The independence of the ordering principle from a restricted axiom of choice

by

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The main purpose of this paper is to show that the axiom of choice for sets of finite sets does not imply the ordering theorem. (This problem has been mentioned by Henkin [2] and by Łoś [3] among others.)

However, the ordering theorem is independent of a much stronger axiom of choice [1]:

THEOREM (S). *The conjunction of the following two statements is consistent with set theory* (theory T below).

(I) *Not every set can be totally ordered.*

(II) *There is a function which assigns to every well-orderable set a well-ordering of it.*

COROLLARIES. The following theorems do not imply the ordering theorem:

(a) *There is a choice function for the class of all well-orderable sets.*

In particular, for every class (set) of finite sets there is a choice function. For every class (set) of countable sets there is a choice function, etc.

(b) *The union of a well-orderable set of well-orderable sets is well-orderable.*

In particular, the union of a countable set of countable sets is countable.

It is perhaps worth mentioning that as a consequence of theorem (S), (II) does not imply the full axiom of choice. However, our proof will show that (II) is satisfied in Mostowski's model too ([4]).

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[1] The question whether such a stronger result could be obtained by means of the original method was raised by A. Tarski and E. Specker.

The theorem (S) refers to a theory T which is essentially the Bernays-Gödel set theory except that we do not assume the axiom of choice and that we replace the ordinary axiom of regularity by an axiom of regularity with respect to a basis set B (which may be taken as a set of sets $x=\{x\}$, cf. [5]).

The proof of (S) will be given in two parts. In the second part, the group theoretical one, we are going to prove theorem (G). For the statement of this theorem we use the following denotations: Let G be a permutation group on a set C. Then, for every $e\subset C$, $H_G(e)$ is that subgroup of G which consists in those permutations which transform e onto itself. $K_G(e)$ is the subgroup of those permutations which leave e pointwise fixed. $[H_1, H_2]$ is the subgroup of G generated by H_1 and H_2. $E(C)$ is the set of the finite subsets of C.

THEOREM (G). *There exists a permutation group G on a denumerable set C satisfying*

(A) $H_G(e)=K_G(e)$ *for every* $e\in E(C)$.

(B) $[K_G(e),\ K_G(e')]=K_G(e\cap e')$ *for all* $e, e'\in E(C)$.

(C) *For every* $e\in E(C)$ *there are* $u, v, w\in C$, u, v, w *different from each other, and* $\sigma, \tau\in K_G(e)$ *such that* $\sigma\colon u\to v\to w$ *and* $\tau\colon w\to u\to v$.

In the first part theorem (S) is proved using the Fraenkel-Mostowski-Specker method ([1], [4], [5]) of interpreting T in some theory T^*: Let T' be the theory which is obtained in assuming beside the axioms of T the strong axiom of choice (choice function for the universal class) and postulating that the basis set B is denumerable. Since the proof of theorem (G) could be formalized within T', it is provable in T' that there is a permutation group G on the basis set B satisfying (A), (B), (C). Now let T^* be that extension of T' which is obtained in introducing a special constant G and an axiom which says that G is a permutation group on B satisfying (A), (B), (C). Then T^* is consistent provided that T is consistent.

First Part

DENOTATIONS. $\overline{\varphi}$ is the (uniquely determined) automorphism of the universal class which corresponds to a permutation φ of B, $H(X)$ is the subgroup of G which is determined by the following condition: $\varphi\in H(X)$ iff $\overline{\varphi}$ transforms X onto itself. $K(X)$: $\varphi\in K(X)$ iff $\overline{\varphi}$ leaves X pointwise fixed. F: $H\in F$ iff $K(e)<H<G$ for some $e\in E(B)$ ("$<$" used for subgroups).

Let $M=\bigcup Q_\alpha$, where α runs over the ordinals and Q_α is recursively defined by $Q_0=B$, $Q_{\xi+1}=\{x\colon x\subset Q_\xi\wedge H(x)\in F\}$, $Q_\lambda=\bigcup\limits_{\xi<\lambda}Q_\xi$ for a limit number λ. Then, for every set x, $x\in M$ iff $x\subset M$ and $H(x)\in F$. Let $\mathfrak{M}$ be defined by

$$\mathfrak{M}(X)\equiv X\subset M\wedge H(X)\in F\quad(\text{“}X\text{ is an }\mathfrak{M}\text{-class”}).$$

Then $\mathfrak{M}(x)$ iff $x\in M$ ("x is an $\mathfrak{M}$-set").

The interpretation in question of T in T^* is given by the following stipulations: The classes of T are the $\mathfrak{M}$-classes of T^*, the sets of T are the $\mathfrak{M}$-sets of T^*, equality and membership relation of T are the same as those of T^* (cf. [5]).

A notion $\mathfrak{A}(X_1, \ldots, X_n)$ is said to be absolute if it is provable in T^* that

$$\mathfrak{M}(X_1)\wedge\ldots\wedge\mathfrak{M}(X_n)\to\big(\mathfrak{A}(X_1, \ldots, X_n)\leftrightarrow\mathfrak{A}^{\mathfrak{M}}(X_1, \ldots, X_n)\big).$$

Here, $\mathfrak{A}^{(\mathfrak{M})}$ is obtained by relativizing all bound variables in $\mathfrak{A}$ to $\mathfrak{M}$.

Generally, we will state the absoluteness of a notion without proof. The proofs that the statements (I), (II) do hold in our interpretation of T are given in colloquial language. We use the prefix "$\mathfrak{M}$-" in referring to a relativized notion.

Proof of (I). We are going to show that the basis set B (which is the basis set in the $\mathfrak{M}$-sense, too) can not be $\mathfrak{M}$-ordered. Since the notion "y is an ordering relation on x" is absolute, it is sufficient to show that no ordering relation on B is an $\mathfrak{M}$-set, i.e. that $H(y)\notin F$ for every ordering relation y on B. Now, every $H\in F$ contains $K(e)$ for some $e\in E(B)$, and because of property (C) of the group G it is clear that no ordering relation on B is left invariant under $K(e)$, q.e.d.

For the proof of (II) we need some lemmas, all of them except one (namely lemma 4) being provable without using the special properties of G. Since lemma 4 will be proved using properties (A) and (B) of G only, (II) is satisfied in Mostowski's model too.

