alence class of V, say C. But then, except in the trivial case $X = \emptyset$, $V(X) = C$, so

$$V(X)\times V(X) = C\times C\subset V.$$

Hence, for any $W\in\mathfrak{U}$, $V(X)\times V(X)\subset W$, $V(X)$ is bounded, and V is conserving.

COROLLARY. *A Hausdorff nonarchimedean uniform space* $(S,\mathfrak{U})$ *is conservative if and only if* $\mathfrak{U}$ *is discrete, i.e.* $\Delta\in\mathfrak{U}$.

Proof. $(S,\mathfrak{U})$ is Hausdorff if and only if $\bigcap\mathfrak{U} = \Delta$.

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The inversion of Peano continua by analytic functions*

by

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1. Introduction. Suppose that f is a function analytic, or even schlicht, in the open disk $|z|<1$. Suppose that A is an arc which has one end point, p, on the unit circle, but which otherwise lies in the open unit disk. Despite the fact that A itself is locally connected at every point, it may very well happen that the image of $A-p$, $f(A-p)$, will have a closure that is not locally connected. This will occur, for example, for any such arc that leads to a point of $|z|=1$ which corresponds under a conformal map to a prime end of the fourth kind [4]. If the map is not schlicht, $f(A-p)$ may even be a closed set, but fail to be locally connected. Thus it is *not* true of analytic functions that, given a Peano continuum (1) P in the plane, and a component C of the intersection of P and the open disk, then the closure of $f(C)$ is always a Peano continuum. In this sense, analytic functions are not "Peano-continuum preserving". They do, of course, preserve local connectedness for Peano continua lying entirely in $|z|<1$, since any continuous map on a Peano continuum preserves this property.

This paper is concerned with the opposite problem: Given a function f into the plane or the extended plane, defined in $|z|<1$, when is such a function Peano-continuum reversing? By this I mean the following: The map $f(z)$, $|z|<1$, is *Peano-continuum reversing* provided that if P is any Peano continuum in the extended plane, and C is a component of $f^{-1}(P)$, then the closure of C, $\bar{C}$, is a Peano continuum.

In this paper, I show that bounded analytic or quasiconformal functions, the elliptical modular functions, and some meromorphic functions of bounded characteristic are all Peano-continuum-reversing. These functions are all special cases of the interior light functions of Stoïlow, and, actually, the theorems of this paper follow by purely topological methods from topological hypotheses. Thus the results are quite general, but that was not an aim of the paper. The fact is that I do not know

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(1) A *Peano continuum* is a compact, metric, connected and locally connected space.

how to shorten the proofs by use of analytic methods even for the special case of bounded analytic functions.

That arcs can be reversed under some conditions is certainly well known to many analysts, particularly those interested in cluster sets. However, there exist analytic functions f, defined in $|z|<1$, and arcs A such that some components of $f^{-1}(A)$ do not have locally connected closures. The first such function I know of is Valiron's 1934 example [8] of a regular unbounded function f in the open unit disk such that f is bounded on some spirals approaching the unit circle, while on other spirals f approaches ∞. For this function, if A is any arc (on the Riemann sphere) with ∞ as one end point, each component of $f^{-1}(A)$ will necessarily spiral to $|z|=1$, and so cannot have a closure locally connected at any point of $|z|=1$. For other examples and references relating particularly to such spirals, see Bagemihl and Seidel [3].

A. J. Lohwater has shown ([5], [6]) that if $w=f(z)$ is a meromorphic function of bounded characteristic that has only a finite number of poles and G_ε is a component of the inverse of the disk $|w|<\varepsilon$, then $\bar{G}_\varepsilon$ is a closed Jordan region. This result has proved quite useful in cluster-set theory, and was the origin of my interest in this problem. G. T. Whyburn has proved general theorems ([9], [10], [11]) about arc inversion for light interior maps of Peano continua, and has also showed [10], essentially, that if f is a light interior map of the open disk into the plane, and C is a component of the inverse of a Peano continuum, then C itself is locally connected.

