Some theorems on vector spaces and the axiom of choice *

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I. Introduction, definitions, and notation. In the study of the theory of infinite dimensional vector spaces there are many theorems, especially those dealing with the existence and cardinality of bases and the relative complementation of subspaces and subsets, all known proofs of which require some form of the axiom of choice. In the case of such a theorem it is natural to ask whether or not the use of the axiom of choice is essential, and if it is essential, is its full strength essential; i.e., is the theorem equivalent to the axiom of choice or to some weakened version of it. In this work we address ourselves to these questions concerning five such propositions to be listed below. We also consider these propositions in the context of axiomatic dependence (abstract linear dependence relations).

The notation is the usual set-theoretic notation accompanying the use of ϵ, $\cup$, $\cap$, and $\setminus$ for the notions of membership, union, intersection, and relative complement. The set of all subsets of a given set X (the power set of X) is denoted by $P(X)$. The empty set is denoted by $\emptyset$.

In the statements of the proposition with which we are concerned the vector space is denoted by V. At times the class of vector spaces over which V ranges is restricted to certain special classes; e.g. the class of all vector spaces over the reals, over the two-element field, over finite fields, etc.

The first proposition with which we are concerned is the Hamel Basis Theorem.

PROPOSITION 1. *The vector space V possesses a basis.*

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The second proposition is a strengthened version of the first.

PROPOSITION 2. *If the subset G of the vector space V generates V then there is a subset of G which is a basis.*

The next three propositions have to do with various forms of relative complementation. If $X \subset V$ then by $[X]$ is denoted the subspace generated by X, or equivalently, the least subspace containing X.

PROPOSITION 3. *For each subspace S of the vector space V there is a subspace S' of V such that $S \cap S' = \{0\}$ and $[S \cup S'] = V$.*

PROPOSITION 4. *If B is a basis for the vector space V then for each subspace S of V there is a subset $\widetilde{S}$ of B such that $S \cap [\widetilde{S}] = \{0\}$ and $[S \cup \widetilde{S}] = V$.*

PROPOSITION 5. *If B is a basis for the vector space V then for each independent subset X of V there is a subset X^* of B such that $X \cup X^*$ is a basis for V.*

Without loss of generality it may be assumed that $X \cap X^* = \emptyset$ for if not X^* may be replaced by $X^* \setminus X$. In the sequel we assume that X and X^* are disjoint.

These theorems have been improved and generalized in many ways by various authors. Some have stated the conclusion functionally and shown that the function can be made to satisfy auxiliary conditions; others have shown that the propositions remain true in more general structures, e.g. generalized vector spaces, exchange lattices, etc. The reader interested in pursuing the more general aspects of these propositions would consult [1]-[6], [8]-[10], and [12]-[14] which contain the results most pertinent to this discussion, and also references to further literature.

It is convenient to define here precisely what we mean by the axiom of choice and the weaker versions of it which arise in the discussion.

By the *axiom of choice* we mean the following:

If $\mathcal{T}$ is any non-empty set of non-empty sets then there exists a function f defined on $\mathcal{T}$ such that for each set $T \in \mathcal{T}$, $f(T) \in \mathcal{T}$.

It is well known that if we require that $\mathcal{T}$ consist of disjoint non-empty sets or that $\mathcal{T}$ be of the form $\mathcal{T} = P(X) \setminus \{\emptyset\}$ for some set X, we obtain equivalent formulations of the axiom of choice.

By imposing the restriction that $\mathcal{T}$ consist of only finite sets we obtain the *axiom of choice for finite sets*; namely,

If $\mathcal{T}$ is such a family of non-empty sets that for each element $T \in \mathcal{T}$, T is a finite set then there is a function f defined on $\mathcal{T}$ such that $f(T) \in T$ for each element $T \in \mathcal{T}$.

The axiom of choice for finite sets is weaker than the axiom of choice in the sense that there are models of set theory in which the axiom of choice for finite sets holds while the axiom of choice does not. This is the case for example in the model of Mostowski [10] since the proposition that every set may be linearly ordered implies the axiom of choice for finite sets (viz. e.g. Sierpiński [14], pp. 412-3).

