f^* and δ. Thus $\{\pi^{\infty-n}, f^*, \delta\}$ is a contravariant functor from the category of closed and bounded pairs in E^∞ to the category of abelian groups. Obviously, $\{\pi^{\infty-n}, f^*, \delta\}$ depends not only on E^∞ but also on the system $\{E^{\infty-n}, R_0^n, Q\}$.

Now let us suppose that there is given another system $\{\bar{E}^{\infty-n}, \bar{R}_0^n, \bar{Q}\}$ such that conditions (i)-(v) of the § 3 are fulfilled and that halfsubspaces $\bar{E}_+^{\infty-n}$, $\bar{E}_-^{\infty-n}$ and a point $\bar{q} \in \bar{Q}$ are distinguished. In this case, the contravariant functor $\{\bar{\pi}^{\infty-n}, \bar{f}^*, \bar{\delta}\}$ is defined.

We shall prove a theorem which says that in this case functors $\{\pi^{\infty-n}, f^*, \delta\}$ and $\{\bar{\pi}^{\infty-n}, \bar{f}^*, \bar{\delta}\}$ are "almost" isomorphic.

THEOREM 5.18. *For each $n \geqslant 0$ there exists an isomorphism*

$$\Omega_n\colon \pi^{\infty-k}(X, A) \to \bar{\pi}^{\infty-k}(X, A),$$

defined for each closed and bounded pair (X, A) in E^∞ and each $k \leqslant n$ and such that:

(i) *if (X, A) and (Y, B) are any two closed and bounded pairs in E^∞ and if $f \in D(X, A; Y, B)$, then the following diagram is commutative:*

$$\begin{tikzcd} \pi^{\infty-k}(Y, B) \arrow[r, "f^*"] \arrow[d, "\Omega_n"] & \pi^{\infty-k}(X, A) \arrow[d, "\Omega_n"] \\ \bar{\pi}^{\infty-k}(Y, B) \arrow[r, "\bar{f}^*"] & \bar{\pi}^{\infty-k}(X, A), \end{tikzcd}$$

(ii) *for each closed and bounded pair in E^∞ the following diagram is commutative:*

$$\begin{tikzcd} \pi^{\infty-k}(A) \arrow[r, "\delta"] \arrow[d, "\Omega_n"] & \pi^{\infty-k+1}(X, A) \arrow[d, "\Omega_n"] \\ \bar{\pi}^{\infty-k}(A) \arrow[r, "\bar{\delta}"] & \bar{\pi}^{\infty-k+1}(X, A). \end{tikzcd}$$

Proof. Let us put $E^{\infty-n} = E_1 + Q$, $\bar{E}^{\infty-n} = E_2 + \bar{Q}$, $E_0 = E_1 \cap E_2$. Obviously, $E^\infty = E_0 + M$, where $\dim M \leqslant 2n+2$. Let $\varphi\colon E^\infty \to E^\infty$ be a linear isomorphism such that

(i) $\varphi(x) = x$ for $x \in E_0$,

(ii) $\varphi(Q) = \bar{Q}$, $\varphi(Q_-) = \bar{Q}_-$,

(iii) $\varphi(E^{\infty-k}) = \bar{E}^{\infty-k}$, $\varphi(E_+^{\infty-k}) = \bar{E}_+^{\infty-k}$, for $k \leqslant n$.

It is easy to see that such a φ always exists. If $f \in D(X, A; P^{\infty-n}, Q^{\infty-n})$, then $\varphi f \in D(X, A; \bar{P}^{\infty-n}, \bar{Q}^{\infty-n})$. Putting

$$\Omega_n([f]) = [\varphi f]$$

we obtain a one-to-one transformation of $\pi^{\infty+n}(X, A)$ onto $\bar{\pi}^{\infty-n}(X, A)$. It is easy to verify that Ω_n is an isomorphism and that (i) and (ii) are true.

§ 6. Duality theorems

In this section we are going to prove some theorems on duality between $\pi^{\infty-n}$ and S-homotopy groups. These theorems are generalizations of the duality theorems proved by E. Spanier and J. H. C. Whitehead in S-theory.

Let X be an arbitrary space; we shall denote by SX the suspension of X. If $f\colon X\to Y$ is a mapping, then we shall denote by $Sf\colon SX\to SY$ the suspension of the mapping f. If $f,g\colon X\to Y$ are two homotopic mappings, then $Sf\sim Sg$; thus a transformation

$$S\colon [X,Y]\to[SX,SY]$$

is defined.

Let $k=n+m+1$, and let $S_k^m=\{x\in S^k,\ x_1=x_2=\ldots=x_n=0\}$. If X is a compact space and $\dim X\leqslant 2m-2$, then we shall identify $[X,S_k^m]$ with $\pi^m(X)$. Thus

$$S\colon \pi^m(X)\to\pi^{m+1}(SX).$$

Similarly, $S\colon \pi_n(X)\to\pi_{n+1}(SX)$. It is known that in these cases S is a homomorphism (see [11], p. 162). We have the following exact sequences:

$$\pi_n(X)\xrightarrow{S}\pi_{n+1}(SX)\xrightarrow{S}\ldots\xrightarrow{S}\pi_{n+l}(S^lX)\xrightarrow{S}\pi_{n+l+1}(S^{l+1}X)\xrightarrow{S}\ldots,$$
$$\pi^m(X)\xrightarrow{S}\pi^{m+1}(SX)\xrightarrow{S}\ldots\xrightarrow{S}\pi^{m+l}(S^lX)\xrightarrow{S}\pi^{m+l+1}(S^{l+1}X)\xrightarrow{S}\ldots$$

The limits of this sequences are denoted by $\Sigma_n(X)$ and $\Sigma_m(X)$ and called the *S-homotopy* and *S-cohomotopy groups*. For more precise definitions and properties of the functor S, see [12] and [13].

