The usefulness of theorem 5 is not limited to the systems described in theorem 7. For example, a well-ordered set is a system $(X, \leqslant)$ satisfying

(WO$_1$) $(X_1 \leqslant)$ is simply ordered,

(WO$_2$) $\bigwedge Y \subseteq X \big[\bigvee x(x \in Y) \to \bigvee y\big(y \in Y \wedge \bigwedge z[z \in Y \to y \leqslant z]\big)\big]$.

Axiom WO$_1$ is equivalent to a set of sentences possessing the nesting property, as shown above, and axiom WO$_2$ is a *second-order* sentence preserved by every equivalence relation on X which preserves the order. A fortiori conditions (1) and (2) of theorem 5 must hold. This implies that *a well-ordered set may be represented as a subdirect product of properly irreducible well-ordered sets.* One may show without difficulty that the irreducible factors are isomorphic to $(\{0, 1\}, \leqslant)$.

References

[1] A. Adam, *On the definitions of direct product in universal algebras*, Publ. Math. Debrecen 6 (1959), pp. 303-310.

[2] M. Benado, *Bemerkungen zur Theorie der Vielverbande, I, Über einen Satz von V. Glivenko*, Math. Nachr. 20 (1959), pp. 1-16.

[3] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Coll. Publ., Revised Edition (1948).

[4] N. Chomsky and G. Miller, *Finite state languages*, Information and Control 1 (1958), pp. 91-112.

[5] M. Dresher and O. Ore, *Theory of multigroups*, Amer. J. Math. 60 (1938), pp. 705-733.

[6] J. Hartmanis, *Generalized partitions and lattice embedding theorems. Lattice theory*, Proc. of Symposia in Pure Math. of the Amer. Math. Soc. 2 (1961), pp. 23-30.

[7] J. Hashimoto, *Direct, subdirect decompositions and congruence relations*, Osaka Math. J. 9 (1957), pp. 87-112.

[8] V. Gluskov, *Abstract theory of automata* (Russian), Uspehi Mat. Nauk 16 (1961), No. 5 (101), pp. 3-62.

[9] W. Krull, *Subdirekte Summendarstellung von Integritätsbereichen*, Math. Ziet. 52 (1950).

[10] R. Lyndon, *Properties preserved in subdirect products*, Pac. J. Math. 9 (1959), pp. 155-164.

[11] M. McPhee, *Axioms for betweenness*, J. London Math. Soc. 37 (1962), pp. 112-116.

[12] G. Pickert, *Bemerkungen zum Homomorphiebegriff*, Math. Zeit. 53 (1950), pp. 375-386.

[13] W. Prenowitz, *A contemporary approach to classical geometry*, The American Mathematical Monthly, Vol. 68, No. 1, Part II (1961).

Reçu par la Rédaction le 12. 3. 1964

Closed subgroups of locally compact Abelian groups

by

K. A. Ross (Rochester)

Let G be an Abelian group, and let $\mathfrak{O}_1$ and $\mathfrak{O}_2$ be two topologies on G such that $\mathfrak{O}_1 \underset{\neq}{\supset} \mathfrak{O}_2$ and $(G, \mathfrak{O}_1)$ and $(G, \mathfrak{O}_2)$ are locally compact topological groups. E. Hewitt [1] has proved that there is an $\mathfrak{O}_1$-continuous character on G that is $\mathfrak{O}_2$-discontinuous. We submit here an outline of a somewhat shorter proof of this result based on an observation about closed subgroups. We then make some further remarks about closed subgroups.

We first given an alternative proof of Lemma (2.1) in [1]:

Let R denote the additive group of real numbers with the usual topology, and let $(R, \mathfrak{O})$ be a locally compact group such that $\mathfrak{O}$ is strictly stronger than the usual topology of R. Then $\mathfrak{O}$ is the discrete topology.

Proof. Let φ denote the identity mapping of $(R, \mathfrak{O})$ onto R; φ is clearly continuous. Let C be the component of the identity in $(R, \mathfrak{O})$. If $C = R$, then $(R, \mathfrak{O})$ is σ-compact and (5.29) [2] shows that φ is a homeomorphism, contrary to our hypothesis. Hence $\varphi(C)$ is a proper connected subgroup of R in the usual topology. Therefore $\varphi(C) = \{0\}$, $C = \{0\}$, and $(R, \mathfrak{O})$ is totally disconnected. By Theorem (7.7) [2], $(R, \mathfrak{O})$ contains a compact open subgroup H. Since $\varphi(H)$ is a compact subgroup of R in the usual topology, we have $\varphi(H) = \{0\}$ and $H = \{0\}$. Consequently, $\{0\}$ is open in $(R, \mathfrak{O})$ and $\mathfrak{O}$ is discrete.