Let C be the class, the elements of which are the transitivity domains in V (universal class) under G, i.e.

$$x\in C\leftrightarrow x\neq 0\wedge\bigwedge y, z\Big(y\in x\to\big[z\in x\leftrightarrow\bigvee\varphi\big(\varphi\in G\wedge\overline{\varphi}(y)=z\big)\big]\Big).$$

Notice that every transitivity domain is a set because the rank of a set is left invariant under the automorphisms of V. C is not an $\mathfrak{M}$-class, but $K(C)=G$. Let W be the class of all well-orderable sets in the $\mathfrak{M}$-sense (i.e. W is the class of all $\mathfrak{M}$-well-orderable $\mathfrak{M}$-sets). Then $H(W)=G$. Let $D=C\cap P(W)$, $P(W)$ being the power class of W.

LEMMA 1. $K(D)=G$, $\mathfrak{M}(D)$ *and* $W=\bigcup D$.

Proof. $K(D)=G$ because of $D\subset C$ and $K(C)=G$. Since every element of D is as an element of C and as a subset of W an invariant set of $\mathfrak{M}$-sets, and therefore an $\mathfrak{M}$-set, we get $\mathfrak{M}(D)$. $W=\bigcup D$, i.e. $W\subset\bigcup D$ is a consequence of $H(W)=G$.

LEMMA 2. *$x \in W$ iff $\mathfrak{M}(x)$ and $K(x) \in F$.*

Proof. 1. Suppose $\mathfrak{M}(x)$ and $K(x) \in F$. Let φ be a one-one mapping from x onto an ordinal (axiom of choice in T^*). Because ordinals are left pointwise fixed under automorphisms of V, we get $H(\varphi) \supset K(x)$, and hence $H(\varphi) \in F$. Ordinals are $\mathfrak{M}$-sets, M is closed under formation of ordered pairs, $\mathfrak{M}(x)$ implies $x \subset M$. Hence $\varphi \subset M$, $\varphi \in M$. Because the notions of an ordinal and of a one-one correspondence are absolute, x is equipotent with an ordinal in the $\mathfrak{M}$-sense. Hence $x \in W$.

2. For the proof of the converse use the above idea and notice that even $H(\varphi) = K(x)$.

Let $W(y, x)$ express that y is a well-ordering relation on x.

LEMMA 3. *$W(y, x)$ is an absolute notion.*

Proof. Suppose $\mathfrak{M}(x)$ and $\mathfrak{M}(y)$. If $W(y, x)$ then obviously $W^{(\mathfrak{M})}(y, x)$. Conversely, if $W^{(\mathfrak{M})}(y, x)$ then $x \in W$ and, using lemma 2, $K(x) \in F$. Hence $H(z) \in F$ for every $z \subset x$ and thus, because of $x \subset M$, every subset of x is an $\mathfrak{M}$-set, i.e. every non-empty subset of x has a first element with respect to y, i.e. $W(y, x)$, q.e.d.

LEMMA 4. *$K(x) \in F$ implies that $K(x) = H(x)$.*

Proof. 1. For every $H \in F$ there is an $e \in E(B)$ such that $H = K(e)$ (cf. also [4]). *Proof*: Let $e \in E(B)$ be minimal with respect to the property $H \supset K(e)$. Let $\varphi \in H$. Then $H = \varphi H \varphi^{-1} \supset \varphi K(e) \varphi^{-1} = K(\varphi e)$. Using (B) of theorem (G) we get $H \supset [K(e), K(\varphi e)] = K(e \cap \varphi e)$. Hence $\varphi e = e$ (minimality of e), i.e. $\varphi \in H(e)$ and because of (A), $\varphi \in K(e)$. Hence $H = K(e)$, q.e.d.

2. If $m \neq 0$, $\varphi \in G$, $H \in F$, $\varphi^m \in H$ then $\varphi \in H$. *Proof*: We may assume that $m > 0$. In virtue of 1. let $H = K(e)$. $\varphi^m \in K(e)$ implies $\varphi \in H\big(e \cup \varphi(e) \cup \varphi^2(e) \cup \ldots \cup \varphi^{m-1}(e)\big) \overset{(A)}{=} K\big(e \cup \ldots \cup \varphi^{m-1}(e)\big) \subset K(e)$.

3. *Proof of the Lemma*: Let $K(x) \in F$, $\varphi \in H(x)$, $y \in x$. We have to show that $\varphi(y) = y$. Let $z = \{\varphi^{2k}(y)\colon \text{integers } k\}$. Then $z \subset x$, $H(z) \supset K(x)$ and hence $H(z) \in F$. Because of $\varphi^2 \in H(z)$, 2. gives $\varphi \in H(z)$, i.e. $\varphi(y) = \varphi^{2m}(y)$ for some m. That is $\varphi^{2m-1} \in H(y)$. Because of $H(y) \supset K(x)$, i.e. $H(y) \in F$, and $\varphi^{2m-1} \in H(y)$ (and $2m-1 \neq 0$), a second application of 2. gives $\varphi \in H(y)$, i.e. $\varphi(y) = y$, q.e.d.

Proof of (II). Let $F(X)$, $D(X, Y)$ respectively express that X is a function, Y is the domain of X. Then, in virtue of the absoluteness of the notions involved (especially $W(y, z)$), it is sufficient to prove:

(*) *There is an A such that $\mathfrak{M}(A)$, $F(A)$, $D(A, W)$, and for all x, y, $\langle y, x\rangle \in A$ implies $W(y, x)$.*

The idea of the proof is the following one: Using the given choice function t for the universal class, pick for every $z \in D$ a pair $\langle y, x\rangle$ such that $x \in z$ and $W(y, z)$, and close the class of these pairs with respect to the whole automorphism group G. Then, looking at the lemmas we proved, we get a choice function as required.

For the detailed proof we define:

$$r(z) = \{\langle y, x\rangle\colon\ x \in z \wedge \mathfrak{M}(y) \wedge W(y, x)\}$$

$$s(z) = C \cap P\big(r(z)\big), \qquad P = \text{power set},$$

$$A = \bigcup_{d \in D} t\big(s(d)\big), \qquad t = \text{choice function}$$

(note that $s(d)$ is not empty).

A satisfies the conditions stated in (*): We are going to prove that

(a) $H(A) = G$.