If X is a compact space and $\dim X\leqslant 2m-2$, then

$$S\colon \pi^{m+k}(S^kX)\to\pi^{m+k+1}(S^{k+1}X)$$

is an isomorphism (Spanier [11]); in this case we shall identify $\Sigma^m(X)$ and $\pi^m(X)$.

If $X,Y\subset S^k$ are subcomplexes of S^k in some triangulation T, $k=n+m+1$, then we shall call Y a *k-dual* to X if Y is a deformation retract of $S^k\setminus X$. In this case there is defined an isomorphism

$$\mathfrak{D}_k\colon \Sigma^m(X)\to\Sigma_n(Y)$$

which has properties analogous to the Alexander-Pontrjagin duality isomorphism (Spanier, Whitehead [13]). In particular, if X_1,X_2 are subcomplexes of S^k, Y_1,Y_2 are their k-duals and $i_1\colon X_1\to X_2$, $i\colon Y_2\to Y_1$ are inclusions, then the following diagram is commutative:

$$\begin{array}{ccc}\Sigma^m(X_2) & \xrightarrow{i_1^*} & \Sigma^m(X_1)\\ \downarrow{\scriptstyle\mathfrak{D}_k} & & \downarrow{\scriptstyle\mathfrak{D}_k}\\ \Sigma_n(Y_2) & \xrightarrow{i^*} & \Sigma_n(Y_1)\end{array}$$

We would like to point out that in this brief exposition of the Spanier-Whitehead theory we have restricted ourselves to the very simple case. The original theory is more general, for example, the notion of k-dual is more general.

Let us denote by $i\colon Y\to(S^k\setminus X)$ the inclusion mapping. Since Y is a deformation retract of $S^k\setminus X$, $i\colon \Sigma_n(Y)\to\Sigma_n(S^k\setminus X)$ is an isomorphism. In what follows we shall identify these groups. We denote by

$$\mathfrak{D}_k\colon \Sigma^m(X)\to\Sigma_n(S^k\setminus X)$$

the composition of $\mathfrak{D}_k$ and i_*.

LEMMA 6.1. *Let T be a triangulation of S^{k+1} such that S^k is a subcomplex. Let X be a subcomplex of S^{k+1} and let Y be k-dual for X, such that $Y_0=Y\cap S^k$ is k-dual for $X_0=X\cap S^k$. If $i\colon Y_0\to Y$ is the inclusion mapping and $\Delta\colon \pi^m(X_0)\to\pi^{m+1}(X)$ is the coboundary operator of the triad $(X, X\cap E_+^{k+1}, X\cap E_-^{k+1})$, then the following diagram is commutative:*

$$\begin{array}{ccc}\pi^m(X_0) & \xrightarrow{\Delta} & \pi^{m+1}(X)\\ \downarrow{\scriptstyle\mathfrak{D}_k} & & \downarrow{\scriptstyle\mathfrak{D}_k}\\ \Sigma_n(Y_0) & \xrightarrow{i_*} & \Sigma_n(Y)\end{array}$$

Proof. Let T_0 be the triangulation induced by T on S^k. Let T_1 be the suspension of T_0. Denote by $T_1^{(p)}$ the p-th baricentric subdivision of T_1. Let $Z^{(p)}$ be a subcomplex of S^{k+1} consisting of all simplexes of $T_1^{(p)}$, whose vertices do not belong to $S^k\setminus X_0$. There exists a p_0 such that $X\subset Z^{(p_0)}$; let us put $Z=Z^{(p_0)}$. Thus Y_0 is $(k+1)$-dual to SX_0 and Z. Let $i_0\colon Y_0\to Y_0$ be the identity mapping and let $i_1\colon SX_0\to Z$, $i_2\colon X\to Z$ be the inclusion mappings. Consider the diagram

$$\begin{tikzcd}
 & & \pi^{m+1}(X) \arrow[d, "i_2^*"] \arrow[dr, "\mathfrak{D}_{k+1}"] & \\
\pi^m(X_0) \arrow[r, "S"] \arrow[d, "\mathfrak{D}_k"] \arrow[urr, bend left] \arrow[dr, phantom, "\mathrm{I}"] & \pi^{m+1}(SX_0) \arrow[d, "\mathfrak{D}_{k+1}"] \arrow[dr, phantom, "\mathrm{II}"] & \pi^{m+1}(Z) \arrow[l, "i_1^*"'] \arrow[d, "\mathfrak{D}_{k+1}"] \arrow[r, phantom, "\mathrm{III}"] & \Sigma_n(Y) \\
\Sigma_n(Y_0) \arrow[r, "(i_0)_*"] & \Sigma_n(Y_0) & \Sigma_n(Y_0) \arrow[l, "(i_0)_*"'] \arrow[ur, "i_*"'] &
\end{tikzcd}$$