Many authors have considered weakened versions of the axiom of choice obtained from the full axiom by putting various restrictions on the domain $\mathcal{T}$ of the choice function. In this work only the above mentioned weakened form of this type is needed. Another less studied method for obtaining weakened versions of the axiom of choice is to put less stringent conditions of the range of the choice function (1). We need here one family of such weakened versions of the axiom of choice.

Let n by any positive integer then by FSn is meant the following proposition:

FSn. *If $\mathcal{T}$ is any non-empty set of non-empty sets then there is a function f defined on $\mathcal{T}$ such that for each element $T \in \mathcal{T}$, $f(T)$ is a finite non-empty subset of T and the number of elements of $f(T)$ is relatively prime to n.*

FS1 requires only that the function yield a finite subset of the argument. It is clear that FSn and the axiom of choice for finite sets together imply the axiom of choice. In an intuitive sense FS1 and the axiom of choice for finite sets are complementary parts of the axiom of choice.

We define now certain vector spaces which repeatedly play important roles in the forthcoming discussion.

Let X be an arbitrary set and F any field; by $L(X, F)$ we denote the vector space over F which consists of all finite linear forms with indeterminants from X and coefficients from F. Symbolically:

$$L(X, F) = \{f_1x_1 + f_2x_2 + \ldots + f_nx_n\colon\ f_i \in F,\ x_i \in X,\ 0 < i \leqslant n,\ n = 1, 2, \ldots\}.$$

The usual conventions are assumed about the order of factors, zero coefficients, addition and scalar multiplication (2). The form $1x$ is written simply x and is identified with the corresponding element of X. Thus X may be considered as a subset of $L(X, F)$, and so considered it is clearly a basis for $L(X, F)$.

$L_0(X, F)$ denotes the subspaces of $L(X, F)$ in which the sum of the coefficients of each linear form are zero. Symbolically,

$$L_0(X, F) = \{f_1x_1 + f_2x_2 + \ldots + f_nx_n \in L(X, F)\colon\ f_1 + f_2 + \ldots + f_n = 0\}.$$

If $x \in X$ it is clear that $B_x = \{x - y\colon\ y \in X \setminus \{x\}\}$ is a basis for $L_0(X, F)$. Further if $Y \subset X$ it is clear that (after the obvious indetifications)

(1) (Added in proof.) See A. Lévy, Fund. Math. 50 (1962), pp. 475-483.

(2) This can, of course, all be made quite precise by defining $L(X, F)$ to be the appropriate subset of the set of all functions from X to F, etc.

$L_0(Y, F)$ is a subspace of $L(Y, F)$, $L_0(X, F)$, and $L(X, F)$; further, $L(Y, F)$ is a subspace of $L(X, F)$.

When the field under consideration is clear from the context the reference to it in the above notation is ommitted; thus $L_0(Y, F) = L_0(Y)$.

If $\{V_a\colon a \in A\}$ is a family of vector spaces over the same field F, then their weak direct product, denoted by $\boldsymbol{\Pi}\{V_a\colon a \in A\}$, is the vector space over F defined by

$$\boldsymbol{\Pi}\{V_a\colon a \in A\} = \{v_1 + v_2 + \ldots + v_n\colon v_i \in V_{a_i},\ a_i \in A,\ 0 < i \leqslant n,\ n = 1, 2, \ldots\}.$$

Again the usual conventions are made concerning zeroes, order of factors, etc. (3).

II. Main results. We begin this section with some lemmas which are not only helpful in the proof of the second theorem, but which are interesting in themselves and also illustrate necessary techniques.

LEMMA 1. *If for some field F in which $1+1 \neq 0$ Proposition 2 holds for all vector spaces over F, then the axiom of choice for finite sets holds.*

Proof. Let $\mathcal{T}_1$ be a given non-empty set of finite non-empty sets. Let $X = \bigcup\{T\colon T \in \mathcal{T}_1\}$ and let $\mathcal{T} = \{\{\langle t, T\rangle\colon t \in T\}\colon T \in P(X)$ where T is finite$\}$, then $\mathcal{T}$ is a family of disjoint finite sets and for each $T \in \mathcal{T}_1$ there is an effective correspondence to an effactively isomorphic element of $\mathcal{T}$.

