2. Fall. $D_1 \cap B_2$ ist nicht zusammenhängend. Dann besteht dieser Durchschnitt aus einer Kante von D_1 und dem dieser Kante gegenüberliegenden Eckpunkt p. L sei der kleinste Teilbogen von B_2, der K und p enthält. Die Endpunkte von L sind p und ein Endpunkt q von K, und die Strecke $[p, q]$ ist eine Kante von D_1. Wir betrachten nun den Bogen $B = (B_2 \setminus L) \cup [p, q]$. Es ist nicht schwer nachzuweisen, daß B zu $\mathfrak{B}_2$ gehört. Um zu zeigen, daß B die Röhre T_0 durchdringt, bemerken wir folgendes: B kann höchsten soviele Teilbogen besitzen, die eine von T_0 verschiedene Röhre eines zu $\mathfrak{X}_{n_0-1}$ gehörenden Würfels durchlaufen, wie B_2 selbst. Da jedoch $r(B_2) = r_0$ war und $r(B) \geqslant r_0$ sein muß, folgt hieraus unmittelbar, daß B ebensoviele Teilbogen enthält, die T_0 durchlaufen, wie B_2 selbst. Da natürlich $d(B) < d(B_2)$ ist, ist der Beweis hiermit beendet.

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On a class of order-types generalizing ordinals

by

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1. Introduction. The class Ω of ordinals T may be characterized by the following properties:

(i) The right half of every Dedekind cut of T has a first term,

(ii) T has a first term.

It does not appear that anyone has investigated those order types for which (i) holds but (ii) does not. In this paper normal forms are given for such types, their possible factorizations are investigated, and various properties generalizing those of ordinals are given.

2. Definitions and notation. Let $T = (T, <)$ be a linearly ordered set. We define a Dedekind cut of T to be a pair (A, B) such that $A \neq \emptyset \neq B$, $A \cup B = T$, and for $a \in A$, $b \in B$, $a < b$. We write $T = A + B$. We say a *cut* (A, B) *is of type* J, G, L or R according as both, neither, only the first or only the second of the following conditions holds:

(i) A has a last term;

(ii) B has a first term.

For each $s \in T$ we define $L(s)$ to be the set of predecessors of s in T, and $L[s] = L(s) \cup \{s\}$. Similarly for $R(s)$ and $R[s]$. A subset A of T is an *initial* (*final*) *segment* of T iff $A = L(t)$ ($A = R[t]$) for some $t \in T$.

Ordinals will be denoted by Greek letters. If

$$\alpha = \omega^{a_r} a_r + \ldots + \omega^{a_0} a_0$$

in Cantor normal form (see [3], p. 320), we shall typically write $\alpha = (\alpha_i, a_i, r)$. We define functions φ and ψ on Ω by setting

$$\varphi 0 = -\infty, \qquad \psi 0 = -\infty,$$
$$\varphi\alpha = \alpha_r \quad (\alpha > 0), \qquad \psi\alpha = \alpha_0 \quad (\alpha > 0).$$

We shall use the following valuation-like properties of φ:

(i) $\varphi(\alpha + \beta) = \max(\varphi\alpha, \varphi\beta)$.

(ii) $\varphi(\alpha\beta) = \varphi\alpha + \varphi\beta$.

(iii) $\alpha + \beta = \beta$ iff $\varphi\alpha < \varphi\beta$.

Expressions such as $\beta = \omega^{\delta} d + \varepsilon$ will be assumed in normal form, i.e. $\varphi\varepsilon < \delta$, $d > 0$.

We will not usually distinguish between an ordered set and its order type.

3. RJ **sets.** We define an ordered set to be RJ iff every cut is of type R or J.

Let us say S is *coinitial* in T iff for every $t \in T$ we may find $s \in S$ such that $s \leqslant t$. Then

THEOREM 1. *The following conditions on* T *are equivalent*:

a) T *is* RJ.

b) *Every final segment of* T *is an ordinal.*

c) *Any subset of* T *is an ordinal or coinitial in* T.

d) *Every proper subset of* T *is* RJ.

e) *If* $s_0 > s_1 > \ldots$ *is a properly descending infinite sequence in* T, *then* $S = \{s_i\}$ *is coinitial in* T.

f) *If* R *is a subset of* T *such that*

(i) R *contains an initial segment of* T,

(ii) $t \in R$ *whenever* $L(t) \subseteq R$,

then $R = T$.

This last is a generalization of the principle of induction. Compare [3], p. 262.

Proof. a→b. Let S be a non-empty subset of a final segment $R[s_0]$ of T. We show that S has a first term. If $s_0 \in S$ this is trivial. If not, set $B = \bigcup \{R[s]: s \in S\}$, $A = T - B$. Then (A, B) is a cut of T. So B has a first term, which is easily seen to be the required first term of S.

b→c. If A is not coinitial in T, then $A \subseteq R[t]$ for some t. Thus A is a subset of an ordinal, so itself an ordinal.

c→d. If A is a proper subset of T, and (B, C) a cut in A, then C is not coinitial in T; hence has a first term.

d→e. If $t < s_i$ for all i, then $\{t\} \cup \{s_{2i}: i = 1, 2, \ldots\}$ is a proper subset of T, and of type $1 + \omega^*$, which is not RJ.

e→f. Set $S = T - R$. Then S has no first term a, since then $L(a) \subseteq R$, contradicting (ii). Thus if S is non-empty we may find an infinite descending chain in it, which is then coinitial in T. So the initial segment of T in R is empty. But then T has a first term t, contradicting (ii). So S is empty.

f→a. Let (A, B) be a cut of T, and suppose B has no first term. Then A satisfies (i) and (ii), yet $A \neq R$.

COROLLARY. *An ordered set is an ordinal iff it is* RJ *and has a first term.*

4. Normal forms for RJ **types.** We shall use the notation $\sum (\lambda_n, l_n)$ or $\sum\limits_{\omega}^{0} (\lambda_n, l_n)$ for

$$\sum_{n \in \omega^*} \omega^{\lambda_n} l_n = \ldots + \omega^{\lambda_2} l_2 + \omega^{\lambda_1} l_1 + \omega^{\lambda_0} l_0 ,$$

where $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$ and each $l_n > 0$. The letter λ will always denote $\lim\limits_n \lambda_n$.

THEOREM 2. *An ordered set is* RJ *iff it is of one of the following forms*:

0. α.

1. $\omega^{\mu}\omega^* + \alpha$: $\varphi\alpha < \mu$.

2. $\Sigma(\lambda_n, l_n)$.

3. $\omega^{\mu}\omega^* + \alpha$: $\varphi\alpha > \mu$.

4. $\Sigma(\lambda_n, l_n) + \alpha$: $\varphi\alpha \geqslant \lambda$.

Proof. It is clear that an ordered set of any of these forms is RJ. Conversely, define a function f on an initial segment of ω inductively by setting $f(0) = s_0$, where $s_0 \in T$ is arbitrary; $f(1) = s_1$, where $s_1 < s_0$ and

$$\beta_1 = \varphi[s_1, s_0) \geqslant \varphi R[s_0] = \beta_0 ,$$

if such an s_1 exists, and in general $f(n) = s_n$, where

$$\beta_n = \varphi[s_n, s_{n-1}) \geqslant \beta_{n-1}$$

provided such an s_n exists. If not, we terminate the construction, and have $\mathfrak{D}f = n$.