(b) *$v \in A$ implies $v = \langle y, x\rangle$, where $\mathfrak{M}(y)$, $\mathfrak{M}(x)$ and $W(y, x)$.*

(c) *For every $x \in W$ there is a y such that $\langle y, x\rangle \in A$.*

(d) *$\langle y, x\rangle \in A$ and $\langle y', x\rangle \in A$ implies $y' = y$.*

Then we get $\mathfrak{M}(A)$ because of (a) and (b), $F(A)$ because of (b) and (d), $D(A, W)$ because of (b) and (c). Furthermore, (b) says that $\langle y, x\rangle \in A$ implies $W(y, x)$.

Proof of (a). It is sufficient to notice that $H(A) \supset \bigcap_{d \in D} H\Big(t\big(s(d)\big)\Big)$ and that $H\Big(t\big(s(d)\big)\Big) = G$ because of $t\big(s(d)\big) \in C$.

Proof of (b). If $v \in A$ then $v \in r(d)$ for some $d \in D$ and hence $v = \langle y, x\rangle$ for some x, y satisfying $x \in d \in D$, $\mathfrak{M}(y)$ and $W(y, x)$. $\mathfrak{M}(x)$ is a consequence of $\mathfrak{M}(D)$ (Lemma 1).

Proof of (c). Let $x \in W$. By Lemma 1, there is a $d \in D$ such that $x \in d$. $t\big(s(d)\big)$ being an element of C is not empty. Let $\langle y', x'\rangle \in t\big(s(d)\big)$ (and hence $\langle y', x'\rangle \in A$). Then $x' \in d$, and because of $d \in D \subset C$ there is a $\varphi \in G$ such that $\overline{\varphi}(x') = x$. Let $y = \overline{\varphi}(y')$. Then $\langle y, x\rangle = \overline{\varphi}(\langle y', x'\rangle)$, and since $H(A) = G$, we get $\langle y, x\rangle \in A$, q.e.d.

Proof of (d). Let $\langle y, x\rangle, \langle y', x\rangle \in A$. Let $d, d' \in D$ be such that $\langle y, x\rangle \in t\big(s(d)\big)$ and $\langle y', x\rangle \in t\big(s(d')\big)$. Then $x \in d \cap d'$ and therefore $d' = d$ (element of C). Since $t\big(s(d)\big) \in C$, there is a $\varphi \in G$ such that $\overline{\varphi}(\langle y, x\rangle) = \langle y', x\rangle$, i.e. $\varphi \in H(x)$ and $\overline{\varphi}(y) = y'$. By (b) above, we have $\mathfrak{M}(x)$, $\mathfrak{M}(y)$ and $W(y, x)$. Since $W(y, x)$ is absolute, it follows that $x \in W$. Hence, by Lemma 2, $K(x) \in F$ and thus $H(x) = K(x)$ by Lemma 4. Furthermore, $W(y, x)$ obviously implies $K(x) = H(y)$. Hence $H(x) = H(y)$, what gives $\varphi \in H(y)$, i.e. $\overline{\varphi}(y) = y = y'$, q.e.d.

Second Part [2]

In this section we prove theorem (G) which was stated in the introduction. The proof consists essentially in constructing a suitable subgroup of a free group and considering the latter as a permutation group acting on the left cosets modulo that subgroup.

Definition. $\mathfrak{U}$ is said to be a *pure* subgroup of $\mathfrak{G}$, write $\mathfrak{U} \underset{p}{<} \mathfrak{G}$, if $\xi^n \in \mathfrak{U}$ implies $\xi \in \mathfrak{U}$ for every $\xi \in \mathfrak{G}$ and for every natural number $n > 0$.

Lemma 1. *If $\mathfrak{G}$ is the free group generated by α, β and $\mathfrak{U} = [\alpha^2\beta, \alpha\beta^2]$, then*

(a) $$\alpha, \beta \notin \mathfrak{U},$$

(b) $$\mathfrak{U} \underset{p}{<} \mathfrak{G}.$$

(a) is an immediate consequence of the fact that the total exponent [3] of any element of $\mathfrak{U}$ as a word in α, β is an integral multiple of 3.

In order to prove (b) we show that the following condition (c) is sufficient for purity:

(c) *For any A with $l(A) \geqslant 3$ there are A_1, A_2, X such that $A = A_1 \cdot A_2$ and such that whenever $UAV = U \cdot A \cdot V \in \mathfrak{U}$ for some U, V then $UA_1X \in \mathfrak{U}$ (and $X^{-1}A_2V \in \mathfrak{U}$).*

Here capital letters denote reduced words in α, β. $l(A)$ denotes the length of A. AB denotes the concatenation of the words A, B and the dot in $A \cdot B$ indicates that AB is a reduced word.

(c) *implies* (b).

Let $W^n = P \cdot Q^n \cdot P^{-1} \in \mathfrak{U}$, where $QQ = Q \cdot Q$. We may assume that $n \geqslant 4$ (otherwise consider $W^{4n} \in \mathfrak{U}$) and that Q is non empty. Let $A = Q^{n-1}$. Then $l(A) \geqslant 3$. Take $A_1, A_2 X$ according to (c). Then, $W^n = P \cdot A \cdot Q \cdot P^{-1} \in \mathfrak{U}$ implies $PA_1X \in \mathfrak{U}$ and $W^n = P \cdot Q \cdot A \cdot P^{-1} \in \mathfrak{U}$ implies $X^{-1}A_2P^{-1} \in \mathfrak{U}$. Hence $W^{n-1} = PA_1XX^{-1}A_2P^{-1} \in \mathfrak{U}$, i.e. $W = W^nW^{-(n-1)} \in \mathfrak{U}$, q.e.d.

In order to verify condition (c) for our group, we state without proof what A_1, A_2, X in the several cases are: Let x, y stand for α or β. If A contains positive and negative letters, i.e. A can be written as $R \cdot x^\varepsilon \cdot y^{-\varepsilon} \cdot S$, then set $A_1 = Rx^\varepsilon$, $A_2 = y^{-\varepsilon}S$, $X = \begin{cases} \beta & \text{if } \varepsilon = 1 \\ \alpha^{-1} & \text{if } \varepsilon = -1 \end{cases}$. If all letters in A are positive, then, because of $l(A) \geqslant 3$, we are in one of the following cases: $A = R \cdot \alpha\alpha \cdot S$, $A = R \cdot \beta\beta \cdot S$, $A = R \cdot \beta\alpha \cdot S$. Let X be empty in all three cases and set $A_1 = R$, $A_2 = \alpha\alpha S$ in the first case, $A_1 = R\beta\beta$, $A_2 = S$ in the second case and $A_1 = R\beta$, $A_2 = \alpha S$ in the last case. If all letters in A are negative we proceed analogously.