in which Δ is a coboundary homomorphism of the triad $(X, X\cap E_+^{m+1}, X\cap E_-^{m+1})$. Let α be any element of $\pi^m(X_0)$, represented by $f_\alpha\colon X_0\to S_k^m$. Let $f\colon Z\to S_{k+1}^{m+1}$ be an extension of f_α such that $f(Z\cap E_+^{m+1})\subset S_{k+1}^{m+1}\cap E_+^{k+1}$, $f(Z\cap E_-^{m+1})\subset S_{k+1}^{m+1}\cap E_-^{m+1}$. Let us put $\beta=[f]$. Obviously $S(\alpha)=i_1^*(\beta)\in\pi^{m+1}(SX_0)$ and $i_2^*(\beta)=\Delta(\alpha)$. Since regions I, II and III are commutative, we have

$$\mathfrak{D}_{k+1}\Delta(\alpha)=\mathfrak{D}_{k+1}i_2^*(\beta)=i_*\mathfrak{D}_k(\alpha).$$

Let X be a compact subset of S^k, $\dim X\leqslant 2m-2$, $m+n+1=k$. Let T be a triangulation of S^k. Let $T^{(p)}$ be the p-th baricentric subdivision of T. Let $X^{(p)}$ be a subcomplex of S^k consisting of all k-dimensional simplexes of $T^{(p)}$ with non-empty intersections with X and their faces. Consider the sequence

$$X^{(1)} \xleftarrow{i_1} X^{(2)} \ldots X^{(p)} \xleftarrow{i_p} X^{(p+1)} \leftarrow \ldots,$$

where i_p are inclusions. It is easily seen that X is homeomorphic in a natural way to the inverse limit of this sequence. Thus, by Theorem 13.4 of [11], $\pi^m(X)$ is isomorphic to the direct limit of the sequence

$$\pi^m(X^{(1)}) \xrightarrow{i_1^*} \pi^m(X^{(2)}) \ldots \pi^m(X^{(p)}) \xrightarrow{i_p^*} \pi^m(X^{(p+1)}) \to \ldots$$

Consider the sequence

$$(S^k \setminus X^{(1)}) \xrightarrow{j_1} (S^k \setminus X^{(2)}) \to \ldots (S^k \setminus X^{(p)}) \xrightarrow{j_p} (S^k \setminus X^{(p+1)}) \ldots,$$

where j_p are inclusions. Passing to the homotopy groups, we obtain the sequence

$$\pi_n(S^k \setminus X^{(1)}) \xrightarrow{j_*} \pi_n(S^k \setminus X^{(2)}) \ldots \pi_n(S^k \setminus X^{(p)}) \xrightarrow{(j)_*} \pi_n(S^k \setminus X^{(p+1)}) \ldots$$

Observe that the direct limit of this sequence isi somorphic in a natural way to $\pi_n(S^k \setminus X)$. Indeed, if Y is a polyhedron, then for every mapping $f\colon Y \to (S^k \setminus X)$ there exist an $\varepsilon > 0$ and a mapping $g\colon Y \to (S^k \setminus X)$ such that f is homotopic to g.

Similarly, $\Sigma_n(S^k \setminus X)$ is the direct limit of the sequence

$$\Sigma_n(S^k \setminus X^{(1)}) \to \Sigma_n(S^k \setminus X^{(2)}) \ldots \Sigma_n(S^k \setminus X^{(p)}) \xrightarrow{(j_p)_*} \Sigma_n(S^k \setminus X^{(p+1)}).$$

Consider the following commutative ladder:

$$\begin{array}{ccccccc}
\pi^m(X^{(1)}) & \xrightarrow{i_1^*} & \pi^m(X^{(2)}) \to & \ldots & \pi^m(X^{(p)}) & \xrightarrow{i_p^*} & \pi^m(X^{p+1}) \\
\downarrow \mathfrak{D}_k & & \downarrow \mathfrak{D}_k & & \downarrow \mathfrak{D}_k & & \downarrow \mathfrak{D}_k \\
\Sigma_n(S^k \setminus X^{(1)}) & \xrightarrow{j_1'} & \Sigma_n(S^k \setminus X^{(2)}) & \ldots & \Sigma_n(S^k X^{(p)}) & \xrightarrow{(j_p)_*} & \Sigma_n(S^k \setminus X^{(p+1)}).
\end{array}$$

Thus, $\mathfrak{D}_k$ is an isomorphism of the direct sequence $\{\pi^m(X^{(p)})\}$ onto $\Sigma_n(S^k \setminus X^{(p)})$; let us denote by

$$\mathfrak{D}_k\colon \pi^m(X) \to \Sigma_n(S^k \setminus X)$$

the limit isomorphism.

Lemma 6.2. *If X is a compact subset of S^{k+1}, $\dim X \leqslant 2m-2$, $m+n+1 = k$, $X_0 = X \cap S^k$ and $i\colon (S^k \setminus X_0) \to (S^{k+1} \setminus X)$ is the inclusion, then the following diagram is commutative:*

$$\begin{array}{ccc}
\pi^m(X_0) & \xrightarrow{\Delta} & \pi^{m+1}(X) \\
\downarrow \mathfrak{D}_k & & \downarrow \mathfrak{D}_{k+1} \\
\Sigma_n(S^k \setminus X_0) & \xrightarrow{i_*} & \Sigma_n(S^{k+1} \setminus X)
\end{array}$$

(Δ is the coboundary operator of the triad $(X, X \cap E_+^{k+1}, X \cap E_-^{k+1})$).