Suppose first f is defined on all ω. Then by Theorem 1e we have

$$T = \sum_{n \in \omega^*} [S_{n+1}, S_n) + R[S_0]$$

$$(\mathrm{A}) \qquad = \sum_{n \in \omega^*} (\omega^{\beta_n} b_n + \gamma_n): \ \beta_0 \leqslant \beta_1 \leqslant \ldots$$

Suppose $B = \{\beta_n\}$ is ultimately constant. Let N be the first index beyond which B is constant. Then for some μ

$$[S_{n+1}, S_n] = \omega^{\mu} b_n + \gamma_n \quad \text{for} \quad n \geqslant N; \quad \beta_{N-1} < \mu .$$

Thus

$$T = \sum_{\omega}^{N} (\omega^{\mu} b_n + \gamma_n) + \sum_{N-1}^{0} (\omega^{\mu_n} b_n + \gamma_n)$$

$$= \sum_{\omega}^{N} (\omega^{\mu} b_n) + a = \omega^{\mu}\omega^* + a$$

for some a such that $\varphi a = \max(\varphi\gamma_N, \beta_{N-1}) < \mu$. So in this case T may be written in the form 1.

If B contains a properly increasing subsequence we amalgamate terms in (A) to reach an expression

$$T = \sum_{n\in\omega^*} (\omega^{\mu_n} m_n + \delta_n)\colon \varphi\delta_n < \mu_n < \mu_{n+1}\,.$$

If $\delta_n = \varkappa_n + \varepsilon_n$: $\varphi\varepsilon_n < \mu_{n-1} \leqslant \psi\varkappa_n$ $(n > 0)$, we may replace each δ_n by $\varkappa_n$ $(n > 0)$ without changing the sum:

$$T = \sum_{n\in\omega^*} (\omega^{\mu_n} m_n + \varkappa_n) + \delta_0\,.$$

On expanding each term $\omega^{\mu_n} m_n + \varkappa_n$ and δ_0 into its own normal form, we find T is expressible in the form 2.

Suppose next f is defined not on all ω, but only on $N+1$. We define a function ξ on a subset of ω inductively by setting

$$T_0 = T,\quad f_0 = f,\quad N_0 = N;\quad t_0 = f(N),\quad \xi(0) = \varphi R[t_0]\,.$$

Suppose we have defined $T_i, f_i, N_i, t_i, \xi(i)$ for all $i < m$, where $m \geqslant 1$. Then we set

$$T_m = T_{m-1} - R[t_{m-1}]\,.$$

We may assume T is not an ordinal, so that T_m is non-empty. We define f_m for T_m exactly as we defined f for T. If $\mathfrak{D}f_m = \omega$ we do not define $\xi(m)$, and set $\mathfrak{D}\xi = m$. If $\mathfrak{D}f_m$ is finite, we set

$$\mathfrak{D}f_m = N_m + 1\,,\quad t_m = f_m(N_m)\,,\quad \xi(m) = \varphi[t_m, t_{m-1})\,.$$

By the definition of f_m, we have $\xi(0) > \xi(1) > \ldots$ So we have constructed a descending sequence of ordinals, defined on $\mathfrak{D}\xi$. Thus $\mathfrak{D}\xi = p$ is finite, and f_p must be defined on all ω. So by what we have already proved

$$T_p = \omega^\mu\omega^* + \beta \quad\text{ or }\quad \sum(\lambda_n, l_n)$$

and $T = \omega^\mu\omega^* + \alpha$ where $\alpha = \beta + R[t_p]$, or $T = \sum(\lambda_n, l_n) + \alpha'$ where $\alpha' = R[t_p]$. Suppose in the second case $\varphi\alpha' < \lambda$. Then $\varphi\alpha' < \lambda_m$ for some m, since λ is of the second kind. But then we can extend the definition of f_p to N_p+2 by setting $f_p(N_p+1)$ to be any point to the left of the final segment $\sum\limits_m^0 (\lambda_n, l_n) + \alpha'$ of T. Thus $\varphi\alpha' \geqslant \lambda$ and in the second case T is of the form 4. A similar argument shows that in the first case $\varphi\alpha > \mu$, so that T is of the form 3.

THEOREM 3. *No RJ type is of more than one of the five forms.*

Proof. Given any properly descending infinite sequence S: $s_0 > s_1 > \ldots$ in T, we define functions RS and φS on ω by setting $RS(i) = R[s_i]$, and $\varphi S(i) = \varphi R[s_i]$.

Consider the assertions A: RS is ultimately constant; B: φS is ultimately constant. If T is of type 1 (2) it satisfies A for no S; B for all (no) S; if T is of type 3 or 4 it satisfies A for all S; B for all S. Furthermore, if T is of type 3 it has an initial segment of type 1, and if T is of type 4 every initial segment is of type 2 or 4. There are no S iff T is an ordinal; otherwise these properties are mutually exclusive, so that the theorem is proved.

THEOREM 4. *The expression for an RJ type T of type* 1, 2, 3 *is unique. If T is of type* 4, *its expression is unique modulo any final segment of* $\sum(\lambda_n, l_n)$.

Proof. Suppose one expression for T is 1, 2, 3 or 4, as in Theorem 2. Let S be a given infinite descending sequence in T.

1. $RS(m)$ is for sufficiently large m of the form $\omega^\mu r + \alpha$. This determines α and μ.

2. Given any n we may find an m such that $\varphi S(m) > \lambda_n$. Then $RS(m) = \gamma + \sum\limits_n^0 (\lambda_n, l_r)$ in normal form. Since normal form is unique, this deterimnes λ_r and l_r for $r \leqslant n$.

3. We may find an N beyond which RS is constant and LS of type 1. The constant must be α and the initial segments of the form $\omega^\mu\omega^* + \delta$. This determines α and (by 1) μ.

4. We determine α as in 3. Let s be such that $R[s] = \alpha$ and $L(s) = \sum(\lambda_n, l_n)$. If also $T = \sum(\lambda'_n, l'_n) + \alpha$, we must have $L(s) = \sum\limits_\omega^n (\lambda'_n, l'_n) + \beta$ (for some m and β) of type 2. On expanding β in normal form and using 2, we see that for some $r \geqslant 0$ and all $n >$ some N,

$$\lambda'_n = \lambda_{n+r}\,,\quad l'_n = l_{n+r}\,,\quad\text{ or }\quad \lambda_n = \lambda'_{n+r}\,,\quad l_n = l'_{n+r}\,.$$

Since any final segment of $\sum(\lambda_n, l_n)$ may be absorbed in α, it is clear that this is best possible.

5. Order-preserving maps. It is well known that an order-preserving map of an ordinal onto itself must be the identity. For RJ sets we have

THEOREM 5. *Let A be an ordered set whose order type T is RJ. Then the group $G(A)$ of order-preserving maps of A onto itself is the trivial group unless T is of type* 3, *in which case it is the infinite cyclic group.*

Proof. 1. Let C be RJ, $\delta \neq 0$. Then $C + \delta = C$ iff $C = \omega^\mu\omega^* + \alpha$ is of type 1, and $\delta = \omega^\mu r + \alpha$ for some $r > 0$.

2. Suppose now f is an order-preserving map of A onto itself, other than the identity. We may assume without loss of generality that $f(b) = a < b$ for some b.

Then $L(a)$ and $L(b)$ have the same RJ order type. But $L(b) = L(a) + [a, b)$, so, by 1, is of type 1. Thus T must be of type 1 or 3.