[2] The author greatly appreciates the help given by Mrs. Verena Dyson in reading this paper.

[3] total exponent = the sum of all exponents occurring in the word.

Lemma 2. *Let $\mathfrak{G}$ be the free product of its free subgroups $\mathfrak{G}_0, \mathfrak{G}_1$, and let $S = \{\xi, \eta_1, \eta_2\}$ be a set of free generators of $\mathfrak{G}_1$. Let $\mathfrak{U}_0 < \mathfrak{G}_0$, $d \in \mathfrak{G}_0$ and let A, B, C be mutually disjoint finite subsets of $\mathfrak{G}_0$ such that*

(i) *If $x, y \in A \cup B \cup C$ and $x \neq y$, then $x^{-1}y \notin \mathfrak{U}_0$,*

(ii) *$x^{-1}dx \in \mathfrak{U}_0$ for every $x \in C$.*

Let $\mathfrak{U}$ be the subgroup of $\mathfrak{G}$ which is generated by $\mathfrak{U}_0$ and the set $M = M_1 \cup M_2 \cup M_3$, where

$$M_1 = \{x^{-1}\xi x\colon\ x \in A \cup C\},$$
$$M_2 = \{x^{-1}\eta_i x\colon\ x \in B \cup C,\ i = 1, 2\},$$
$$M_3 = \{x^{-1}d^{-1}\eta_1\xi\eta_2 x\colon\ x \in A \cup C\}.$$

Then

(a) $\mathfrak{U} \cap \mathfrak{G}_0 = \mathfrak{U}_0$,

(b) $\mathfrak{U}_0 <_p \mathfrak{G}_0$ *implies* $\mathfrak{U} <_p \mathfrak{G}$.

For the proof of this lemma it is convenient to consider the following set N: $N = N_1 \cup N_2 \cup N_3 \cup N_4$, where $N_1 = M_1$, $N_2 = M_2$, $N_3 = \{x^{-1}d^{-1}\eta_1\xi\eta_2 x\colon\ x \in A$ and $y^{-1}dx \notin \mathfrak{U}_0$ for every $y \in B\}$, $N_4 = \{y^{-1}\xi\eta_2 x\colon\ x \in A,\ y \in B,\ y^{-1}dx \in \mathfrak{U}_0\}$.

Then, under the hypothesis of lemma 2, we have

(c) $\mathfrak{U}$ *is generated by* $\mathfrak{U}_0$ *and* N.

Proof. 1. $M_3 \subset [\mathfrak{U}_0, N]$: Let $\varphi = x^{-1}d^{-1}\eta_1\xi\eta_2 x$, where $x \in A \cup C$. If $x \in C$, then $\varphi = (x^{-1}dx)^{-1} \cdot x^{-1}\eta_1 x \cdot x^{-1}\xi x \cdot x^{-1}\eta_2 x$, where $x^{-1}dx \in \mathfrak{U}_0$, $x^{-1}\xi x \in N_1$, $x^{-1}\eta_i x \in N_2$. If $x \in A$, then either $\varphi \in N_3$ or $\varphi = (y^{-1}dx)^{-1} \times \times y^{-1}\eta_1 y \cdot y^{-1}\xi\eta_2 x$ for some $y \in B$, where $y^{-1}dx \in \mathfrak{U}_0$, $y^{-1}\eta_1 y \in N_2$ and $y^{-1}\xi\eta_2 x \in N_4$.

2. $N_4 \subset [\mathfrak{U}_0, M]$: Let $\varphi = y^{-1}\xi\eta_2 x$, where $x \in A$, $y \in B$ and $y^{-1}dx \in \mathfrak{U}_0$. Then $\varphi = (y^{-1}\eta_1 y)^{-1} \cdot y^{-1}dx \cdot x^{-1}d^{-1}\eta_1\xi\eta_2 x$, where $y^{-1}\eta_1 y \in M_2$ and $x^{-1}d^{-1}\eta_1\xi\eta_2 x \in M_3$.

In the following, a string $x_0\varrho_1x_1 \ldots \varrho_kx_k$, $x_i \in \mathfrak{G}_0$ and $\varrho_i \in S^{\pm 1}$ [4] is said to be a $\mathfrak{G}$-*word*, if $\varrho_i \neq \varrho_{i+1}^{-1}$ or $x_i \neq 1$ for $i = 1, 2, \ldots, k-1$. A string $u_0\varphi_1u_1 \ldots \varphi_ku_k$, $u_i \in \mathfrak{U}_0$ and $\varphi_i \in N^{\pm 1}$, is said to be a $\mathfrak{U}$-*word*, if $\varphi_i \neq \varphi_{i+1}^{-1}$ or $u_i \neq 1$ for $i = 1, \ldots, k-1$. The $\mathfrak{G}$-word $x_0\varrho_1x_1 \ldots \varrho_kx_k$ is said to be $\mathfrak{G}$-*cyclically-reduced*, if $\varrho_1 \neq \varrho_k^{-1}$ or $x_0 \neq x_k^{-1}$. Analogous the notion "$\mathfrak{U}$-*cyclically-red*".

Because $\mathfrak{G}$ is the free product of $\mathfrak{G}_0$ and $\mathfrak{G}_1$, every element of $\mathfrak{G}$ has a unique representation as a $\mathfrak{G}$-word. Moreover, $\mathfrak{U}$ is the free product of $\mathfrak{U}_0$ and the subgroup of $\mathfrak{G}$ generated by N in the following strong sense:

(d) *If $u_0\varphi_1u_1 \ldots \varphi_ku_k$ is a $\mathfrak{U}$-word and $\phi_1, \phi_2, \ldots, \phi_k$ are the $\mathfrak{G}$-words corresponding to $\varphi_1, \varphi_2, \ldots, \varphi_k$, respectively, then under composition $u_0\phi_1u_1 \ldots \ldots \phi_ku_k$ no occurence of an S-symbol can be cancelled.*

[4] Recall that $S = \{\xi, \eta_1, \eta_2\}$.