We now construct a choice function defined for the elements of $\mathcal{T}$; because of the effective correspondence between elements of $\mathcal{T}_1$ and $\mathcal{T}$ and because of the effective isomorphism between corresponding sets, the choice function on $\mathcal{T}$ uniquely determines a choice function on $\mathcal{T}_1$. A choice function on $\mathcal{T}$ is uniquely determined through iteration by a function π defined on $\mathcal{T}$ which satisfies the following conditions:

(1) If T is a one element set, $\pi(T)$ equals the single element.

(2) If T has more than one element then $\emptyset \neq \pi(T) \neq T$ and $\pi(T) \subset T$.

We now define the function π. Let $T \in \mathcal{T}$ and suppose T has $n \geqslant 2$ elements. We consider the vector space $V = \boldsymbol{\Pi}\{L_0(T)\colon T \in \mathcal{T}\}$. Since $G_T = \{x - y\colon x, y \in T,\ x \neq y\}$ is a generating set for $L_0(T)$, it follows that $G = \bigcup\{G_T\colon T \in \mathcal{T}\}$ is a generating set for V. Since, by hypothesis, Proposition 2 holds for V, there is a basis B contained in G. It is easy to see that $B_T = B \cap G_T$ is a basis for $L_0(T)$.

The space $L_0(T)$ is an $n-1$ dimensional vector space since, for any $x \in T$, the $n-1$ element set $\{x - y\colon x \neq y \in T\}$ is a basis. Since the dimension is finite, we know without the aid of any choice principle that every basis has the same number of elements; hence B_T has $n-1$ elements. Since each element of B_T is a pair of elements of T, there are $2(n-1)$ occurences, counting according to multiplicity, of elements of T in some element of B_T. If $n \neq 1, 2$, then n does not divide $2(n-1)$ hence not all the elements of T occur with the same multiplicity. In this case define $\pi(T)$ to be those elements of T which occur with minimal frequency. If $n = 2$, then B_T consists of one term $x - y$. Since $1+1 \neq 0$, it follows that $+1 \neq -1$, and we can define $\pi(T)$ to be the element which has coefficient $+1$. If $n = 1$ we define $\pi(T)$ to be the single element of T. We have constructed a function which satisfies Conditions (1) and (2), and the lemma follows.

COROLLARY. *If F is a field of characteristic 2 and Priopostion 2 holds for all vector spaces over F, then: if $\mathcal{T}$ is any non-empty set of sets each having an odd finite number of elements then there is a choice function on $\mathcal{T}$, and if $\mathcal{T}$ is any non-empty set of finite non-empty sets there is a function which assigns to each $T \in \mathcal{T}$ a one or two element subset.*

Proof. In the above proof of the lemma if n is odd, then either $\pi(T)$ or $T \setminus \pi(T)$ is an odd set, hence we may assume that $\pi(T)$ is an odd subset of T. In this manner we avoid the possibility of obtaining a two-element set in the iteration of π in the case T has an odd number of elements; since the case $n = 2$ is the only time the fact that $1+1 \neq 0$ is used in the proof of the lemma, the first part of the corollary follows. The second part of the corollary is immediate since the function π always yields a proper non-empty subset of a given finite set if it has more than 2 elements.

LEMMA 2. *If for some field F Proposition 3 holds for every vector space over F, then FS1 holds.*

Proof. Let $\mathcal{T}$ be a non-empty set of non-empty sets; without loss of generality we assume they are disjoint. Consider the vector space V defined by

$$V = \boldsymbol{\Pi}\{L(T)\colon T \in \mathcal{T}\},$$

and a subspace V_0 of it defined by

$$V_0 = \boldsymbol{\Pi}\{L_0(T)\colon T \in \mathcal{T}\}.$$

According to Proposition 3, there exists a subspace V_0' such that $V_0 \cap V_0' = \{0\}$ and $[V_0 \cup V_0'] = V$; thus, every element in V can be expressed uniquely as a sum $v + v'$ with $v \in V_0$ and $v' \in V_0'$.