3. Let us identify each $a \in A$ with the cut $(L(a), R[a])$. On identifying A with its order type $\omega^\mu\omega^* + \alpha$ we may write the cuts unambiguously in the symbolic form

$$\text{(i)} \quad a = \big(\omega^\mu(\omega^* - r) + \beta,\ -\beta + \omega^\mu r + \alpha\big) \qquad (\varphi\beta < \mu,\ r \geqslant 0;\ \beta < \alpha \text{ if } r = 0)$$

or

$$\text{(ii)} \quad a = \big(\omega^\mu(\omega^* + r) + \beta,\ -(\omega^\mu r + \beta) + \alpha\big) \qquad (\varphi\beta < \mu,\ r \geqslant 1,\ T \text{ of type } 3).$$

4. $T \neq \omega^\mu\omega^*$. For if $b = \big(\omega^\mu(\omega^* - n) + \delta,\ -\delta + \omega^\mu n\big)$, $\varphi\delta < \mu$, then $a = f(b)$ is of type (i), where we must have $r = n$; $\beta = \delta$. So $a = b$.

5. f is determined by its effect on $c = (\omega^\mu\omega^*, \alpha)$. For if $f(c) = a$, then $R[a]$ is of type α, and f on $R[a]$ is determined by the properties of ordinals quoted above, while $L(a)$ is of type $\omega^\mu\omega^*$, so that f on $L(a)$ is determined by 4.

6. Suppose now $f(c) = a$ given by (i) or (ii) above. Then $\beta = 0$ and $\varphi\alpha > \mu$, so that T is of type 3. The mapping $f \to r$ (if a is of type (ii)) or $f \to -r$ (if a is of type (i)) sets up an isomorphism between $G(A)$ and the infinite cyclic group.

This isomorphism is brought out graphically if we write

$$T = \omega^\mu\omega^* + \alpha = \omega^\mu(\omega^* + \omega) + (-\omega^{\mu+1} + \alpha).$$

COROLLARY. *An ordered group H whose order type T is RJ is trivial or the infinite cyclic group.*

For its regular representation is a subgroup of $G(T)$.

In contrast to Theorem 5 we have

THEOREM 6. *If T and T' are RJ, there exists an order-preserving map f of T into T' iff the appropriate condition in the diagram is satisfied:*

<table>
<tr><td>T' of type</td><td></td><td>12</td><td>34</td><td>0</td></tr>
<tr><td>T of type</td><td>12</td><td colspan="2">$\lambda \leqslant \lambda'$</td><td rowspan="2">*</td></tr>
<tr><td></td><td>34</td><td rowspan="2">$\varphi\alpha < \lambda'$</td><td>$\begin{cases} \alpha \leqslant \alpha' \\ \lambda \leqslant \lambda' \end{cases}$</td></tr>
<tr><td></td><td>0</td><td colspan="2">$\alpha \leqslant \alpha'$</td></tr>
</table>

Here we take $\lambda = \mu + 1$ if T is of type 1 or 3, and $T = \alpha$ if T is of type 0.

If $T' = T$ it is easy to construct $f \neq \text{id.}$ so that $f(t) \leqslant t$ for all $t \in T$ iff T is not of type 0, and $g \neq \text{id.}$ so that $g(t) \geqslant t$ for all $t \in T$ iff T is not finite or ω^* (compare [2]).

6. Factorization of RJ sets. We shall find it convenient to reclassify RJ sets T in seven types, and associate with each T ordinals α and λ as follows:

(B)

Group I: $T = \omega^\mu\omega^* + \alpha$: $\lambda = \mu + 1$.

type:
1. $\varphi\alpha < \mu$.
2. $\psi\alpha < \lambda \leqslant \varphi\alpha$.
3. $\lambda \leqslant \psi\alpha$.

Group II: $T = \sum(\lambda_n, l_n) + \alpha$: $\lambda = \lim\limits_n \lambda_n$.

type:
4. $\alpha = 0$.
5. $\psi\alpha < \lambda \leqslant \varphi\alpha$.
6. $\lambda \leqslant \psi\alpha$.

If T is an ordinal, we assign it to type 0, and take

$$\alpha = T, \quad \lambda = \varphi T.$$

It is desirable to have $\alpha(T) \neq 0$. We therefore redefine $\alpha(T)$ to be ω^{λ_0} (T of type 4) and ω^μ (T of type 1, $\alpha(T) = 0$). We shall say T is RJi iff T is RJ and of type i ($i = 0, \ldots, 6$). We may regard types 1, 2, 3 as degenerate cases of types 4, 5, 6, respectively.

THEOREM 7. *Every RJ set is of one and only one of the seven types. Its expression in the form* (B) *is unique (modulo any final segment of Σ if T is of type* 5 *or* 6). *In particular, α and λ are well-defined functions of T.*

This follows from Theorems 2, 3, 4.

THEOREM 8. *Let $T = PQ$ be a non-trivial factorization of an ordered set $T \neq \emptyset$. Then T is RJi iff P is an ordinal and Q is RJi ($i = 0, \ldots, 6$).*

Proof. In one direction this needs only a direct verification. Conversely, let P and Q be ordered sets. Then the Dedekind cuts of PQ are precisely the pairs (PA, PB) where (A, B) is a cut of Q, and the pairs $\big(PL(a) + C,\ D + PR(a)\big)$, where $a \in Q$ and (C, D) is a cut of P.

Suppose now $T = PQ$ is RJ. If (C, D) is a cut of type L or G in P, then so is $\big(PL(a) + C,\ D + PR(a)\big)$ in T. Such cuts exist since $P \neq 0, 1$; $Q \neq 0$. Thus P is RJ. If P has no first term, then every cut (PA, PB) in T is of type L or G. Such cuts exist, since $P \neq 0$, $Q \neq 0, 1$. So P is an ordinal.

If the cut (A, B) of Q is of type G, then so is (PA, PB). If (A, B) is of type L, then (PA, PB) is of type L or G. So Q is RJ. That Q is of the same type as T follows from Theorem 7 and the first part of this theorem.

6.1. We first examine the factorization of $RJ1, 2, 3, 5$ types.

THEOREM 9. *Let T of type 1 or 2 or 3 or 5 be given by*

$$\omega^\mu\omega^*+\alpha, \quad \omega^\mu\omega^*+\delta+\beta,$$
$$\omega^\mu\omega^*+\alpha, \quad \sum(\lambda_n, l_n)+\delta+\beta,$$

respectively, where $0 \leqslant \varphi\beta < \lambda \leqslant \psi\delta$, *and* $\delta = (\delta_i, d_i, r)$; $\alpha = (a_i, a_i, s)$. *Then T has a unique irreducible right factor S, given by*

$$\omega^*, \quad \omega^{-\varphi\beta+\mu}\omega^*+\delta'+1,$$
$$\omega^*+\alpha', \quad \sum(-\varphi\beta+\lambda_n, l_n)+\delta'+1,$$

respectively, where $\delta' = (-\varphi\beta+\delta_i, d_i, r)$: $\alpha' = (-\mu+a_i, a_i, s)$. *We have $T = \gamma S$ iff*

$$\gamma = \omega^\mu m+\alpha \quad (\text{for some } m > 0), \quad \gamma = \beta,$$
$$\varphi\gamma = \mu, \quad \gamma = \beta, \ \textit{respectively}.$$

COROLLARY. *The irreducible $RJ1, 2, 3, 5$ sets are precisely those of the forms*

$$\omega^*, \quad \omega^\mu(\omega^*+\omega\sigma)+1,$$
$$\omega^*+\omega\sigma, \quad \sum(\lambda_n, l_n)+\omega^\lambda\sigma+1,$$

respectively, where $\sigma > 0$.