Since every ϕ_i contains S-symbols, it is sufficient to prove (d) in case of an $\mathfrak{U}$-word $\varphi u\psi$. Let us consider one (the least boring one) of the many cases: Let $\varphi \in N_2^{-1}$, $\psi \in N_3$. Assume that an S-symbol can be cancelled. Then $\varphi = x^{-1}\eta_1^{-1}x$, where $x \in B \cup C$, and $\psi = x'^{-1}d^{-1}\eta_1\xi\eta_2 x'$, where $x' \in A$ and $y^{-1}dx' \notin \mathfrak{U}_0$ for every $y \in B$. Our assumption yields $xux'^{-1}d^{-1} = 1$, and hence $x^{-1}dx' \in \mathfrak{U}_0$. By definition of N_3, this gives $x \notin B$, i.e. $x \in C$. According to hypothesis (ii) stated in Lemma 2, we get $x^{-1}dx \in \mathfrak{U}_0$ and hence $x^{-1}x' \in \mathfrak{U}_0$. (i) yields $x = x'$, which contradicts the hypothesis $A \cap C = 0$.

Obviously, (c) and (d) together imply the assertion (a) of Lemma 2. Another consequence of (d) is:

(e) *The $\mathfrak{G}$-word representing an element of $\mathfrak{U}$ is $\mathfrak{G}$-cyclically-reduced if and only if the corresponding $\mathfrak{U}$-word is $\mathfrak{U}$-cyclically-reduced.*

Therefore, such a word will be referred to from now on simply as to be *cyclically reduced.*

For the proof of part (b) of Lemma 2 we consider the following mapping f from $\mathfrak{G}$ into $\mathfrak{G}_0$: Let $\omega \in \mathfrak{G}$ be represented by the $\mathfrak{G}$-word Ω. Then $f(\omega)$ is the element of $\mathfrak{G}_0$ which is obtained by first replacing in Ω every part of the form $(\xi\eta_2)^\varepsilon$ by d^ε $(\varepsilon = \pm 1)$, and then cancelling all remaining occurrences of S-symbols. (Note that two parts of the form $(\xi\eta_2)^\varepsilon$ never do overlap.)

(f) *f maps $\mathfrak{U}$ into $\mathfrak{U}_0$ and $f(x\omega y) = xf(\omega)y$ for all $x, y \in \mathfrak{G}_0$, $\omega \in \mathfrak{G}$.*

Proof. The second assertion is an immediate consequence of the definition of f. The first one is proved by induction with respect to the length k of the $\mathfrak{U}$-word $\Omega = u_0\varphi_1u_1 \dots \varphi_k u_k$: If $k = 0$, then $\Omega = u_0$, $f(\Omega) = \Omega \in \mathfrak{U}_0$. If $k > 0$, then $\Omega = u_0\varphi_1\Omega'$, where $\Omega' = u_1\varphi_2 \dots \varphi_k u_k$ and, by assumption, $f(\Omega') \in \mathfrak{U}_0$. In the simple case where $f(\Omega) = f(u_0)\cdot f(\varphi_1)f(\Omega')$, we only have to show that $f(\varphi_1) \in \mathfrak{U}_0$. But $f(\varphi) = 1 \in \mathfrak{U}_0$ if $\varphi \in N_1 \cup N_2 \cup N_3$, and $f(\varphi) = y^{-1}dx \in \mathfrak{U}_0$ if $\varphi \in N_4$. Moreover, in the case $\varphi \in N^{-1}$ it is sufficient to notice that $f(\omega^{-1}) = f(\omega)^{-1}$ for every $\omega \in \mathfrak{G}$. In the troublesome case, namely, $f(\Omega) \neq f(u_0)f(\varphi_1)f(\Omega')$, we have the following situation: the $\mathfrak{G}$-word $\overline{\Omega} = u_0\phi_1\overline{\Omega'}$ (ϕ_1, Ω' being the $\mathfrak{G}$-words corresponding to φ_1, Ω', respectively) has a part $(\xi\eta_2)^\varepsilon$ which involves the last S-symbol occuring in ϕ_1 and the first S-symbol occurring in $\overline{\Omega'}$. Hence, inspection of the set $N^{\pm 1}$ gives $\varphi_1, \varphi_2 \in N_1^{\pm 1} \cup N_2^{\pm 1}$, i.e. $\varphi_1 = x^{-1}\varrho^\varepsilon x$, $\varphi_2 = y^{-1}\varrho'^\varepsilon y$, $x, y \in A \cup B \cup C$, $\varrho, \varrho' \in S$. $\varrho^\varepsilon xu_1y^{-1}\varrho'^\varepsilon = (\xi\eta_2)^\varepsilon$ implies $xu_1y^{-1} = 1$, i.e. $x^{-1}y \in \mathfrak{U}_0$. Hence, by hypothesis (i) of the lemma, $x = y$ and therefore $u_1 = 1$. Since one of ϱ, ϱ' is ξ^ε and the other is η_2^ε, one of φ_1, φ_2 belongs to N_1^ε and the other to N_2^ε; thus one of x, y is in $A \cup C$ and the other in $B \cup C$. Finally, because of $x = y$ and $A \cap B = 0$, we get $x \in C$. Now, we have $\overline{\Omega} = u_0x^{-1}\varrho^\varepsilon\varrho'^\varepsilon xu_2\phi_3 \dots$ and $\overline{\Omega'} = x^{-1}\varrho'^\varepsilon xu_2\phi_3 \dots$, and the indicated occurence of the S-symbol ϱ'^ε does not belong to a part $(\xi\eta_2)^\varepsilon$ of $\overline{\Omega'}$ (no overlapping of such parts). Hence $f(\Omega') = u_2f(\phi_3 \dots)$. On the other hand, $f(\Omega) = u_0x^{-1}d^\varepsilon xu_2f(\phi_3 \dots)$, which gives us $f(\Omega) = u_0x^{-1}d^\varepsilon xf(\Omega')$. But by hypothesis (ii) of the lemma, $x \in C$ implies $x^{-1}d^\varepsilon x \in \mathfrak{U}_0$. Hence, because of $f(\Omega') \in \mathfrak{U}_0$, $f(\Omega) \in \mathfrak{U}_0$, q.e.d.