We construct now a function π defined on $\mathcal{T}$ satisfying the conditions of FS1. Let T be an arbitrary element of $\mathcal{T}$ and s and t be elements of T regarded as elements of V. We can write in a unique manner $s = v + v'$ and $t = w + w'$ where $v, w \in V_0$ and $v', w' \in V_0'$. Since v, w, and $s - t$ are in the subspace V_0, it follows that $v' - w'$ is also. On the

(3) Here again this can be made rigorous by defining the weak direct product as the appropriate set of functions from A into $\cup\{V_a\colon a \in A\}$.

other hand, since v' and w' are in the subspace V_0', it follows that their difference is also. Hence,

$$(v'-w') \in V_0 \cap V_0' = \{0\}\,.$$

Thus for each set T of $\mathcal{T}$ there is a well-defined element of V_0', say w_T, such that for each $t \in T$ there is an element $v_t \in V_0$ such that $t = w_T + v_t$. It follows that the sum of the coefficients of the terms from T in the form w_T is one. If we define $\pi(T)$ by

$$\pi(T) = \{t\colon\ t \text{ has a nonzero coefficient in } w_T\}$$

it is clear that $\pi(T)$ is a non-empty finite subset of T. Since $\pi(T)$ satisfies the conditions of FS1, the lemma is established.

By putting certain restriction on the field we are able to arrive at a stronger conclusion. The next lemma is an example along these lines.

LEMMA 3. *If for some field F of finite characteristic p, which is orderable as a set, Proposition 3 holds for all vector space over F then* FSp *holds.*

Proof. The proof of Lemma 3 parallels that of Lemma 2 until we obtain the element w_T. Suppose f_i, $i = 1, 2, \ldots, k$, are the non-zero coefficients occuring in w_T as coefficients of elements of T, and n_i are their respective frequencies of occurence as such coefficients. We know that the sum $n_1f_1+n_2f_2+\ldots+n_kf_k = 1$. It follows that not all the n_i are divisible by p. Since the set of f_i is finite, the order on the field induces a well-order on the f_i. Let f_j be the least f_i for which n_i is not a multiple of p. If we now define π by $\pi(T) = \{t \in T\colon$ the coefficient of t in w_T is $f_j\}$, it is clear that π satisfies FSp. Lemma 3 is thus established.

LEMMA 4. *If for some field F Proposition 5 holds for every vector space over F then the axiom of choice for finite sets holds.*

Proof. Let $\mathcal{T}$ be a given non-empty set of finite non-empty sets. Without loss of generality we may assume that the elements of $\mathcal{T}$ are disjoint. Let $B = \bigcup\{T\colon\ T \in \mathcal{T}\}$. Let the vector space V be defined by $V = L(B)$. Let $X = \{t_1+t_2+\ldots+t_k\colon\ \{t_1, t_2, \ldots, t_k\} \in \mathcal{T}\}$, then regarding X as a subset of V it is clear that X is an independent subset. On applying Proposition 5 to the independent set X with respect to the basis B we obtain a set X^* disjoint from X such that $X^* \subset B$ and $X \cup X^*$ is a basis. It is not difficult to verify that, regarding $L(T)$ as a subspace of V, $L(T) \cap (X \cup X^*)$ is a basis of $L(T)$. If $T = \{t_1, t_2, \ldots, t_k\}$, then any basis of $L(T)$ must have k elements. Since $L(T) \cap X = \{t_1+t_2+\ldots+t_k\}$, it follows that $L(T) \cap X^*$ has $k-1$ elements. However $(L(T) \cap X^*) \subset (L(T) \cap B) = T$ thus the set $T \setminus (L(T) \cap X^*)$ contains a single element. If we define f by $f(T) =$ the unique element in the set $T\setminus(L(T) \cap X^*)$, then f is a choice function on $\mathcal{T}$. Thus the axiom of choice for finite sets holds and the lemma is proved.

We are now in a position to prove Theorem 1.