Proof. Let T be a triangulation of S^{k+1} such that S^k is a subcomplex. Let $X^{(p)}$ be the sequence of polyhedrons defined above, and let us put $X_0^{(p)} = X^{(p)} \cap S^k$. Denote by $i_p\colon X^{(p+1)} \to X^{(p)}$, $j^p\colon (S^{k+1} \setminus X^{(p)}) \to (S^{k+1} \setminus X^{(p+1)})$, $l_p\colon (S^k \setminus X_0^{(p)}) \to (S^{k+1} \setminus X^{(p)})$ the natural inclusions and consider the following diagram:

$$\begin{tikzcd}
\pi^m(X_0^{(p)}) \arrow[rrr, "\Delta"] \arrow[dr, "i_p^*"] \arrow[ddd, "\mathfrak{D}_k"'] \arrow[drr, phantom, "\mathrm{I}"] \arrow[ddr, phantom, "\mathrm{II}"] & & & \pi^{m+1}(X^{(p)}) \arrow[dl, "i_{p+1}^*"'] \arrow[ddd, "\mathfrak{D}_{k+1}"] \arrow[ddl, phantom, "\mathrm{IV}"] \\
& \pi^m(X_0^{(p+1)}) \arrow[r, "\Delta"] \arrow[d, "\mathfrak{D}_k"] \arrow[dr, phantom, "\mathrm{III}"] & \pi^{m+1}(X^{(p+1)}) \arrow[d, "\mathfrak{D}_{k+1}"] & \\
& \Sigma_n(S^k \setminus X_0^{(p+1)}) \arrow[r, "l_*^{p+1}"] & \Sigma_n(S^{k+1} \setminus X^{(p+1)}) & \\
\Sigma_n(S^k \setminus X_0^{(p)}) \arrow[rrr, "l_*^p"] \arrow[ur, "j_*^p"] \arrow[urr, phantom, "\mathrm{V}"] & & & \Sigma_n(S^{k+1} \setminus X^{(p)}) . \arrow[ul, "j_*^p"']
\end{tikzcd}$$

We assert that all regions are commutative: I by Lemma 2.5, III by Lemma 6.1; the commutativity of II and IV is a property of $\mathfrak{D}_k$ (see [13]) and V is manifestly commutative. Thus, by Theorem VIII. 3.14 of [3], the lemma is proved.

Let $l_k\colon R^k \to S^k$ be a mapping which maps R^k onto $S^k \setminus \{\overline{p}\}$ homeomorphically and is such that for $n \leqslant k$

$$l_k(R_+^n) = E_+^n \setminus \{\overline{p}\}, \quad l_k(R_-^n) = E_-^n \setminus \{\overline{p}\}, \quad l_k(S^{k-1}) = S_k^{k-1}.$$

Corollary 6.3. *If X is a compact subset of R^{k+1}, $Y = l_{k+1}(X)$, $X_0 = X \cap R^k$, $Y_0 = l_k(X_0)$, $k \leqslant 2m-2$, then the following diagram is commutative:*

$$\begin{array}{ccc}
\pi^m(Y_0) & \xrightarrow{l_k^*} & \pi^m(X_0) \\
\downarrow \Delta & & \downarrow \Delta \\
\pi^{m+1}(X) & \xrightarrow{l_{k+1}^*} & \pi^{m+1}(X).
\end{array}$$

Obviously, $(l_k)_*\colon \Sigma_p(R^k \setminus X) \to \Sigma_p(S^k \setminus Y)$ is an isomorphism for $p < n-2$.

Definition 6.4. Let X be a compact subset of R^k, $k \leqslant 2m-2$; putting

$$D_k = (l_k)_*^{-1} \mathfrak{D}_k (l_k^*)^{-1}$$

we obtain an isomorphism

$$D_k\colon \pi^m(X) \to \Sigma_n(R^k \setminus X).$$

Lemma 6.5. *If X is a compact subset of R^{k+1}, $X_0 = X \cap R^k$, $k \leqslant 2m-2$, $k = n+m+1$ and $i\colon (R^k \setminus X_0) \to (R^{k+1} \setminus X)$ is the inclusion mapping, then the following diagram is commutative:*

$$\begin{array}{ccc}
\pi^m(X_0) & \xrightarrow{\Delta} & \pi^{m+1}(X) \\
\downarrow D_k & & \downarrow D_k \\
\Sigma_n(R^k \setminus X_0) & \xrightarrow{i_*} & \Sigma_n(R^{k+1} \setminus X).
\end{array}$$

Proof. Let $Y = l_{k+1}(X)$, $Y_0 = l_k(X_0) = Y \cap S^k$; let $i_1\colon (S^k \setminus Y_0) \to (S^{k+1} \setminus Y)$ be the inclusion mapping. Consider the following diagram:

$$\begin{tikzcd}
\pi^m(X_0) \arrow[rrr, "\Delta"] \arrow[ddd, "D_k"'] & & & \pi^{m+1}(X) \arrow[ddd, "D_{k+1}"] \\
& \pi^m(Y_0) \arrow[ul, "l_k^*"'] \arrow[r, "\Delta"] \arrow[d, "\mathfrak{D}_k"] & \pi^{m+1}(Y) \arrow[ur, "l_{k+1}^*"] \arrow[d, "\mathfrak{D}_k"] & \\
& \Sigma_n(S^k \setminus Y_0) \arrow[r, "(i_1)_*"] \arrow[dl, "(l_k)_*"'] & \Sigma_n(S^{k+1} \setminus Y) \arrow[dr, "(l_{k+1})_*"] & \\
\Sigma_n(R^k \setminus X_0) \arrow[rrr, "i_*"] & & & \Sigma_n(R^{k+1} \setminus X).
\end{tikzcd}$$

In this diagram all regions are commutative. Thus the exterior square is commutative.