Hewitt's theorem follows from the following lemma.

LEMMA 1. *Let G, $\mathfrak{O}_1$, and $\mathfrak{O}_2$ be as before. There exists a subgroup H of G that is $\mathfrak{O}_1$-closed but not $\mathfrak{O}_2$-closed.*

Proof. Let φ be the [continuous] identity mapping of $(G, \mathfrak{O}_1)$ onto $(G, \mathfrak{O}_2)$. Arguing as in the proof of Theorem (3.3) [1] and noting that invoking Theorem (2.2) [1] is unnecessary, we find that there is a subgroup J of G such that the topology $\mathfrak{O}_1$ on J is strictly stronger than the topology $\mathfrak{O}_2$ on J, and such that either

(1) $(J, \mathfrak{O}_2)$ is topologically isomorphic with R,

or

(2) $(J, \mathfrak{O}_2)$ is compact.

Suppose that (1) holds. By Lemma (2.1) [1], $(J, \mathfrak{O}_1)$ is a discrete group. If H is a proper subgroup of J that is dense in $(J, \mathfrak{O}_2)$ [a copy of the rationals, say], then H is $\mathfrak{O}_1$-closed but not $\mathfrak{O}_2$-closed.

Suppose that (2) holds. Let M be a subgroup of J that is open, closed, and compactly generated in the $\mathfrak{O}_1$-topology. The group $(J, \mathfrak{O}_1)$ is not σ-compact, since otherwise (5.29) [2] would imply that φ is a homeomorphism on J. It follows that J/M is [uncountably] infinite. Let $\{x_1M, x_2M, \ldots\}$ denote a countably infinite subgroup of J/M, and let $H = \bigcup_{k=1}^{\infty} x_k M$. Clearly H is open, closed, and σ-compact in the $\mathfrak{O}_1$-topology; also H is not $\mathfrak{O}_1$-compact. Assume that H is $\mathfrak{O}_2$-closed. Then H is locally compact in the $\mathfrak{O}_2$-topology and (5.29) [2] implies that $(H, \mathfrak{O}_1)$ and $(H, \mathfrak{O}_2)$ are homeomorphic. Since $H \subset J$, H is $\mathfrak{O}_2$-compact, while not being $\mathfrak{O}_1$-compact. This contradiction shows that H is not $\mathfrak{O}_2$-closed.

THEOREM (Hewitt [1]). *There exists an $\mathfrak{O}_1$-continuous character that is $\mathfrak{O}_2$-discontinuous.*

Proof. Let H be a subgroup of G that is $\mathfrak{O}_1$-closed and not $\mathfrak{O}_2$-closed. Let H_0 be the $\mathfrak{O}_2$-closure of H. By (24.12) [2], the character identically one on H can be extended to an $\mathfrak{O}_1$-continuous character on H_0 that is not identically one. This character can be extended to an $\mathfrak{O}_1$-continuous character on G. Plainly such a character is not $\mathfrak{O}_2$-continuous.

As easy counting argument shows that G has at least $2^{\aleph_1}$ $\mathfrak{O}_1$-continuous characters; from the above it follows that at least $2^{\aleph_1}$ of them are $\mathfrak{O}_2$-discontinuous.

We next make an elementary observation:

PROPOSITION. *Every infinite locally compact Abelian group has a nontrivial proper closed subgroup.*

Proof. Assume that the proposition is false; then there is a group G for which every element different from the identity generates a dense subgroup. Obviously G cannot be the discrete group of integers. Since G is monothetic, G must be compact (9.1) [2]. The character group X of G is discrete and hence contains a nontrivial proper subgroup Y. The character group of X/Y must then be a nontrivial proper closed subgroup of G; see (23.25) [2].

We now consider the question: to what extent is a locally compact Abelian group determined by its family of closed subgroups? For the remainder of this paper, $(G, \mathfrak{O}_1)$ and $(G, \mathfrak{O}_2)$ will denote locally compact Abelian groups such that *a subgroup of G is $\mathfrak{O}_1$-closed iff it is $\mathfrak{O}_2$-closed.*

One can easily construct examples where $\mathfrak{O}_1 \neq \mathfrak{O}_2$. (Let T denote the circle group, and let σ be a discontinuous isomorphism of T onto T. Let $\mathfrak{O}_1$ be the usual compact topology for T and $\mathfrak{O}_2$ consist of images of sets in $\mathfrak{O}_1$ under σ. Then clearly $\mathfrak{O}_1 \neq \mathfrak{O}_2$ and the only $\mathfrak{O}_1$-closed or $\mathfrak{O}_2$-closed subgroups are finite.) The writer has been unable to produce an example for which the groups $(G, \mathfrak{O}_1)$ and $(G, \mathfrak{O}_2)$ are not topologically isomorphic. In spite of this, he feels that Theorems 1 and 2 below are of some interest.