(g) *Let $\mathfrak{U}_0 <_p \mathfrak{G}_0$. Then, if $\varphi^n = a_1\Omega b_1a_2\Omega b_2 \dots a_n\Omega b_n$, where $a_i, b_i \in \mathfrak{G}_0$ and $b_1a_2 = b_2a_3 = \dots = b_na_1$ and $a_i\Omega b_i \in \mathfrak{U}$ for $i = 1, 2, \dots, n$, then $\varphi \in \mathfrak{U}$.*

Proof. The invariance of the products b_ia_{i+1} implies $\varphi^n = (a_1\Omega b_n)^n$ and, since $\mathfrak{G}$ is a free group, $\varphi = a_1\Omega b_n$. By (f), $a_i\Omega b_i \in \mathfrak{U}$ implies $a_if(\Omega)b_i = f(a_i\Omega b_i) \in \mathfrak{U}_0$. Hence $\big(a_1f(\Omega)b_n\big)^n = a_1f(\Omega)b_1a_2f(\Omega)b_2 \dots a_nf(\Omega)b_n \in \mathfrak{U}_0$ (we use again the invariance of b_ia_{i+1}). Since $a_1f(\Omega)b_n \in \mathfrak{G}_0$ and $\mathfrak{U}_0 <_p \mathfrak{G}_0$, we get $a_1f(\Omega)b_n \in \mathfrak{U}_0$, and since $a_1f(\Omega)b_1 \in \mathfrak{U}_0$, we obtain $b_1^{-1}b_n \in \mathfrak{U}_0$. Since $\varphi = a_1\Omega b_n = a_1\Omega b_1\cdot b_1^{-1}b_n$ and $a_1\Omega b_1 \in \mathfrak{U}$ (hypothesis), we get $\varphi \in \mathfrak{U}$, q.e.d.

Proof of (b). We are going to show that the general case reduces to the special case treated in (g):

Assume $\mathfrak{U}_0 <_p \mathfrak{G}_0$ and let $\varphi \in \mathfrak{G}$ be such that $\varphi^n \in \mathfrak{U}$ for some $n > 0$. Choose $\tau \in \mathfrak{U}$ such that the $\mathfrak{U}$-word $u_0\psi_1u_1 \dots \psi_ku_k$ which represents $\sigma^n = \tau^{-1}\varphi^n\tau$ is cyclically reduced (cf. (e)) and such that the following holds:

(*) *Either $\psi_1 \in N_3^{\pm 1}$, or $\psi_1 \in N_4^{\pm 1}$ and $\psi_i \notin N_3^{\pm 1}$ for all i, or $\psi_1 \in N_1^{\pm 1} \cup N_2^{\pm 1}$ and $\psi_i \notin N_3^{\pm 1} \cup N_4^{\pm 1}$ for all i* ([5]).

We will show that under these conditions σ^n satisfies the hypothesis of (g). Then (b) will be proved: We conclude $\sigma \in \mathfrak{U}$ and hence, because of $\tau \in \mathfrak{U}$, $\varphi = \tau\sigma\tau^{-1} \in \mathfrak{U}$.

We may assume that $k > 0$, i.e. $\sigma^n \notin \mathfrak{U}_0$. Otherwise we get $\sigma \in \mathfrak{U}$ because of $\mathfrak{U}_0 <_p \mathfrak{G}_0$. Now, let $x\Omega x'$ be the $\mathfrak{G}$-word which represents σ, where $x, x' \in \mathfrak{G}_0$ and $\Omega = \varrho\overline{\Omega}\varrho'$ for some $\varrho, \varrho' \in S^{\pm 1}$ (or $\Omega = \varrho$, in which case we identify $\varrho' \equiv \varrho$ and set $\overline{\Omega} =$ empty). Using subscripts we distinguish different occurrences of ϱ, ϱ': $\sigma^n = x\varrho_1\overline{\Omega}\varrho_1'x'x\varrho_2\overline{\Omega}\varrho_2'x' \dots \overline{\Omega}\varrho_n'x'$. This is the $\mathfrak{G}$-word representing σ^n, for σ^n and hence σ is cyclically-reduced. Determine l_i, $i = 1, \dots, n$, such that ϱ_i' occurs in ψ_{l_i}. We define a_i, b_i by

(A) $(x\Omega x')^{i-1}x\Omega b_i = u_0\psi_1u_1 \dots \psi_{l_i}u_{l_i}$, $i = 1, \dots, n$

and

(B) $b_1a_2 = b_2a_3 = \dots = b_na_1 = x'x$.

It remains to prove that

1. $a_1\Omega b_1a_2\Omega b_2 \dots a_n\Omega b_n = \sigma^n$,
2. $a_i\Omega b_i \in \mathfrak{U}$, $i = 1, \dots, n$,
3. $a_i, b_i \in \mathfrak{G}_0$, $i = 1, \dots, n$.

[5] The operation $\tau^{-1}(\dots)\tau$ allows us to shift any of the ψ_i's to the very left of the word.

Since $l_n = k$, definition (A) gives $b_n = [(x\Omega x')^{n-1}x\Omega]^{-1}\sigma^n$ and therefore $b_n = x'$. (B) gives $a_1 = x$. Thus 1. is obvious. 2. follows easily by induction, using (A). In order to obtain 3., we first prove that

(**) *ϱ_i' is the last S-symbol in ψ_{l_i}.*

For, since ϱ is the first S-symbol in ψ_1, condition (*) on ψ_1 implies that ϱ occurs in no ψ_i otherwise but as first S-symbol (in order to verify this look at the definition of $N^{\pm 1}$). Now, if ϱ_i' were not the last S-symbol in ψ_{l_i}, then the symbol ϱ_{i+1} (which is a symbol ϱ) did occur in ψ_{l_i} otherwise than as first S-symbol, which is a contradiction.

By (**), $(x\Omega x')^{i-1}x\Omega$, which is the same as $x\varrho_1\overline{\Omega}\varrho_1'x' \ldots x\varrho_i\overline{\Omega}\varrho_i'$, is that initial segment of the $\mathfrak{G}$-word corresponding to $u_0\psi_1 \ldots \psi_{l_i}u_{l_i}$, which ends with the last S-symbol of ψ_{l_i}. Hence, by definition (A), $b_i \in \mathfrak{G}_0$ and therefore, by definition (B), $a_{i+1} \in \mathfrak{G}_0$ (set $a_{n+1} = a_1$). This completes the proof of (b) and thus of Lemma 2.

LEMMA 3. *There exists a group $\mathfrak{F}$ and a subgroup $\mathfrak{H}$ of $\mathfrak{F}$ such that*

(A) $\mathfrak{H} <_p \mathfrak{F}$.