Let $\varphi\colon R^k \to R^k$, $k = n+m+1$, be a homeomorphism such that $\varphi(R^n) = R^n$ and $\varphi(S^m_{k-1}) = S^m_{k-1}$. Let X be a compact subset of R^k, $\dim X \leqslant 2m-2$ and $Y = \varphi(X)$. If α is any element of $\pi^m(X)$ with a representative $f_\alpha\colon X \to S^m_{k-1}$, we shall put

$$\varphi^0_X(\alpha) = [\varphi f_\alpha \varphi^{-1}] \in [Y, S^m_{k-1}] = \pi^m(Y)\,;$$

thus

$$\varphi^0_X\colon \pi^m(X) \to \pi^m(Y)$$

is an isomorphism.

Anologously we define the isomorphism

$$\varphi^X_0\colon \Sigma_n(R^k \setminus X) \to \Sigma_n(R^k \setminus Y)\,.$$

Lemma 6.6. *If X is a compact subset of R^k, $\dim X \leqslant 2m-2$, $k = n+m+1$, then the diagram*

$$\begin{tikzcd}
\pi^m(X) \arrow[r, "\varphi^0_X"] \arrow[d, "D_k"] & \pi^m(Y) \arrow[d, "D_k"] \\
\Sigma_n(R^k \setminus X) \arrow[r, "\varphi^X_0"] & \Sigma_n(R^k \setminus Y)
\end{tikzcd}$$

is commutative.

Proof. The diagram is commutative for the subcomplexes of S^k (see [13]). Thus the commutativity of the diagram follows from the definition of D_k.

The subspace R^n_0 of E is linearly isomorphic to R^n; we shall suppose that there is given an isomorphism $\mu_n\colon R^n_0 \to R^n$. Let L be an n-admissible subspace of E^∞, $\dim L = k = n+m+1$. We shall say that the (linear) isomorphism $\varphi\colon L \to R^k$ is *n-admissible* if φ is an extension of μ_n and

$$\varphi(L \cap E^{\infty - n}) = \{x \in R^k,\ x_1 = x_2 = \ldots = x_n = 0\}\,.$$

Definition 6.7. Suppose that X is a closed and bounded subset of E^∞, L an n-admissible subspace of E^∞, $\dim L = k = n+m+1$, $\tau\colon L \to R^k$ an n-admissible linear isomorphism $Y = \tau(X \cap L)$ and $f \in ND^n_L(X)$. We shall denote by

$$\psi_\tau\colon \pi^{\infty-n}_L(X) \to \pi^m(Y)$$

the isomorphism defined by the rule

$$\psi_\tau([f]_L) = [\tau f_L \tau^{-1}] \in [Y, S^m_{k-1}] = \pi^m(Y)\,.$$

Lemma 6.8. *Let X be a closed and bounded subset of E^∞, L an n-admissible subspace of E^∞, $\dim L = k = m+n+1$, and $\tau_1, \tau_2\colon L \to R^k$ any two n-admissible linear isomorphisms; then*

$$(\tau_1)^{-1}_* D_k \psi_{\tau_1} = (\tau_2)^{-1}_* D_k \psi_{\tau_2}$$

where $(\tau_i)^{-1}_*\colon \Sigma_n(R^k \setminus Y_i) \to \Sigma_n(L \setminus X)$, $Y_i = \tau_i(L \cap X)$ for $i = 1, 2$.

Proof. Let us put $\varphi = \tau_1\tau_2^{-1}$, $\varphi\colon R^k \to R^k$. By Lemma 6.6, $D_k\varphi^0_{Y_2} = \varphi^{Y_2}_0 D_k$. Since $\varphi^0_{Y_2} = \psi_{\tau_1}(\psi_{\tau_2})^{-1}$ and $\varphi^0_{Y_2} = (\tau_1)_*(\tau_2)^{-1}_*$, we have $D_k\psi_{\tau_1}(\psi_{\tau_2})^{-1} = (\tau_1)_*(\tau_2)^{-1}_* D_k$. Thus

$$(\tau_1)^{-1}_* D_k \psi_{\tau_1} = (\tau_2)^{-1}_* D_k \psi_{\tau_2}\,.$$

Definition 6.9. Let X be a closed and bounded subset of E^∞, L an n-admissible subspace of E^∞, $\dim L = k = n+m+1$ and $\tau\colon L \to R^k$ an n-admissible isomorphism. Denote by

$$\Delta_n\colon \pi^{\infty-n}_L(X) \to \Sigma_n(L \cap X)$$

the isomorphism defined by the rule

$$\Delta_n = \tau^{-1}_* D_k \psi_\tau\,.$$

By Lemma 6.8 this definition is independent of τ.