Proof. If T is of type 5 we assume, as we may, that $\lambda_0 > \varphi\beta$. We may check directly that $T = \gamma S$ whenever γ and S are as given, and that S is of the form given in the corollary. If $T = \varepsilon S'$ with S' irreducible, then S' is of the same type as T, so has a right factor of the form given in the corollary, so is itself of this form. We may then check directly that $S' = S$ and ε is of the form given for γ. The same argument applied to S shows that S is irreducible.

6.2. $RJ4$ and 6 types do not in general have irreducible right factors. We therefore introduce two weaker notions:

DEFINITION. We say an RJ type is *reduced* iff for every right factor S of T we have $\alpha(S) = \alpha(T)$ and $\lambda(S) = \lambda(T)$. We say that T is (right) *quasi-irreducible* (q. i.) iff every right factor S of T is at the same time a right multiple.

If S is a right factor of T, $\alpha(S) \leqslant \alpha(T)$ and $\lambda(S) \leqslant \lambda(T)$. It follows that T is reduced if it is q. i.

We shall need two other notions:

DEFINITION. We define ordinal functions τ and θ of T by setting $\lambda(T) = \tau+\omega^\theta$ in normal form (i.e. $\tau = 0$ or $\psi\tau \geqslant \theta$). Note that for $RJ4$ and 6 types $\theta \neq 0$, since λ is of the second kind.

DEFINITION. We say that an increasing sequence $\Lambda = \{\lambda(n)\}$ of ordinals is *left step-periodic* iff for some *step* α and *period* $p \neq 0$ we have $\lambda(n+p) = \alpha+\lambda(n)$ for all *sufficiently large* n. It may be shown that the set of periods is a semi-ideal in the set of positive integers, so that Λ has a fundamental period. We say Λ is *step-periodic* iff it is both left and right step-periodic (defined in the corresponding manner).

6.3. Suppose now $T = \sum(\lambda_n, l_n)$ is of type 4. We say β is an *end-segment* of T or of $\varrho = (\lambda_i, l_i, r)$ iff for some e

$$\beta = \omega^{\lambda_N}d+(\lambda_i, l_i, N-1), \quad de = l_N. \tag{1}$$

LEMMA 1. *The left factors of ϱ or T are precisely those β such that either $\varphi\beta < \lambda_0$ or β is an end-segment of ϱ or T.*

Proof. If ϱ or $T = \beta U$ and U is of the second kind, then $\varphi\beta < \lambda_0$ and for every such β we have $\beta U = \varrho$ or T, where

$$U = \sum^{0}(-\varphi\beta+\lambda_n, l_n) \tag{2}$$

(the sum being finite or infinite).

If U is of the first kind, and $U = \sum^{0}(\mu_n, m_n)$, $\beta = \omega^\delta d+\varepsilon$, then

$$\varrho \text{ or } T = \sum^{1}(\delta+\mu_n, m_n)+\omega^\delta dm_0+\varepsilon.$$

Since $\delta < \delta+\mu_1$, this is in normal form as it stands, so that β satisfies (1) for $e = m_0$ and some N. Also $\varphi\beta = \lambda_N \geqslant \lambda_0$. Conversely, with β as in (1), we have $\beta U = T$, where

$$U = \sum^{N+1}(-\lambda_N+\lambda_n, l_n)+e, \quad de = l_N. \tag{3}$$

This lemma suggests that $RJ4$ types may be regarded as "infinitely long" ordinals, and that by continuing to "peel off" right factors we can obtain an "infinite" normal factorization for them parallel with the normal factorization (n. f.) of ordinals (see [3], p. 340, or [1], p. 88).

DEFINITION. Let $\Pi = \{\pi_1, \pi_2, \ldots\}$ be a sequence of non-zero ordinals and T of type 4. Then we write $T \sim \Pi$ iff every final segment γ of T is a final segment of every product $\pi_1\pi_2\ldots\pi_n$ for $n \geqslant$ some $N = N(\gamma)$.

Given ordinals $\gamma = (a_i, c_i, r)$ and $\delta = (\delta_i, d_i, s)$, we say γ is *strictly longer* than δ iff $r > s$ and $c_i = d_i$ and $\gamma_i = \delta_i$ for $0 \leqslant i \leqslant s$.

LEMMA 2. *Given $\alpha, \beta > 0$, α is an end-segment of $\alpha\beta$ iff β is of the first kind, and $\alpha\beta$ is strictly longer than α iff β is infinite of the first kind.*

LEMMA 3. *Given a sequence Π there exists a $T \sim \Pi$ iff*

a) *The π_i are not ultimately all finite;*

b) *only finitely many π_i are of the second kind.*

If T exists, it is unique.

Proof. The first part follows from Lemma 2, and the second from the fact that T is determined by any coinitial sequence of end-segments.

LEMMA 4. *If $T \sim \Pi$, then every product $\pi(m) = \pi_1\pi_2 \dots \pi_m$ is an end-segment of T iff*

c) *π_i $(i > 1)$ is of the first kind.*

Proof. Let σ be an end-segment of T of length greater than length $\pi(m)$. Suppose σ is a final segment of $\pi(n)$, where $n > m$. Then $\pi(m)$ is an end-segment of $\pi(n) = \pi(m)\pi_{m+1}\dots\pi_n$ by an iteration of Lemma 2, hence of σ, hence of T. The converse follows from Lemma 2.

Suppose now Π satisfies

d) π_i $(i > 1)$ is either finite >1 or infinite of the first kind and irreducible;

e) π_1 satisfies d or is of the form ω^α $(\alpha > 0)$;

f) No two adjacent π_i are both finite.

Then we say T (or ϱ) has the *amalgamated normal factorization* (anf) $\pi_1\pi_2 \dots$ (or $\pi_1\pi_2 \dots \pi_n$) iff $T \sim \Pi$ $(\varrho = \pi_1\pi_2 \dots \pi_n)$. Note that conditions a, b, c are satisfied.

LEMMA 5. *Every $\varrho = (\lambda_i, l_i, N)$ has the unique anf*

$$\varrho = \omega^{\lambda_0} l_0 \tau_1 l_1 \tau_2 l_2 \dots \tau_N l_N\,, \tag{4}$$

where $\tau_i = \omega^{\varkappa(i)}+1$: $\varkappa(i) = -\lambda_{i-1}+\lambda_i$, and we omit l_i if $l_i = 1$, and ω^{λ_0} if $\lambda_0 = 0$.

Proof. It is clear that this is an anf. It may be obtained from the n. f. by amalgamating adjacent factors which are both finite or both of the second kind. Suppose now $\varrho = \pi_1\pi_2\dots\pi_s$ is any anf. On refining each π_i to its own n. f. we clearly obtain an n. f. for ϱ. On reamalgamating, we return to $\pi_1\pi_2\dots\pi_s$, but also we must obtain (4), since the n. f. is unique. Thus the two anfs are in fact the same.