THEOREM 1. *The following are equivalent:*

(1) *The axiom of choice.*

(2) *For some field F Propositions 1 and 5 hold for all vector spaces over F.*

(3) *For some field F Propositions 3 and 5 hold for all vector spaces over F.*

Proof. We first show that (2) implies (3). It is sufficient to show that Propositions 1 and 5 imply Proposition 3 when restricted to vector spaces over F. Let S be a subspace of V. Since S is itself a vector space over F, it has a basis, say X. X is then an independent subset of V. Again by Proposition 1, V has a basis B. By now applying Proposition 5 to V for the basis B and the independent set X we obtain a set X^* such that $X \cap X^* = \emptyset$ and $X \cup X^*$ is a basis of V. If we define S' by $S' = [X^*]$ it is easy to show that $S \cap S' = \{0\}$ and also that $[S \cup S'] = V$; i.e., S' satisfies the conditions of Proposition 3.

We now show that (3) implies (1). The statement to be proved follows immediately from Lemmas 2 and 4.

Since the proof that (1) implies (2) is well known, the truth of the theorem is established.

THEOREM 2. *The following are equivalent:*

(1) *The axiom of choice.*

(2) *For some field F such that either $1+1 \neq 0$ or $1+1 = 0$ and F is orderable as a set, Propositions 2 and 3 hold for all vector spaces over F.*

Proof. For the case in which $1+1 \neq 0$ it follows immediately from Lemmas 1 and 2 that (2) implies (1). In the other case Lemma 3 yields a function which is a finite odd subset and by the corollary to Lemma 1 this situation may be handled. The reverse implication is to be found in the literature cited, e.g., [3].

THEOREM 3. *The following are equivalent:*

(1) *The axiom of choice.*

(2) *For some field F Proposition 4 holds for all vector spaces over F.*

Proof. Let $\mathcal{T}$ be a given non-empty set of non-empty sets which, without loss of generality, are disjoint. As in the proof of Lemma 2, we define V and V_0 by

$$V = \boldsymbol{\Pi}\{L(T)\colon\ T \in \mathcal{T}\} \quad \text{and} \quad V_0 = \boldsymbol{\Pi}\{L_0(T)\colon\ T \in \mathcal{T}\}\,.$$

It is clear that V_0 is a subspace of V and that $B = \bigcup\{T\colon\ T \in \mathcal{T}\}$ is a basis of V. We apply now Proposition 4 to the subspace V_0 with respect to the basis B and obtain a subset $\widetilde{V}_0$ of B. Let $T \in \mathcal{T}$; then $W = T \cap \widetilde{V}_0$ has at most one element, for let s, t belong to W then $(s-t) \in ([\widetilde{V}_0] \cap V_0) = \{0\}$, and hence $s = t$. On the other hand W is non-empty since for any element $t \in T$, $t = v + \widetilde{v}$ where $v \in V_0$ and $\widetilde{v} \in [\widetilde{V}_0]$. The sum of the coefficients of terms from T in v is zero since $v \in V_0$, hence the sum of the coefficients of terms from T in $\widetilde{v}$ must be one. It follows that $\widetilde{V}_0$ must contain terms from T and hence that $W \neq \emptyset$. We may now define $f(T)$ to be the unique element of W. Since f is clearly a choice function, the truth of the theorem is established.

Before investigating what additional strength is acquired by the assumption of the truth of Propositions 1-5 in the case of generalized vector spaces, it is perhaps appropriate to make a few comments on the results thus far obtained.

It can be noticed, particularly in Lemma 1, and Theorem 2, that vector spaces over fields of characteristic 2 seem to be a separate case from those over fields with other characteristics. It would be interesting to know if there is something inherently different in that case or if it is only an accident of the methods of proof.

It is also interesting that while both Propositions 2 and 5 imply the axiom of choice for finite sets (Lemmas 1 and 4) at the same time their conjunction implies the axiom of choice. This would indicate that one of Lemmas 1 or 4 is unnecessarily weak. It may well be that it is Lemma 1, in as much as Dr. J. D. Halpern has communicated to the author a proof that the truth of Proposition 2 in all vector spaces over all infinite fields of a given characteristic is equivalent to the axiom of choice.