Lemma 6.10. *If X is a closed and bounded subset of E^∞, L, M are two n-admissible subspaces of E^∞, $L \subset M$ and $i\colon (L \cap X) \to (M \cap X)$ is the inclusion mapping, then the following diagram is commutative:*

$$\begin{tikzcd}
\pi^{\infty-n}_L(X) \arrow[r, "\sigma^M_L"] \arrow[d, "\Delta_n"] & \pi^{\infty-n}_M(X) \arrow[d, "\Delta_n"] \\
\Sigma_n(L \setminus X) \arrow[r, "i_*"] & \Sigma_n(M \setminus X).
\end{tikzcd}$$

Proof. It is sufficient to prove this in the case of $\dim M = \dim L + 1 = k+1 = n+m+2$. Let $\tau_0\colon L \to R^k$, $\tau\colon M \to R^{k+1}$ be two n-admissible isomorphisms such that τ is an extension of τ_0. Let $Y = \tau(M \cap X)$,

$Y_0 = \tau_0(L \cap X) = Y \cap R^k$ and let $i_1\colon (R^k \setminus Y_0) \to (R^{k+1} \setminus Y)$ be the inclusion mapping. Consider the diagram:

$$\begin{tikzcd}
\pi_L^{\infty-n}(X) \arrow[rrr, "\sigma_L^M"] \arrow[dr, "\psi_{\tau_0}"] \arrow[ddd, "\Delta_n"'] & & & \pi_M^{\infty-n}(X) \arrow[dl, "\psi_\tau"'] \arrow[ddd, "\Delta_n"] \\
& \pi^m(Y_0) \arrow[r, "\Delta"] \arrow[d, "D_k"] \arrow[r, phantom, shift left=5, "\mathrm{I}"] & \pi^{m+1}(Y) \arrow[d, "D_{k+1}"] & \\
& \Sigma_n(R^k \setminus Y_0) \arrow[r, "(i_1)_*"] \arrow[r, phantom, shift right=4, "\mathrm{V}"] \arrow[u, phantom, shift right=8, "\mathrm{II}"] & \Sigma_n(R^{k+1} \setminus Y) \arrow[u, phantom, shift left=8, "\mathrm{IV}"] \arrow[ul, phantom, "\mathrm{III}"] & \\
\Sigma_n(L \setminus X) \arrow[rrr, "i_*"] \arrow[ur, "(\tau_0)_*"] & & & \Sigma_n(M \setminus X) \arrow[ul, "\tau_*"']
\end{tikzcd}$$

We assert that all regions are commutative:

I: since τ is an extension of τ_0,

II, IV: by the definition of Δ_n,

III: by Lemma 6.5,

V: this is an elementary property of Σ_n.

Thus the exterior square is commutative.

It follows from Lemma 6.10 that Δ_n is an isomorphism of the direct sequence $\{\pi_L^{\infty-n}(X), \sigma_L^M\}$ onto the direct sequence $\{\Sigma_n(L\setminus X), \tau_*\}$. It is easy to verify that the direct limit of the first is isomorphic to $\Sigma_n(E^\infty \setminus X)$ and the direct limit of the second is isomorphic to $\pi^{\infty-n}(X)$.

Definition 6.11. Denote by

$$\Delta_n^\infty\colon \pi^{\infty-n}(X) \to \Sigma_n(E^\infty \setminus X)$$

the limit isomorphism.

Theorem 6.12. *If X is a closed and bounded subset of E^∞, then $\Delta_n^\infty\colon \pi^{\infty-n}(X) \to \Sigma_n(E^\infty\setminus X)$ is an isomorphism. If X_1, X_2 are closed and bounded subsets of E^∞, $X_1 \subset X_2$ and $i\colon X_1 \to X_2$, $j\colon (E^\infty \setminus X_2) \to (E^\infty \setminus X_1)$ are inclusions, then the following diagram is commutative:*

$$\begin{array}{ccc}
\pi^{\infty-n}(X_2) & \xrightarrow{i_*} & \pi^{\infty-n}(X_1) \\
\downarrow \Delta_n^\infty & & \downarrow \Delta_n^\infty \\
\Sigma_n(E^\infty \setminus X_2) & \xrightarrow{j_*} & \Sigma_n(E^\infty \setminus X_1).
\end{array}$$

Proof. The first part was proved above. An inspection of the proof of the first part shows that the second part is true.

§ 7. Applications

Theorem 7.1. *If (X, A) is a closed and bounded pair in E^∞ and $f_0 \in D^n(A)$, then f_0 is extendable to a compact field $f \in D^n(X)$ if and only if $[f_0] \in \operatorname{Im} i^*$, where $i\colon A \to X$ is the inclusion mapping.*

Proof. The condition is manifestly necessary. We shall prove the sufficiency. If $[f_0] \in \operatorname{Im} i^*$, then there exits a $g \in D^n(X)$ such that $f_0 \approx gi$. Thus, by Theorem 3.13, f_0 can be extended to a compact field $f \in D^n(X)$.

From the exactness of the sequence

$$\pi^{\infty-n}(X) \xrightarrow{i_*} \pi^{\infty-n}(A) \xrightarrow{\delta} \pi^{\infty-n+1}(X, A)$$

we obtain

Corollary 7.2. *If (X, A) is a closed and bounded pair in E^∞, $f_0 \in D^n(A)$, then f_0 can be extended to $f \in D^n(X)$ if and only if $\delta([f_0]) = 0$.*

In particular, if $\pi^{\infty-n+1}(X, A) \approx 0$, then each compact field $f_0 \in D^n(A)$ can be extended to a field $f \in D^n(X)$.