(B) *For any mutually disjoint finite subsets A, B, C of $\mathfrak{F}$ and for every $d \in \mathfrak{F}$ satisfying*

(i) *$x, y \in A \cup B \cup C$ and $x \neq y$ implies $x^{-1}y \notin \mathfrak{H}$,*

(ii) *$x^{-1}dx \in \mathfrak{H}$ for every $x \in C$,*

there are $\xi, \eta_1, \eta_2 \in \mathfrak{F}$ such that

(a) *$x^{-1}\xi x \in \mathfrak{H}$ for every $x \in A \cup C$,*

(b) *$x^{-1}\eta_i x \in \mathfrak{H}$ for every $x \in B \cup C$, $i = 1, 2$,*

(c) *$x^{-1}d^{-1}\eta_1\xi\eta_2 x \in \mathfrak{H}$ for every $x \in A \cup C$.*

(C) *For some $\alpha, \beta \in \mathfrak{F}$ we have $\alpha^2\beta, \alpha\beta^2 \in \mathfrak{H}$ and $\alpha, \beta \notin \mathfrak{H}$.*

Proof. Let $\mathfrak{F}$ be the free group with $\{\alpha, \beta, \xi^{(1)}, \eta_1^{(1)}, \eta_2^{(1)}, \xi^{(2)}, \eta_1^{(2)}, \eta_2^{(2)} \ldots\}$ as an enumerable set of free generators. Consider the subgroups $\mathfrak{F}_0, \mathfrak{F}_1, \mathfrak{F}_2, \ldots$ defined by $\mathfrak{F}_0 = [\alpha, \beta]$, $\mathfrak{F}_{l+1} = [\mathfrak{F}_l, \xi^{(l+1)}, \eta_1^{(l+1)}, \eta_2^{(l+1)}]$. Enumerate all quadruples $\tau = \langle A, B, C, d\rangle$, where A, B, C are mutually disjoint finite subsets of $\mathfrak{F}$ and $d \in \mathfrak{F}$, in such a way that every τ is counted infinitely many times and such that $A_l \cup B_l \cup C_l \cup \{d_l\} \subset \mathfrak{F}_l$ for every $\tau_l = \langle A_l, B_l, C_l, d_l\rangle$. Define $\mathfrak{H}_0 = [\alpha^2\beta, \alpha\beta^2]$. If A_l, B_l, C_l, d_l satisfy the conditions (i), (ii) stated in Lemma 3 (taking $\mathfrak{H}_l$ in place of $\mathfrak{H}$), then let $\mathfrak{H}_{l+1} = [\mathfrak{H}_l, M_1^{(l)} \cup M_2^{(l)} \cup M_3^{(l)}]$, where

$$M_1^{(l)} = \{x^{-1}\xi^{(l+1)}x\colon\ x \in A_l \cup C_l\},$$

$$M_2^{(l)} = \{x^{-1}\eta_i^{(l+1)}x\colon\ x \in B_l \cup C_l,\ i = 1, 2\},$$

$$M_3^{(l)} = \{x^{-1}d_l^{-1}\eta_1^{(l+1)}\xi^{(l+1)}\eta_2^{(l+1)}x\colon\ x \in A_l \cup C_l\}.$$

Otherwise let $\mathfrak{H}_{l+1} = \mathfrak{H}_l$. In both cases $\mathfrak{H}_l < \mathfrak{F}_l$ implies $\mathfrak{H}_{l+1} < \mathfrak{F}_{l+1}$ (note that $A_l \cup B_l \cup C_l \cup \{d_l\} \subset \mathfrak{F}_l$). In the first case the hypothesis of Lemma 2 is satisfied (taking $\mathfrak{H}_l, \mathfrak{H}_{l+1}, \mathfrak{F}_l, \mathfrak{F}_{l+1}$ for $\mathfrak{U}_0, \mathfrak{U}, \mathfrak{G}_0, \mathfrak{G}$). Hence we have

(a) $\mathfrak{H}_{l+1} \cap \mathfrak{F}_l = \mathfrak{H}_l$,

(b) $\mathfrak{H}_l <_p \mathfrak{F}_l$ implies $\mathfrak{H}_{l+1} <_p \mathfrak{F}_{l+1}$.

Since (a) and (b) obviously do hold in the second case too—note that $\mathfrak{H}_l <_p \mathfrak{F}_l$ implies $\mathfrak{H}_l <_p \mathfrak{F}_{l+1}$ because $\mathfrak{F}_l$ is a free factor of $\mathfrak{F}_{l+1}$—(a) and (b) are satisfied for every l. Finally we define $\mathfrak{H} = \bigcup \mathfrak{H}_l$ and show that $\mathfrak{F}$ and $\mathfrak{H}$ have the required properties.

Proof of (A). By Lemma 1 (b), $\mathfrak{H}_0 <_p \mathfrak{F}_0$. By (b) we get $\mathfrak{H}_l <_p \mathfrak{F}_l$ for every l. This implies $\mathfrak{H} <_p \mathfrak{F}$.

Proof of (B). Given A, B, C, d. Because C is finite, (ii) is satisfied with respect to $\mathfrak{H}_r$ for some r. Since every quadruple τ is counted infinitely many times, there exists an $l > r$ such that $\tau_l = \langle A, B, C, d\rangle$. Because of $\mathfrak{H}_r < \mathfrak{H}_l$, (ii) is satisfied with respect to $\mathfrak{H}_l$ too, and (i) is satisfied with respect to $\mathfrak{H}_l$ because of $\mathfrak{H}_l < \mathfrak{H}$. Hence we are in the first case (above) where the sets $M_1^{(l)}, M_2^{(l)}, M_3^{(l)}$ are in $\mathfrak{H}_{l+1}$ and thus in $\mathfrak{H}$. That is, (a), (b), (c) are satisfied with respect to $\xi^{(l+1)}, \eta_1^{(l+1)}, \eta_2^{(l+1)}$.

Proof of (C). $\alpha^2\beta, \alpha\beta^2 \in \mathfrak{H}$ because of the definition of $\mathfrak{H}_0$, and $\alpha, \beta \notin \mathfrak{H}$ because of Lemma 1 (a) and property (a) above.

LEMMA 4. *There exists a permutation group F on a set D such that the following conditions are satisfied:*

(A) *$K_F(e) = H_F(e)$ for every $e \in E(D)$.*

(B) *$[K_F(e), K_F(e')] = K_F(e \cap e')$ for all $e, e' \in E(D)$.*

(C) *There are $u, v, w \in D$, u, v, w different from each other, and $\sigma, \tau \in F$ such that $\sigma\colon u \to v \to w$ and $\tau\colon w \to u \to v$.*

(Note that assertion (C) of this lemma is weaker than the corresponding assertion of theorem (G).)