THEOREM 10. (i) *Every RJ4 type T has a unique anf Π.*

(ii) *If $T = \beta U$ is any factorization of T, then for some m*

$$\beta = \pi_1\pi_2\dots\pi_{m-1}\sigma \text{ in anf}; \quad U \sim \tau\pi_{m+1}\pi_{m+1}\dots \text{ in anf}; \quad \pi_m = \sigma\tau\,.$$

(iii) *If $T \sim \pi_1\pi_2\dots$ (not necessarily anf), then $\beta T \sim \beta\pi_1\pi_2\dots$*

Proof. (i) Consider

$$\omega^{\lambda_0} l_0 \tau_1 l_1 \tau_2 l_2 \dots = \pi_1\pi_2\dots\,, \tag{5}$$

where $\tau_n = \omega^{k(n)}+1$: $k(n) = -\lambda_{n-1}+\lambda_n$, and we omit ω^{λ_0} if $\lambda_0 = 0$, and l_n if $l_n = 1$. By Lemma 3 some $T_0 \sim$ this, so by Lemma 4 every product $\pi(n)$ is an end-segment of T_0. But it is clear from equation (4) that $\pi(n)$ is an end-segment of T. Thus T and T_0 coincide on a coinitial sequence of end-segments, so are equal. It is clear that (5) is an anf for T.

If $T \sim \sigma_1\sigma_2\dots$ is any anf for T, then $\sigma(r) = \sigma_1\sigma_2\dots\sigma_r$ defines $\sigma(r)$ by its anf. By Lemma 4, $\sigma(r)$ is an end-segment of T, so of $\pi(n)$ for some $n = n(r)$. By an easy application of equations (1) and (4), $\sigma(r)$ has the anf $\pi_1\pi_2\dots\pi_{m-1}d$ $(d|\pi_m$ or $=1)$. By Lemma 5, $m = r$ or $r+1$, and $\pi_i = \sigma_i$ $(i \leqslant r)$. Since r is arbitrary, the given anf is unique.

(ii) If $T = \beta U$, then by Lemma 1 either $\varphi\beta < \lambda_0$ or β is an end-segment. If $\varphi\beta < \lambda_0$ we may take $m = 1$, $\sigma = \beta$, $\tau = \omega^\mu$, where $\mu = -\varphi\beta + \lambda_0$. The corresponding form for U is (2), and its anf by part (i) is

$$\tau l_0 \tau_1' l_1 \tau_2' l_2 \dots$$

where $\varkappa'(n) = -(-\varphi\beta+\lambda_{n-1})+(-\varphi\beta+\lambda_n) = -\lambda_{n-1}+\lambda_n = \varkappa(n)$. If β is an end-segment it is given by (1) and has the anf (4), with d for l_N. We take $\tau = e$. The corresponding form for U is given by (3), and its anf by part (i) is $e\tau_{N+1}' l_{N+1}\tau_{N+2}'\dots$, where $\varkappa'(n) = -(-\lambda_N+\lambda_{n-1})+(-\lambda_N+\lambda_n) = -\lambda_{n-1}+ +\lambda_n = \varkappa(n)$. Thus in each case β and U are of the form stated.

(iii) It is clear from the working of Lemma 1 that an end-segment of βT is either an end-segment of β or of the form $\beta\pi$, where π is an end-segment of T. The result follows.

THEOREM 11. *Each RJ4 type T has a unique normal factorization: that is, there is a unique Π such that $T \sim \Pi$ and*

g) *each π_i is irreducible;*

h) *no π_i of the second kind follows one of the first kind;*

k) *adjacent π_i which are both of the second kind, or both finite, occur in non-increasing order.*

Proof. Note first that it is legitimate in an expression

$$T \sim \pi(1)\pi(2)\dots$$

both to refine individual terms—thus $T \sim \Sigma$, where

$$\pi(i) = \sigma(n_i+1)\sigma(n_i+2)\dots\sigma(n_{i+1})\text{—}$$

and to amalgamate blocks of adjacent terms—thus $T \sim Z$ where

$$\zeta(i) = \pi(n_i+1)\pi(n_i+2)\dots\pi(n_{i+1})\,.$$

We may obtain one n. f. for T by refining the terms of the anf to their own n. f. s. Unicity is proved by the reverse of the argument used in Lemma 5.

In terms of the type of factorization introduced here for RJ 4 types, and also of their normal form, we may regard them as the natural generalization of ordinals, and take the irreducible RJ4 types to be the irreducible ordinals, and these alone. We now consider ordinary factorization for RJ 4 types.

6.4. LEMMA 6. *An RJ4 type T is reduced iff $\tau(T) = 0 = \alpha(T)$.*

Proof. We may choose a β given by (1) such that $\lambda_N > \tau$. Then $\alpha(U) = 0$, and $\lambda(U) = \lim(-\lambda_N+\lambda_n) = -\lambda_N+\tau+\omega^\theta = \omega^\theta = \lambda$ iff $\tau = 0$. If $\alpha(T) = 0$, then T is of the first kind, and so is any right factor U so that $\alpha(U) = 0$.

LEMMA 7. *T of type 4 is q. i. iff for some β we have $T\sim\beta\beta\beta\ldots$, and T is irreducible iff we may take β irreducible.*

Proof. If T is q. i. it is reduced, so of the first kind. Let $\varrho > 1$ be a left factor. Then $T = \varrho T'$, $T' = \sigma T$, and $T = \beta T$ where $\beta > 1$ is of the first kind. If $\beta = \pi_1\pi_2\ldots\pi_n$ in anf, then an easy argument from Theorem 10 (iii) shows that T has anf $\pi_1\pi_2\ldots\pi_n\pi_1\pi_2\ldots\pi_n\pi_1\ldots$ (with π_n and π_1 amalgamated if they are both finite). Thus $T\sim\beta\beta\beta\ldots$ Conversely, if $T\sim\beta\beta\beta\ldots$ and U is a right factor, then, by Theorem 10 (ii), $U\sim\tau\beta\beta\beta\ldots$ for some right factor τ of β, and then $\tau T = U$ by Theorem 10 (iii), so that T is q. i.

If T is irreducible it follows from Theorem 10 (ii) that every right factor τ of β is also a left factor. Lemmas 1 and 5 applied to the anf for β then show that β is the n-th power of an irreducible ordinal.

Knowing the structure of reduced, q. i. and irreducible RJ 4 types, and applying Lemma 1 and Theorem 10 (iii), we have

THEOREM 12. *If $T = \sum(\lambda_n, l_n)$ is of type 4, exactly one of* (i), (ii), (iii) *holds*:

(i) *The following equivalent conditions are satisfied*:

a) *T has a unique irreducible right factor.*

b) *$T\sim\gamma\beta\beta\beta\ldots$ for some irreducible β.*

c) *The terms l_n are ultimately all* 1, *and for some N (and hence all sufficiently large N) $\{-\lambda_N+\lambda_{N+n}\}$ is step-periodic of period* 1.

d) $T = \sum(\delta+an, 1)+\gamma$: $\varphi\gamma = \delta$.

(ii) *The following equivalent conditions are satisfied*:

a) *T has a finite number $m > 1$ of q. i. right factors, the right factors of each being precisely the others.*

b) *$T\sim\gamma\beta\beta\beta\ldots$ for some β not a power of an irreducible ordinal.*

c) *The terms l_n are ultimately periodic of period p say, and for some N (and hence all sufficiently large N) $\{-\lambda_N+\lambda_{N+n}\}$ is left step-periodic of period q, say, and for no M does $pql_n = 1$ for all $n \geqslant M$.*

d) $T = \sum_{n\in\omega^*}\omega^{\delta+an}\varepsilon+\gamma$: $\varphi\gamma = \delta$; $\varphi\varepsilon < a$, *and we cannot take* $\varepsilon = 1$.