LEMMA 2. *Let $H \subset G$. If G is $\mathfrak{O}_1$-compact and H is $\mathfrak{O}_2$-open, then H is $\mathfrak{O}_1$-open.*

Proof. Let φ be the natural mapping of G onto G/H. Let $\mathfrak{H}$ be any subgroup of G/H. Then $\mathfrak{H} = \varphi(K)$ for some subgroup $K \supset H$ of G. Since H is $\mathfrak{O}_2$-open, K is $\mathfrak{O}_2$-open. Therefore K is $\mathfrak{O}_2$-closed and hence K is $\mathfrak{O}_1$-closed. Since $(G, \mathfrak{O}_1)$ is compact, φ is a closed mapping in the $\mathfrak{O}_1$-topology by (5.18) [2]. Hence $\mathfrak{H} = \varphi(K)$ is closed in $(G/H, \mathfrak{O}_1)$. In other words, every subgroup of $(G/H, \mathfrak{O}_1)$ is closed. Consequently we have from Lemma 1 that the locally compact Abelian group $(G/H, \mathfrak{O}_1)$ is discrete. Hence H is $\mathfrak{O}_1$-open.

The following shows that for locally compact Abelian groups, the closed subgroups alone determine whether the group is compact.

THEOREM 1. *The group $(G, \mathfrak{O}_1)$ is compact iff $(G, \mathfrak{O}_2)$ is compact.*

Proof. Assume, on the contrary, that $(G, \mathfrak{O}_1)$ is compact and that $(G, \mathfrak{O}_2)$ is not. Since $(G, \mathfrak{O}_2)$ contains an open, closed, and compactly generated subgroup, (9.8) [2] shows that there is a subgroup J of G such that $(J, \mathfrak{O}_2)$ is topologically isomorphic with $R^a \times F$ for some integer $a \geqslant 0$ and some compact group F.

Suppose first that $a > 0$. Then clearly there exists a countably infinite subgroup H of G that is $\mathfrak{O}_2$-closed. Hence H must also be $\mathfrak{O}_1$-closed. By (4.26) [2], we see that $(H, \mathfrak{O}_1)$ is discrete and this contradicts the compactness of $(G, \mathfrak{O}_1)$.

Suppose now that $a = 0$, so that $J = F$. Since F is $\mathfrak{O}_2$-open, Lemma 2 shows that F is $\mathfrak{O}_1$-open. Since $(G, \mathfrak{O}_1)$ is compact, it follows that G/F is finite. Since F is $\mathfrak{O}_2$-compact, this in turn implies that $(G, \mathfrak{O}_2)$ is compact—a contradiction.

If one of the topologies in Theorem 1 is 0-dimensional, a stronger conclusion results.

THEOREM 2. *If $(G, \mathfrak{O}_1)$ is compact and 0-dimensional, then $\mathfrak{O}_1 = \mathfrak{O}_2$.*

Proof. By Theorem 1, $(G, \mathfrak{O}_2)$ is also compact. By (7.7) [2], the family $\mathfrak{H}$ of subgroups of G that are $\mathfrak{O}_1$-open is a basis at the identity e. By Lemma 2, each H in $\mathfrak{H}$ is also $\mathfrak{O}_2$-open. Since $\bigcap\{H\colon H \in \mathfrak{H}\} = \{e\}$, an elementary compactness argument shows that $\mathfrak{H}$ is a basis at e for the $\mathfrak{O}_2$-topology. (Compare the proof of (8.5) [2].) It follows that $\mathfrak{O}_1 = \mathfrak{O}_2$.

The author wishes to thank Professor Edwin Hewitt for the opportunity of reading a pre-publication copy of his paper.

Note. As remarked in [1], the theorem of Hewitt is included in a more general result by I. Glicksberg, *Uniform boundedness for groups*, Canad. J. Math. 14, pp. 269-276; see Corollary 2.4.

References

[1] E. Hewitt, *A remark on characters of locally compact Abelian groups*, Fund. Math. 56 (1964), pp. 55-64.

[2] — and Kenneth A., Ross, *Abstract harmonic analysis*, Volume I. Grundlehren der Math. Wiss., Band 115, Springer-Verlag, Heidelberg, 1963.

THE UNIVERSITY OF ROCHESTER

Reçu par la Rédaction le 17. 1. 1963