Proof. Let $\mathfrak{F}, \mathfrak{H}$ be as in Lemma 3. Let D be the set of the left cosets of $\mathfrak{F} \bmod \mathfrak{H}$. We denote the coset containing x by $\overline{x}$. To every $y \in \mathfrak{F}$ corresponds the permutation y^* of D which maps $\overline{x}$ onto $\overline{yx}$. We have

$$(*) \qquad y^*(\overline{x}) = \overline{z} \quad \text{iff} \quad z^{-1}yx \in \mathfrak{H}.$$

Since the correspondence $y \to y^*$ is a homomorphism, we have for any word $W(y_1, \ldots, y_K)$:

$$(**) \qquad \big(W(y_1^*, \ldots, y_K^*)\big)(\overline{x}) = \overline{z} \quad \text{iff} \quad z^{-1}W(y_1, \ldots, y_K)x \in \mathfrak{H}.$$

Let F be the group of the y^*'s, $y \in \mathfrak{F}$. Then F satisfies (A), (B), (C):

Proof of (A). We have to show that $H_F(e) \subset K_F(e)$ for any $e \in E(D)$. Now, if $e \in E(D)$ and $\varphi \in H_F(e)$, then $\varphi^n \in K_F(e)$ for a suitable $n > 0$.

But assertion (A) of Lemma 3, namely $\mathfrak{H} <_p \mathfrak{F}$, is translated by $(**)$ into the following: If $\varphi \in F$, $a \in D$, $n > 0$, then $\varphi^n(a) = a$ implies $\varphi(a) = a$. Hence $\varphi \in K_F(e)$, q.e.d.

Proof of (B). Given $e, e' \in E(D)$, we have to show that $K_F(e \cap e') \subset [K_F(e), K_F(e')]$. Let $\varphi \in K_F(e \cap e')$ and choose $d \in \mathfrak{F}$ such that $d^* = \varphi$. We denote the set of the cosets corresponding to the elements of a subset X of $\mathfrak{F}$ by $\bar{X}$. Now, picking out one representative of each coset belonging to one of the sets $e - e'$, $e' - e$, $e \cap e'$, we get mutually disjoint finite subsets A, B, C of $\mathfrak{F}$ such that $e = \bar{A} \cup \bar{C}$ and $e' = \bar{B} \cup \bar{C}$. A, B, C, d satisfy the conditions (i), (ii) stated in Lemma 3. (i) expresses that different elements of $A \cup B \cup C$ represent different cosets. For the proof of (ii), let $x \in C$. Then $\bar{x} \in \bar{C} = e \cap e'$ and since $d^* = \varphi \in K_F(e \cap e')$, $d^*(\bar{x}) = \bar{x}$. $(*)$ gives $x^{-1}dx \in \mathfrak{H}$. By Lemma 3 (B) and translation $(**)$ there are $\xi, \eta_1, \eta_2 \in \mathfrak{F}$ such that $\xi^* \in K_F(e)$, $\eta_i^* \in K_F(e')$ and $\psi = d^{*-1}\eta_1^* \xi^* \eta_2^* \in K_F(e)$. Hence $\varphi = d^* = \eta_1^* \xi^* \eta_2^* \psi^{-1} \in [K_F(e), K_F(e')]$, q.e.d.

Proof of (C). Take $\alpha, \beta \in \mathfrak{F}$ according to Lemma 3 (C). Define $u = \bar{\beta}$, $v = \overline{\alpha\beta}$, $w = \overline{\alpha^2\beta}$ and $\sigma = \alpha^*$, $\tau = \beta^*$. Then $w = \bar{1}$ because of $\alpha^2\beta \in \mathfrak{H}$ and $v = \overline{\beta^2}$ because of $\beta^{-2}\alpha\beta = (\alpha\beta^2)^{-1}\alpha^2\beta \in \mathfrak{H}$. Since $\alpha^*\colon \bar{\beta} \to \overline{\alpha\beta} \to \overline{\alpha^2\beta}$ and $\beta^*\colon \bar{1} \to \bar{\beta} \to \overline{\beta^2}$ we get $\sigma\colon u \to v \to w$ and $\tau\colon w \to u \to v$; and since $\alpha, \beta \notin \mathfrak{H}$, it follows that u, v, w are different from each other, q.e.d.

Proof of Theorem (G). Let F, D be as in Lemma 4. Take a denumerable family $\{\langle F_\iota, D_\iota\rangle\}_{\iota \in I}$ of isomorphic copies of $\langle F, D\rangle$, where the D_ι's are mutually disjoint. Then we define $\langle G, C\rangle$ to be the weak direct product of the $\langle F_\iota, D_\iota\rangle$'s; i.e. $C = \bigcup D_\iota$ and the elements of G are those permutations of C which act on every component D_ι as an element of F_ι, being the identity element of F_ι for almost every $\iota \in I$.

Looking at the constructions we used, it is clear that C is denumerable. Then, it is easily seen that the properties (A), (B) of $\langle F, D\rangle$ are carried over to $\langle G, C\rangle$ (in order to get (B) one uses essentially the fact that the considered direct product is the weak one). Moreover, statement (C) of Lemma 4 is translated into statement (C) of theorem (G). For, if $e \in E(C)$ then $e \cap D_\iota = 0$ for some $\iota \in I$. Now, let $\sigma_\iota, \tau_\iota \in F_\iota$ and $u, v, w \in D_\iota$ according to Lemma 4 (C), and let σ, τ be those elements of G which coincide with σ_ι, τ_ι on D_ι and which are the identity on the remaining components. Then $\sigma, \tau \in K_G(e)$ and $\sigma\colon u \to v \to w$, $\tau\colon w \to u \to v$, q.e.d.

References

[1] A. Fraenkel, *Über eine abgeschwächte Fassung des Auswahlaxioms*, Journ. Symb. Log. 2 (1937), pp. 1-25.

[2] L. A. Henkin, *Problems*, Journ. Symb. Log. 17 (1952), p. 160.

[3] J. Łoś and C. Ryll-Nardzewski, *Effectiveness of the representation theory for Boolean algebras*, Fund. Math. 41 (1954-55), pp. 49-56.

[4] A. Mostowski, *Über die Unabhängigkeit des Wohlordnungssatzes vom Ordnungsprinzip*, Fund. Math. 32 (1939), pp. 201-252.

[5] E. Specker, *Zur Axiomatik der Mengenlehre*, Zeitschr. f. math. Logik und Grundlagen d. Math. 3 (1957), pp. 193-199.

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