(iii) *The following equivalent conditions are satisfied*:

a) *T has a countable infinity of distinct reduced right factors, none of them q. i.*

b) *The terms in any Π such that $T\sim\Pi$ are not ultimately periodic.*

c) *For no N is it true that both $\{l_{N+n}\}$ is periodic and $\{-\lambda_N+\lambda_{N+n}\}$ is left step-periodic.*

Notes. In cases (i) and (ii) T may have reduced right factors which are not q. i. The left factor corresponding to a given right factor of T is unique iff the right factor is not q. i. In (ii) we may take β to have no n-th root ($n > 1$), and in this case the q. i. right factors of T are precisely the $S(\tau)\sim\tau\beta\beta\beta\ldots$ one for each right factor τ of β. We have $S(\tau) = \gamma S(\tau')$ iff $\gamma = \tau\beta^n\sigma'$, where $\sigma'\tau' = \beta$. In (iii) the reduced right factors are precisely those U given by (2) for which $\lambda_N \geqslant \tau(T)$. Further properties of the set of factors of T of type 4 are given in section 7.

6.5. We now consider RJ 6 types. We assume a standard notation

$$T = \sum(\lambda_n, l_n)+a:\quad \lim_n\lambda_n = \lambda\,,$$

$$U = \sum(\mu_n, m_n)+\gamma:\quad \lim_n\mu_n = \mu\,.$$

All assertions and formulas are given modulo an appropriate omission or change in some final segment of Σ, and the corresponding renumbering. In particular, if $\delta < \lambda$ we may always assume $\delta < \lambda_n$ for all n.

LEMMA 8. *If T is of type 6, then $T = \beta U$ iff $\varphi\beta < \lambda$ and*

$$U = \Sigma(-\varphi\beta+\lambda_n, l_n)+(-\varphi\beta+a_i, a_i, r)\,. \tag{6}$$

Proof. If $U = V+1$, $\beta U = \beta V+\beta$ is in normal form, so that, for some i, $\varphi\beta = a_i \geqslant \psi a \geqslant \lambda = \varphi\beta+\mu$, a contradiction, since μ is a limit ordinal. Thus U is of the second kind, and

$$\beta U = \sum(\varphi\beta+\mu_n, m_n)+(\varphi\beta+\gamma_i, c_i, s)\,.$$

The result follows at once.

LEMMA 9. *T of type 6 is reduced iff $\tau = 0$.*

Proof. Let $\tau = 0$ and $T = \beta U$. Then $\varphi\beta < \lambda = \omega^\theta$; $\varphi\varphi\beta < \theta$, so that

$$\mu = \lim_n(-\varphi\beta+\lambda_n) = -\varphi\beta+\lambda = -\varphi\beta+\omega^\theta = \omega^\theta = \lambda\,.$$

Next, $a_i \geqslant \lambda$, so that $\varphi a_i \geqslant \theta$. Thus $\theta \leqslant \varphi a_i = \varphi(\varphi\beta+\gamma_i) = \max(\varphi\varphi\beta, \varphi\gamma_i) = \varphi\gamma_i$, since $\varphi\varphi\beta < \theta$. So $\varphi\varphi\beta < \theta \leqslant \varphi\gamma_i$. But then $a_i = \varphi\beta+\gamma_i = \gamma_i$, and $\gamma = a$ by Lemma 8.

Suppose conversely $\tau \neq 0$. Then $T = \omega^\tau S$, where

$$S = S(T) = \sum(-\tau+\lambda_n, l_n)+(-\tau+a_i, a_i, r) \tag{7}$$

and $\lim(-\tau+\lambda_n) = -\tau+\lambda = \omega^\theta \neq \lambda$, so that T is not reduced.

LEMMA 10. *Each T of type 6 has at least one reduced right factor S. If $T = \beta U$ for some reduced U, then $\tau \leqslant \varphi\beta < \lambda$, and $\mu = \omega^\theta$.*

Proof. We have exhibited one such S in (7). If conversely $T = \beta U$ for some β and reduced U, we have $\mu = \omega^\sigma$, say, by Lemma 9. Then $\varphi\beta+\mu = \lambda$, $\psi(\varphi\beta+\mu) = \psi\lambda = \theta$. But $\psi(\varphi\beta+\mu) = \psi\mu = \sigma$, so that $\sigma = \theta$ and $\varphi\beta+\omega^\theta = \tau+\omega^\theta$. Since the right side is in normal form, we must have $\varphi\beta = \tau+\pi$: $\varphi\pi < \theta$, so that $\tau \leqslant \varphi\beta < \tau+\omega^\theta = \lambda$.

We now assign T to type 6a or 6b according as θ is of the second or first kind respectively.

LEMMA 11. *If U is a right factor of T, then U is of type* 6a (6b) *iff T is of type* 6a (6b).

6.6. LEMMA 12. *Suppose now T is of type* 6a. *Then the S of Lemma* 10 *is unique, and $\beta S = T$ for every β such that $\tau \leqslant \varphi\beta < \lambda$.*

Proof. Suppose $T = \beta U$ for some β and reduced U. By Lemma 10, $\varphi\beta = \tau + \pi$: $\pi < \omega^{\theta}$, so that $\varphi\pi < \theta$. Since θ is of the second kind, we may choose ϱ so that $\varphi\pi < \varrho < \theta$. Now $\mu = \omega^{\theta}$, so that $\varphi\mu_n \geqslant \varrho$ for all n. Thus $\lambda_n = \varphi\beta + \mu_n = \tau + \pi + \mu_n = \tau + \mu_n$.

Next, since U is of type 6, $\varphi\gamma_i \geqslant \theta > \varphi\pi$, so that $a_i = \varphi\beta + \gamma_i = \tau + \pi + \gamma_i = \tau + \gamma_i$. Together with Lemma 8 this shows that $U = S = S(T)$, given by (7).

Conversely if $\tau \leqslant \varphi\beta < \lambda$ and $\beta = \tau + \pi$ with $\varphi\pi < \varrho < \theta$, then we may write $\lambda_n = \tau + \delta_n$ with $\varphi\delta_n > \varrho$, so that if $\beta S = U$, then

$$\mu_n = \tau + \pi + (-\tau + \lambda_n) = \tau + \pi + \delta_n = \tau + \delta_n = \lambda_n ,$$

and $\gamma_i = \tau + \pi + (-\tau + a_i) = a_i$ similarly. Together with Lemma 8 this shows $U = T$.

THEOREM 13. *An RJ* 6a *type T has a unique irreducible right factor S, given by* (7). *We have $T = \beta S$ iff $\tau \leqslant \varphi\beta < \lambda$.*

Proof. By Lemmas 10 and 12 there exists a β and a unique reduced S such that $T = \beta S$. If S' is a right factor of S, it is reduced. Since S' is also a right factor of T, $S' = S$ by Lemma 12. Thus S is irreducible. The second part follows from Lemmas 10 and 12.

COROLLARY. *The following conditions on T of type* 6a *are equivalent:*

(i) *T is irreducible.*

(ii) *T is reduced.*

(iii) $\tau(T) = 0$.

(iv) $T = \Sigma(\lambda_n, l_n) + \omega^{\omega^{\omega\sigma}}\delta$,

where $\sigma \geqslant 1$, $\delta \geqslant 1$, and $\lim\limits_n \lambda_n = \omega^{\omega\gamma}$.