If Proposition 3 is assumed to hold over any sufficiently large class of vector spaces, for example all vector spaces over finite prime fields, then we can obtain as a consequence the conjunction of FSp for all primes p. The relationship between this conjunction and the axiom of choice is still open, but the difference between them is not great. The author plans to elucidate elsewhere some of the more interesting consequences of the propositions FSp and various conjunctions thereof.

III. Generalized vector spaces. The characterization of abstract dependence used in this section is (except for minor modifications) the same as the system used in [3], [5], [8], and [9]. This system is in fact equivalent to many of the other seemingly different systems, but we do not concern ourselves with that problem here.

Let V be any set, X, Y, and Z subsets of V, and x and y elements of V. A relation $\Subset$ between the subsets of V is an abstract linear dependence relation if it satisfies the following:

L1. $Y \subset X$ implies $Y \Subset X$.

L2. $X_\alpha \Subset Y$ for every $\alpha \in A$ implies $\bigcup\{X_\alpha\colon \alpha \in A\} \Subset Y$.

L3. $X \Subset Y$ and $Y \Subset Z$ imply $X \Subset Z$.

L4. $\{x\} \Subset X \cup \{y\}$ and $\{x\} \not\Subset X$ imply $\{y\} \Subset X \cup \{x\}$.

L5. $\{x\} \Subset X$ implies there exists a finite subset $F \subset X$ such that $\{x\} \Subset F$.

The ordered pair $\langle V, \Subset\rangle$ is called a *generalized vector space*. When only one dependence relation is being considered on the set V, this couple is denoted simply by V.

In the usual interpretation $X \Subset Y$ is read X depends on Y, or X is dependent upon Y. A set X is called *independent* if for no $x \in X$ is it true that $\{x\} \Subset X\setminus\{x\}$; X is called *dependent* otherwise. A basis is an independent set B for which $V \Subset B$. A subspace is a set S for which $\{x\} \Subset S$ implies $x \in S$.

For the proofs that Propositions 1-5 hold in generalized vector spaces the reader should consult the papers cited earlier, especially [3], [5], and [8].

We do not develop any of the theory of generalized vector spaces here, but other than the fact that Propositions 1-5 hold, the results we use are easily verified from the L1-L5.

THEOREM 4. *The following are equivalent:*

(1) *The axiom of choice.*

(2) *Proposition* 1 *holds for generalized vector spaces.*

Proof. We prove here only that (2) implies (1). Let $\mathcal{T}$ be a given non-empty set of sets which we suppose are disjoint and non-empty. Let $V = \bigcup\{T\colon T \in \mathcal{T}\}$. We define a dependence relation on V in the following manner: For all $X, Y \in P(V)$, $X \Subset Y$ *if and only if for each* $T \in \mathcal{T}$, $(X \cap T) \neq \emptyset$ *implies* $(Y \cap T) \neq \emptyset$.

L1-L3 follow immediately from the definition. We now show that $\Subset$ satisfies L4. Let $\{x\} \Subset X \cup \{y\}$ and $\{x\} \not\Subset X$. Since $x \in V$, there is a unique $T \in \mathcal{T}$ for which $x \in T$. Since $\{x\} \cap T \neq \emptyset$, it follows that $(X \cup \{y\}) \cap T \neq \emptyset$. Since $\{x\} \cap T' = \emptyset$ for all $T' \in \mathcal{T}\setminus\{T\}$ and $\{x\} \not\Subset X$, it follows that $X \cap T = \emptyset$. Therefore $y \in T$. It follows that $\{y\} \Subset X \cup \{x\}$. L5 can be verified as follows: since $\{x\} \Subset X$, it follows that $X \cap T \neq \emptyset$ where $x \in T$. Let $y \in (X \cap T)$; then by putting $F = \{y\}$ it is clear that L5 is satisfied.