Theorem 7.3. *If X is a closed and bounded subset of E^∞, then $\pi^\infty(X)$ is a free abelian group and there exists a one-to-one correspondence between the generators of $\pi^\infty(X)$ and the bounded components of $E^\infty \setminus X$.*

This theorem is a slight generalization of the theorem proved by Granas in [7] (Theorem IX. 12), which may be formulated as follows: X disconnects E^∞ if and only if $\pi^\infty(X) \not\approx 0$.

Proof. By Theorem 6.12, $\pi^\infty(X)$ is isomorphic to $\Sigma_0(E^\infty \setminus X)$. By the Hurewicz theorem $\Sigma_0(E^\infty \setminus X)$ is isomorphic to $\widetilde{H}_0(E^\infty \setminus X)$—the reduced singular homology group. It is easy to verify that $\widetilde{H}_0(E^\infty \setminus X)$ is free abelian and that the generators of $\widetilde{H}_0(E^\infty \setminus X)$ are in one-to-one correspondence with the set of bounded components of $E^\infty \setminus X$.

For completeness we would like to give more details. Let $\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ be the set of all bounded components of $E^\infty \setminus X$. For any $\alpha \in \mathfrak{A}$ let us put $X_\alpha = \bigcup_{\beta\in\mathfrak{A}, \alpha\neq\beta} U_\beta$. Let $i_\alpha\colon X \to X_\alpha$, $i^\alpha\colon (E^\infty \setminus X_\alpha) \to (E^\infty \setminus X)$ be the inclusions. $\Sigma_0(E^\infty \setminus X_\alpha)$ is infinite cyclic and $i_*^\alpha\colon \Sigma_0(E^\infty \setminus X_\alpha) \to \Sigma_0(E^\infty \setminus X)$ is a monomorphism for each $\alpha \in \mathfrak{A}$ and $\{i_*^\alpha\}$ is an injective representation of $\Sigma_0(E^\infty \setminus X)$ as the direct sum of the groups $\Sigma_0(E^\infty \setminus X_\alpha)$. Thus, by Theorem 6.12, we obtain

Corollary 7.4. *For each $\alpha \in \mathfrak{A}$, $\pi^\infty(X^\alpha)$ is an infinite cyclic group and $\{i_\alpha^*\}$, $i_\alpha^*\colon \pi^\infty(X_\alpha) \to \pi^\infty(X)$ is an injective representation of $\pi^\infty(X_\alpha)$ as the direct sum of $\pi^\infty(X_\alpha)$.*

Corollary 7.5. *If X and Y are two closed and bounded subsets of E^∞, $f\colon X \to Y$ is a homeomorphism and $f \in D^n(X, Y)$, then $E^\infty \setminus X$ and $E^\infty \setminus Y$ have the same number of components.*

This is an immediate consequence of Theorems 5.9 and 7.3.

If X is a closed and bounded subset of E^∞ and $f \in D(X, E^\infty)$, U is a component of $E^\infty \setminus X$ and $x_0 \notin f(X)$, then there is defined a number $d(f, U, x_0)$—the Leray-Schauder degree of f. Since $x_0 \notin f(X)$, we have $g(x) = f(x) - x_0 \neq 0$ for $x \in X$ and thus $g \in D^n(X)$. Let $y_0 \in U_\alpha$, and $f_0(x) = x - y_0$ for $x \in X_\alpha$, $f_0 \in D^n(X_\alpha)$. Then $\pi^\infty(X_\alpha)$ is isomorphic to Z—the group of integers—and $[f_0]$ is a generator of $\pi^\infty(X_\alpha)$. Thus there exists a (uniquely determined) integer m such that $[g] = a + m i_\alpha^*([f_0]) \in \pi^\infty(X)$. It is easy to see that $m = d(f, U_\alpha, x_0)$ and it is possible to deduce some properties of $d(f, U_\alpha, x_0)$ from the algebraic properties of $\pi^\infty(X)$.

Definition 7.6. Let X, Y be closed and bounded subsets of E^∞. A mapping $d \in DH(X, E^\infty)$ will be called a *compact deformation* of X onto Y if $d_0(x) = x$ and $d_1(X) = Y$.

Theorem 7.7. *Let X, Y be closed and bounded subsets of E^∞ and d a compact deformation of X onto Y. Let W be a finite dimensional polyhedron such that $W \subset (E^\infty \setminus X) \cap (E^\infty \setminus Y)$. Denote by $i\colon W \to (E^\infty \setminus X)$ and $i_1\colon W \to (E^\infty \setminus Y)$ the inclusion mappings. If there exists an element $a \in \Sigma_n(W)$ such that $i_*(a) \neq 0$ and $(i_1)_*(a) = 0$, then there exist $x_0 \in X$ and $t_0 \in I$ such that $d(x_0, t_0) \in W$.*