The simplest example is

$$T = \ldots + \omega^{\omega^3} + \omega^{\omega^2} + \omega^{\omega} + \omega + \omega^{\omega^{\omega}} .$$

6.7. Suppose now $S = \Sigma(\lambda_n, l_n) + \gamma$ is reduced of type 6b. Set $\theta = \xi + 1$. Then $\lambda_n = \omega^{\xi}u_n + \delta_n$ in normal form, where $\{u_n\}$ is non-decreasing and $\lim\limits_n u_n = \omega$. We write $S = (\xi, u_n, \delta_n, l_n, \gamma)$ as standard notation. By Lemma 8 we have $S = \beta U$ for some β and U iff $\varphi\varphi\beta \leqslant \xi$ and

$$U = S(u) = (\xi, -u + u_n, \delta_n, l_n, \gamma) \tag{8}$$

where $\varphi\beta = \omega^{\xi}u + \delta$ $(u \geqslant 0)$ in normal form.

It follows that the set of all right factors and reduced right multiples of S is precisely the set

$$E(S) = \{S(u):\ -\infty < u < \infty\} .$$

If no two $S(u)$ are equal, then $E(S)$ is infinite of type $\omega^* + \omega$, each member a right factor of all its predecessors, and S is not q. i. If for some $v \neq u$ we have $S(v) = S(u)$, then

$$\beta S(u+r) = S(u) = S(v) = \beta S(v+r) ,$$
$$S(u-r) = \beta S(u) = \beta S(v) = S(v-r),$$

where $\varphi\beta = \omega^{\xi}r$. Since we may left-cancel β, we easily see that $E(S)$ is finite. If its cardinal is t, then $S(u) = \beta S(v)$ whenever $\varphi\beta = \omega^{\xi}r + \delta$, where $r \equiv v - u \pmod t$, so that S is q. i. In this situation t is the smallest integer m such that for some $d > 0$ we have

$$u(n+d) = u(n) + m , \quad \delta(n+d) = \delta(n) , \quad l(n+d) = l(n)$$

for all n (where we write $u(n)$ for u_n, etc.). Thus $\{u_n\}$ is step-periodic, and $\{l_n\}$ and $\{\delta_n\}$ are periodic, and if their fundamental periods are p, q, r, and m is the fundamental step of $\{u_n\}$, then

$$t = dm/p:\ d = [p, q, r] .$$

We may take it that

$$\begin{gathered} u(d-1) < u(d) = u(0) + t < 2t , \\ \delta(nd+s) = \delta(s) , \\ l(nd+s) = l(s) , \\ u(nd+s) = u(s) + nt , \\ (0 \leqslant s \leqslant d-1;\ n \geqslant 0) . \end{gathered} \tag{9}$$

On setting $u(0) = j$, $v(i) = u(i) - u(0)$, we find

LEMMA 13. *S of type* 6b *is q. i. iff $\{l_n\}$ is periodic and $\{\lambda_n\}$ is left step-periodic. In this case we may write*

$$S = S(t-j) = \sum_{n \in \omega^*} \omega^{\pi(n)}\delta + \gamma , \tag{10}$$

where $\pi(n) = \omega^{\xi}(j + tn)$, $\varphi\psi\gamma > \xi > \varphi\delta$.

In terms of the discussion above, we have

$$\delta = \big(\omega^{\xi}v(i) + \delta(i),\ l_i,\ d-1\big) .$$

The right factors of S are just the sets $S(k)$, $0 \leqslant k < t$.

In particular, S is irreducible iff $t = 1$, so that $m = 1$ and $p = d$. Then $u(d) = u(0) + 1 < 2$, by (9). But the $u(n)$ are non-decreasing, so that

$0 = u(0) = \ldots = u(d-1) < u(d) = 1$. Thus $u(nd+s) = n$ $(0 \leqslant s \leqslant d-1)$. In this case we may simplify and renumber in (9) to obtain

LEMMA 14. *S of type* 6b *is irreducible iff $\{\lambda_n\}$ is left step-periodic of period p, say, and corresponding step ω^ξ, and $\{l_n\}$ is periodic of period dividing p. We may write*

$$S = \sum_{n \in \omega^*} \omega^{\omega^\xi n}\delta + \gamma, \quad \varphi\delta < \xi < \varphi\psi\gamma. \tag{11}$$

Knowing the structure of reduced, q. i., and irreducible RJ 6b types, and applying Lemma 10, we have

THEOREM 14. *If T is of type* 6b *we may write $\lambda_n = \tau + \omega^\xi u_n + \delta_n$, where $\{u_n\}$ is non-decreasing with limit ω. Subject to appropriate changes and omissions of a finite number of the λ_n, and the appropriate renumbering, exactly one of the following holds:*

(i) *T has a unique irreducible right factor S, given by* (7). *Equivalently, $\{-\lambda_0 + \lambda_n\}$ is left step-periodic of fundamental period p, say, and a corresponding step of the form ω^ξ, and $\{l_n\}$ is periodic of period dividing p. We have $T = \beta S$ iff $\tau \leqslant \varphi\beta < \lambda$.*

(ii) *T has a finite number $t > 1$ of q. i. right factors, the right factors of each being precisely the others. Equivalently $\{-\lambda_0 + \lambda_n\}$ is left step-periodic and $\{l_n\}$ is periodic, and if d is their least common period, then $\{\lambda_n\}$ has a step $\omega^\xi t$ corresponding to the period d. The q. i. right factors are $S(j)$ $(0 \leqslant j < t)$ given by* (10), *where $\gamma = (-\tau + a_i, a_i, r)$, and $T = \beta S(j)$ iff $\varphi\beta = \tau + \omega^\xi(wt+j) + \varepsilon$ in normal form. Also $S(j) = \beta S(k)$ iff $\varphi\beta = \omega^\xi(wt-j+k) + \varepsilon$ $(wt+k \geqslant j)$ in normal form.*

(iii) *T has an infinite chain $\{S(u)\colon u \geqslant 0\}$ of reduced right factors, not q. i., the right factors of each being precisely its successors. $S(u)$ is given by* (8), *where $\gamma = (-\tau + a_i, a_i, r)$. Equivalently, either $\{l_n\}$ is not periodic, or $\{-\lambda_M + \lambda_{M+n}\}$ is not step-periodic for any M. $T = \beta S(u)$ iff $\varphi\beta = \tau + \omega^\xi u + \varepsilon$ and $S(v) = \beta S(u)$ iff $\varphi\beta = \omega^\xi(u-v) + \varepsilon$.*

Summary. RJ 1, 2, 3, 5, 6a types have a unique irreducible right factor. The corresponding left factor is unique for types 2 and 5. RJ 4 and 6b types have either

(i) a unique irreducible right factor, or

(ii) a finite number of q. i. right factors, the right factors of each being just the others, or

(iii) a countable infinity of reduced right factors, none of them q. i.

The corresponding left factor is unique only for case (iii) of type 4.

Condition (iii) holds in particular if $\{\varphi\lambda_n\}$ is not ultimately constant. We may think of such "bad" order types as a countable sum of rapidly increasing ordinals, just as the "bad" Liouville transcendentals are an infinite sum of rapidly decreasing rationals.