By Proposition 1 there is a basis B for V. Since $V \Subset B$, it is clear that for all $T \in \mathcal{T}$, $B \cap T \neq \emptyset$. Let $s, t \in (B \cap T)$. Then since B is independent, it follows that $\{t\} \not\Subset B\setminus\{t\}$. Since $T' \cap \{t\} = \emptyset$ for $T' \neq T$ and $T' \in \mathcal{T}$, we see that $(B\setminus\{t\}) \cap T = \emptyset$. Since $s \in (B \cap T)$, it follows that $s = t$. Thus we can define $f(T)$ to be the uniquely defined element of T

in $B \cap T$. This function is clearly a choice function for $\mathcal{T}$. The desired implications is established.

It is clear that we can substitute Proposition 2 for Proposition 1 in Theorem 4.

THEOREM 5. *The following are equivalent:*

(1) *The axiom of choice.*

(2) *Proposition 3 holds for generalized vector spaces.*

Proof. We prove only that (2) implies (1). As mentioned in the introduction it is sufficient to construct a choice function for an arbitrary set of the form $P(X)\backslash\emptyset$ for some set X. If p is any prime number we denote by p^* the Galois Field with p elements. For any set $Y \in P(X)$ we denote by $L(Y, p^*)^*$ the vector space defined by

$$L(Y, p^*)^* = \{\langle x, p, Y\rangle: x \in L(Y, p^*)\}$$

with addition and scalar multiplication on the first member of the triple. Thus $L(Y, p^*)^* \cap L(Y', q^*)^* = \emptyset$ unless $Y = Y'$ and $p = q$. $L_0(Y, p^*)^*$ denotes the subspace of $L(Y, p^*)^*$ which corresponds to $L_0(Y, p^*)$.

By defining the appropriate generalized vector space we are able to simultaneously apply Proposition 3 to all the spaces $L(Y, p^*)^*$ for $Y \in P(X)$ and p a prime. This process enables us to define on $P(X)$ a function which simultaneously satisfies FSp for all primes p. Such a function however is a choice function on $P(X)$. The details are given below.

Let $V = \bigcup\{L(Y, p^*)^*: p \text{ a prime}, Y \in P(X)\}$. Let R and S be subsets of V. We define the relation $\Subset$ by $R \Subset S$ if and only if for every prime p, and every set $Y \in P(X)$, $R \cap L(Y, p^*)^*$ is dependent on $S \cap L(Y, p^*)^*$ in the vector space $L(Y, p^*)^*$: more symbolically,

$$R \Subset S \text{ if and only if } R \cap L(Y, p^*)^* \subset [S \cap L(Y, p^*)^*] \text{ for all primes } p \text{ and all } Y \in P(X).$$

The verification of L1-L3 and L5 present no difficulty. To verify L4 we assume that $\{r\} \Subset R \cup \{s\}$ and $\{r\} \not\Subset R$. It follows that for precisely one pair p, Y it is true that $r \in L(Y, p^*)^*$ and that for the identical p and Y, $s \in L(Y, p^*)^*$. Since $L(Y, p^*)^*$ is a vector space, it satisfies L4 and r and s may be exchanged in it. Since $\{s\} \cap L(Z, q^*)^* = \emptyset$ unless $q = p$ and $Z = Y$, it follows that $\{s\} \Subset R \cup \{r\}$. Thus we see that V is a generalized vector space.

Let $V_0 = \cup\{L_0(Y, p^*)^*: Y \in P(X) \text{ and } p \text{ prime}\}$. It is straightforward to verify that V_0 is a subspace of the generalized vector space V, when the dependence relation on V_0 is taken to be that induced upon it as a subset of V. We now apply Proposition 3 to the subspace V_0 and obtain a new subspace V_0' such that $V_0 \cap V_0' = [\emptyset]$ and $[V_0 \cup V_0'] = V$, where $[\emptyset]$, the subspace generated by the empty set, is given by