Proof. Let $Z_0 = d(X \times I)$. Suppose, on the contrary, that $Z_0 \cap W = \Lambda$. Since W is a polyhedron, then there exists an open and bounded subset U of E^∞ such that $Z_0 \cap U = \Lambda$ and W is a deformation retract of U. Since Z_0 is a bounded subset of E^∞, then there exists an $r > 0$ such that $Z_0 \cup U \subset V_r = \{x \in E^\infty, \|x\| \leqslant r\}$. Let $Z = V_r \setminus U$. Denote by $j_1\colon Y \to Z$, $j\colon X \to Z$ the inclusion mappings and replace $\Sigma_n(U)$ by $\Sigma_n(W)$; then, by Theorem 6.12, we have the following commutative diagrams:

$$\begin{array}{ccc} \pi^{\infty-n}(Z) & \xrightarrow{j^*} & \pi^{\infty-n}(X) \\ \downarrow \Delta_n^\infty & & \downarrow \Delta_n^\infty \\ \Sigma_n(W) & \xrightarrow{i_*} & \Sigma_n(E^\infty \setminus X), \end{array} \qquad \begin{array}{ccc} \pi^{\infty-n}(Z) & \xrightarrow{j_1^*} & \pi^{\infty-n}(Y) \\ \downarrow \Delta_n^\infty & & \downarrow \Delta_n^\infty \\ \Sigma_n(W) & \xrightarrow{(i_1)_*} & \Sigma_n(E^\infty \setminus Y). \end{array}$$

Since Δ_n^∞ is an isomorphism, we have $\operatorname{Ker} j^* = (\Delta_n^\infty)^{-1}(\operatorname{Ker} i_*)$, $\operatorname{Ker} j_1 = (\Delta_n^\infty)^{-1}(\operatorname{Ker}(i_1)_*)$. On the other hand, $j \approx j_1 d_1$ and hence $j^* = d_1^* j_1^*$ and $\operatorname{Ker} j_1^* \subset \operatorname{Ker} j^*$, which is a contradiction.

Theorem 7.7 is a generalization of the Sweeping theorem proved in [4]. Let $d(x, t)$ be a compact deformation of a bounded and closed subset X of E^∞ onto Y. Let x_1 be a point in the bounded component of $E^\infty \setminus X$ and unbounded component of $E^\infty \setminus Y$. Let x_2 be a point in the unbounded component of $E^\infty \setminus d(X \times I)$. Since $\Sigma_0(E^\infty \setminus X)$ and $\Sigma(E^\infty \setminus Y)$ are isomorphic to the reduced singular homology groups, the assumptions of Theorem 7.7 are fulfilled. Thus there exist points $x_0 \in X$ and $t_0 \in I$ such that $d(x_0, t_0) = x_1$.

Corollary 7.8. *Let d be a compact deformation of a closed and bounded subset X onto Y. Let $\varphi\colon S^n \to (E^\infty \setminus X) \cap (E^\infty \setminus Y)$ be a mapping; denote by $\eta\colon H_n(S^n) \to H_n(E^\infty \setminus X)$ and $\zeta\colon H_n(S^n) \to H_n(E^\infty \setminus Y)$ the homomorphisms induced by φ. If there exists an $a \in H_n(S^n)$ such that $\eta(a) \neq 0$ and $\zeta(a) = 0$, then there exist $x_0 \in X$ and $t_0 \in I$ such that $d(x_0, t_0) \in \varphi(S^n)$.*

It is sufficient to observe that $\Sigma_n(S^n) \approx H_n(S^n)$ (by the Hurewicz theorem) and the Corollary is an immediate consequence of Theorem 7.7.

Let us put

$$S^{\infty-p} = \{x \in E^{\infty-p}, \|x\| = 1\}.$$

Theorem 7.9. $\pi^{\infty-n}(S^{\infty-p}) \approx \Sigma^m(S^{m+n-p})$.

Proof. If L is an n-admissible subspace of E^∞, $\dim L = n+m+1$, then $\pi_L^{\infty-n}(S^{\infty-p})$ is isomorphic to $\Sigma^m(S^{n+m-p})$. Thus $\pi_L^{\infty-n}(S^{\infty-p})$ is a direct sequence in which all homomorphisms are isomorphisms.

Since $\Sigma_q(S^m) \approx \Sigma^m(S^q)$ (see [7], p. 225), we have $\pi^{\infty-n}(S^{\infty-p}) \approx \Sigma_{n+m-p}(S^m)$. Thus, since some stable homotopy groups of spheres are known (see [7], p. 332), we obtain

Corollary 7.10. *We have*

$$\begin{array}{lll}
\pi^{\infty-n}(S^{\infty-p}) \approx 0 & \textit{for} & p > n, \\
\pi^{\infty-n}(S^{\infty-p}) \approx Z & \textit{for} & p = n, \\
\pi^{\infty-n}(S^{\infty-p}) \approx Z_2 & \textit{for} & n-p = 1, \\
\pi^{\infty-n}(S^{\infty-p}) \approx Z_2 & \textit{for} & n-p = 2, \\
\pi^{\infty-n}(S^{\infty-p}) \approx Z_{24} & \textit{for} & n-p = 3, \\
\pi^{\infty-n}(S^{\infty-p}) \approx 0 & \textit{for} & n-p = 4 \textit{ and } n-p = 5, \\
\pi^{\infty-n}(S^{\infty-p}) \approx Z_2 & \textit{for} & n-p = 6, \\
\pi^{\infty-n}(S^{\infty-p}) \approx Z_{240} & \textit{for} & n-p = 7, \\
\pi^{\infty-n}(S^{\infty-p}) \approx Z_2 + Z_2 & \textit{for} & n-p = 8
\end{array}$$

(here Z is the infinite cyclic group and Z_q the cyclic group of order q).

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