7. Arithmetic of RJ types. We first extend the definition of φ, ψ to arbitrary RJ types by setting

$$\varphi T = \max(\lambda T, \varphi a T), \quad \psi T = \min(\lambda T, \psi a T).$$

This agrees with the definition of RJ 0 types. We refer to types $4j$ and $6j$ $(j = \text{(i), (ii), (iii)})$ in accordance with Theorems 12 and 14.

7.1. THEOREM 15. *The following relations hold whenever the expressions make sense (i.e. C, D are any RJ types; γ, δ are of type* 0):

1. $\varphi(C+\gamma) = \max(\varphi C, \varphi\gamma)$,
2. $\varphi(\gamma C) = \varphi\gamma + \varphi C$,
3. $\psi(C+\gamma) = \psi\gamma$ $(\gamma \neq 0)$,
4. $\psi(\gamma C) = \begin{cases} \psi\gamma & (C \text{ of the first kind}), \\ \varphi\gamma + \psi C & (C \text{ of the second kind}). \end{cases}$

Cancellation theorems:

5. $C + \gamma = D + \gamma$ *iff* $C(\text{say}) = D + \varepsilon$ *for some ε such that* $\varphi\varepsilon < \varphi\gamma$.
6. $C + \gamma = C + \delta$; $\gamma < \delta$ *iff C is of type* 1 *and* $\delta = \omega^\mu m + a(C) + \gamma$, *where* $m > 0$ if $a(C) \neq \mu(C)$.
7. $\gamma C = \gamma D$, $C \neq D$ *iff* $\gamma = 0$.
8. $\gamma C = \delta C$, *C of the first kind,* $\gamma < \delta$, *iff either*

   (i) $C = \omega^*$, $\gamma = \omega^\beta m + \varepsilon$, $\delta = \omega^\beta n + \varepsilon$, $m < n$, *or*

   (ii) $C \sim \beta\beta\beta\ldots$ *is q. i. of type* 4 *with β not an m-th power* $(m > 1)$,

   and $\delta = \gamma\beta^n$, $n > 0$.
9. $\gamma C = \delta C$, *C of the second kind,* $\gamma < \delta$, *iff* $\varphi(-\varphi\gamma + \varphi\delta) < \varphi\psi C$ *and in addition, if C is of type* 6b, *then C is q. i., and* $-\varphi\gamma + \varphi\delta = \omega^\xi ta + \varepsilon$, *where $a \geqslant 0$ and t and ξ are as in Theorem* 14 (ii).

7.2. The sets $L(T)$ and $R(T)$ of left and right factors of an RJ set T have a well-defined structure which we now consider. For given ordered sets A, B, let us write $A <_l B$ iff A is a left divisor of B. Then $<_l$ is a transitive relation, but need not be a partial order. Let us say a class $\mathcal{A}$ of order types is *almost linearly ordered* by $<_l$ iff $<_l$ is a partial order on $\mathcal{A}$ such that whenever $A, B \in \mathcal{A}$ are incomparable, then for some $m, n > 0$ and $C \in \mathcal{A}$ we have $A = Cm$; $B = Cn$. We make the corresponding definitions for $<_r$.

It will be convenient to exclude T from $L(T)$ and 1 from $R(T)$.

THEOREM 16L. *If T is RJ we have* $L(T) = L_1(T) \cup L_2(T)$, *where*

1. $L_1(T) <_l L_2(T)$,
2. *$L_1(T)$ is the set of all ordinals $a \neq 0$ such that $\varphi a < \psi T$ (so that $L_1(T)$ is empty iff T is of the first kind).*
3. *$L_2(T)$ is:*

   a) *empty iff T is of type* 3 *or* 6;

   b) *finite, non-empty iff T is of type* 0, 2, 5;

   c) *countably infinite iff T is of type* 1, 4.

4. *$L_2(T)$ is almost linearly ordered by $<_l$.*
5. *Every member of $L_2(T)$ has the left factor $\omega^{\psi T}$.*

COROLLARY. *$(L_2(T), <_l)$ is a distributive lattice, and $(L(T), <_l)$ is a lattice. It is distributive iff $\psi T \leqslant 1$.*

The proof is by consideration of cases, using property 4. If $\psi T \geqslant 2$, the lattice is not even modular, since then $a = 2$, $b = \omega$, $c = \omega+1$ are left factors of T with $a <_l b$, and $a \vee (c \wedge b) = 2$ while $(a \vee c) \wedge b = \omega$. Compare [1], p. 81.

THEOREM 16R. *If T is RJ we have $R(T) = R_1(T) \cup R_2(T)$, where*

0. *$R_i(T)$ is the set of right factors of T corresponding to the left factors in $L_i(T)$ $(i = 1, 2)$.*
1. *$R_2(T) <_r R_1(T)$.*
2. *$R_1(T)$ is empty iff T is of the first kind, and otherwise is a finite set of RJ sets of the same type and kind as T, and is linearly ordered by $<_r$.*
3. *$R_2(T)$ is a set of RJ sets which is countably infinite iff T is of type 4(iii) or 6b(iii), and otherwise finite.*
4. *$(R_2(T), <_r)$ is*
   a) *a linearly ordered set if T is of type 3, 6,*
   b) *an (almost) linearly ordered set followed by a finite loop iff T is of type 4(ii) or 6b(ii),*
   c) *an almost linearly ordered set in all other cases.*

COROLLARY. *$(R(T), <_r)$ is a lattice iff T is not of type 4 (ii) or 6b (ii), and is then moreover distributive.*

The "almost" in both parts of Theorem 16 can be omitted iff one of the following holds:

(i) T is of type 0(4) and every $a_i(\lambda_n)$ is a prime power (where the a_i are the coefficients of $a = a(T)$);

(ii) T is of type 2 or 5 and a_i is a prime power for every i such that $a_i < \lambda$.

7.3. Goldbach's hypothesis may be stated in the form: any J-set is expressible as a sum of $\leqslant 3$ irreducible sets. Here a J-set is one whose only cuts are of type J. It is of course either finite or one of ω, ω^*, $\omega^*+\omega$. For RJ sets we have

THEOREM 17. *Suppose T is RJ not of type 1 or 4, and $a = a(T) = (a_i, a_i, r)$. Then T is expressible as a finite sum of irreducible sets iff one of the following relations holds:*

(i) $T = \sum(n, 1) + \omega^{\omega}$,

(ii) $a_0 = \omega^{\xi}$, *but not* $T = \sum(\lambda_n, l_n) + \omega^{\delta}$: $\delta = \omega^{\xi}$, *unless* (i).

(iii) $a_0 = 0$; $a_0 \neq 2$.

(iv) $a_0 = 0$; $a_0 = 2$ *and either* $T = 2$ *or* $a_1 = \omega^{\xi}$ $(\xi \geqslant 0)$.

If T is of type 1 or 4 it is so expressible iff $T = \omega^*$ or $T = \sum_{\omega}^{m}(\gamma n, 1) + a$: $\gamma(m-1) \leqslant \varphi a < \gamma m$, and a satisfies one of (ii), (iii), (iv) above.

The smallest number of summands needed is at least $\sum_{1}^{r} a_i$ (with a as above if T is of type 4), and so is unbounded for every type (except trivially for type 1, where T must be ω^*).

It is true however that if T is RJ and not of type 1 or 4, it may always be expressed in the form $P_1+P_2+\ldots+P_n-P_0$, where P_i $(0 \leqslant i \leqslant n)$ is irreducible. We may even prescribe $P_0 =$ either 2 or 3.

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