$$[\emptyset] = \{\langle 0, Y, p\rangle: p \text{ a prime } Y \in P(X)\}.$$

We first show that $Z = V_0' \cap L(Y, p^*)^*$ is at most a one dimensional subspace of $L(Y, p^*)^*$. Let $r, s \in Z$; and let f be any element of the field p^*, then $\{fr\} \Subset \{r\} \Subset V_0'$ and since V_0' is a subspace, it follows that $fr \in V_0'$. Thus if $r \in Z$ and $f \in p^*$, $fr \in Z$. Therefore after possible multiplication by the appropriate scalar we may suppose that the sum of the coefficients of r and s are the same. Since $\{r-s\} \Subset \{r, s\} \Subset V_0'$ and V_0' is a subspace, it follows that $r-s \in V_0'$. But the sum of the coefficients of the terms in $r-s$ is zero, hence $r-s \in L_0(Y, p^*)^*$. Since $V_0 \cap V_0' \cap L(Y, p^*)^* = \{\langle 0, Y, p\rangle\}$, it follows that $r = s$ and $V_0' \cap L(Y, p^*)^*$ is at most one dimensional.

On the other hand, if $t' \in Y$ then $t = \langle t', Y, p\rangle$ belongs to $L(Y, p^*)^*$, but $\{t\} \not\Subset V_0$ since it is not in $[V_0 \cap L(Y, p^*)^*]$, while $\{t\} \Subset (V_0 \cup V_0')$. From the definition of $\Subset$ we have that $t \in [(V_0 \cup V_0') \cap L(Y, p^*)^*]$, and therefore $V_0' \cap L(Y, p^*)^*$ is non-empty. It follows that $V_0' \cap L(Y, p^*)^*$ is a one-dimensional subspace of $L(Y, p^*)^*$ which is not contained in $L_0(Y, p^*)^*$.

Let r be the unique element of $V_0' \cap L(Y, p^*)^*$ for which the sum of the coefficients is one. Let n_i be the number of terms of r which have i for their coefficient, $i = 1, 2, \ldots, p-1$. Since the sum $1n_1 + 2n_2 + \ldots + (p-1)n_{(p-1)} = 1$, there is a least value of i for which n_i is not divisible by p; define $\pi(Y, p)$ to be the terms having that coefficient. It is clear that $\pi(Y, p)$ is a finite subset of Y the number of elements of which is relatively prime to p.

Let Y be any subset of X with more than one element and let p be the least prime dividing the cardinal of Y (if Y is infinite, $p = 2$). Either $\pi(Y, p)$ is a one element set or there is a least prime p' dividing the cardinal of the finite set $\pi(Y, p)$. The set $\pi(\pi(Y, p), p')$ is a proper finite subset of $\pi(Y, p)$. We continue in this well-defined manner and after a finite number of steps we obtain a single element subset of Y. It is now clear that the function π leads in a well-defined manner to a choice function for $P(X)$, and the proof of the theorem is complete.

We have now shown that any of Propositions 1-4 for generalized vector spaces is equivalent to the axiom of choice. The problem with applications of Proposition 5 seems to be in the restricted nature of its domain, independent sets. It is relatively easy to create proper subspaces of various vector spaces, but it appears that it is necessary to name some particular element if one is to name an advantageous non-maximal independent set.

IV. Concluding remarks. The fact that the truth of any of Propositions 2-5 for all vector spaces over some field is independent from the axioms of set theory without the axiom of choice is now apparent, for: (a) Proposition 4 is equivalent to the axiom of choice, (b) Proposition 3 is independent since, according to Lemma 2, it, together with the axiom of choice for finite sets implies the axiom of choice, while the axiom of choice for finite sets alone does not, [11]; (c) Propositions 2 and 5 since they each imply (the field having characteristic 2 being excluded for Proposition 2) the axiom of choice for finite sets, and this is known to be independent of the usual axioms of set theory ([6], pp. 53-53); (d) Proposition 2 in the case the field has characteristic 2 implies the axiom of choice for families of finite sets of odd cardinal and this can be shown to be independent of the usual axioms of set theory.

By the axioms of set theory is meant any of the common systems which possess the axiom of foundation, for example, that used by Mostowski in [11]. All the work done in this paper can be carried out in this system. By a proposition being independent of an axiom system is meant that there is a model for the system in which the proposition does not hold.

There seems to be a lack of information relating the axiom of choice with the proposition which states that any two bases have the same cardinal. It is hoped that this void can soon be filled.

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