2. Results. We will consider a function f defined in the open disk D consisting of all numbers z with $|z|<1$; we let C denote the boundary of D. We assume:

(1) *The function f is a light interior map of D into the Riemann sphere, S^2.*

A meromorphic function satisfies (1), since poles are not singularities from this viewpoint. In addition, for $k=2,3,\ldots$, f will be assumed to satisfy one or another of the successively stronger conditions:

(k) *Given an arc of C, there are k points of the arc such that at each of these points the radial limit, $\lim_{r\to 1} f(re^{i\theta})$, exists, and such that these radial limits are all different.*

We define the *range* of f at the point $e^{i\theta}$ of C to be the set $R(f, e^{i\theta})$ composed of all points w in S^2 that are taken on by f in every neighborhood of $e^{i\theta}$. We will prove the following theorems:

THEOREM A. *If f is bounded and satisfies* (1) *and* (2), *and P is a Peano continuum in E^2, then each component of $f^{-1}(P)$ has a closure that is a Peano continuum.*

THEOREM B. *If f satisfies* (1) *and* (2), *then either f is Peano-continuum reversing or there exist an arc ab of C, and two points w_1 and w_2, of S^2 such that for each point $e^{i\theta}$ in ab, the set $R(f, e^{i\theta})$ contains $S^2-w_1-w_2$.*

THEOREM C. *If f satisfies* (1) *and* (3), *either f is Peano-continuum reversing or there exist an arc ab of C such that for each point $e^{i\theta}$ in ab, the range $R(f, e^{i\theta})$ is all of S^2.*

COROLLARY D. *If f is a bounded analytic or quasi-conformal function satisfying* (1) *and* (2), *then f is Peano-continuum reversing.*

COROLLARY E. *If f is a meromorphic function of bounded characteristic defined in D and there is some value that is taken on by f only a finite number of times in D, then f is Peano-continuum reversing.*

These two corollaries follow because each of the three classes of functions of the corollaries have the Fatou property that the radial limit exists almost everywhere on C, and the Riesz property that the radial limit is not constant on any set of positive measure, so that all three satisfy (1) and (n) for each n. The required proofs for quasi-conformal functions are in Agmon [1].

The four theorems have proofs that are identical up to a certain point, so that it will be easiest to present one argument and at the appropriate points pause to complete the proof of each. This will be done in the next section.

The most useful result concerning continua that are not locally connected is the following, due to R. L. Moore ([9], p. 18).

LEMMA 1 (Moore). *If M is a locally compact connected metric space which is not locally connected at some point p, then there exist an open set U containing p and an infinite sequence of distinct components $N_1, N_2, N_3, \ldots$ of $\bar{U}$ that converges to a non-degenerate subcontinuum N of $\bar{U}$ that contains p.*

LEMMA 2 (Stoïlow) ([2]). *If f is a light interior map from a 2-manifold M into a 2-manifold, then each point of M lies in a neighborhood on which f is topologically equivalent to $w=z^n$ for some positive integer n.*

From these two results, Whyburn has proved ([10], [11], p. 80) theorems stronger than the following, for which I will indicate a proof.

LEMMA 3 (Whyburn). *If f satisfies condition* (1) *in the disk D and P is a Peano continuum in S^2, then $f^{-1}(P)$ is locally connected.*

Proof. From Lemma 2 it follows that the points of D at which f is not a local homeomorphism form a discrete, countable set. Since local connectedness is preserved under homeomorphisms, the only points where $f^{-1}(P)$ can fail to be locally connected lie in this discrete set.

([2]) For a proof of this lemma, and generalizations, see Whyburn's books [9], [11], particularly, [11], 5.1, p. 88.

It now follows from Lemma 1 that each *component* of $f^{-1}(P)$ is locally connected. Hence if $f^{-1}(P)$ is not locally connected at a point z_0, every neighborhood of z_0 intersects infinitely many components of $f^{-1}(P)$. Now let U be a neighborhood of z_0, with $\overline{U} \subset D$, on which f is equivalent to z^n, $n > 1$, and let $C_1, C_2, \ldots$ be a convergent sequence of components of $f^{-1}(P)$ having z_0 in their limit. If infinitely many sets C_j intersect $\overline{U} - U$, the limit set of $\{C_j\}$ is non-degenerate, and $f^{-1}(P)$ is not locally connected at any of the uncountably many points of U lying in this limit set, a contradiction. On the other hand, if infinitely many sets C_j lie in U, they are all mapped onto P by f. For if C_j is such a component, and is not mapped onto P, there is an open set $V \subset C_j$ such that $f(V) \cap P$ is a proper subset of P, such that $\overline{V}$ is compact in D, and such that $(\overline{V} - V) \cap f^{-1}(P)$ is empty. Since $f(V)$ is open, we get a contradiction. Then f on U is infinite-to-one, not n-to-one.

This lemma shows that the only points where the closure of a component of $f^{-1}(P)$ could fail to be locally connected must lie in C.

3. Proofs. Suppose now that X is a component of $f^{-1}(P)$, and that $\overline{X}$ is not locally connected. Since by Lemma 3, X itself is locally connected, we can conclude that use of Lemma 1 gives a non-degenerate continuum K in C, an open set U whose closure contains K, and a sequence $K_1, K_2, K_3, \ldots$ of components of $X \cap \overline{U}$ whose limit set is K. Since Lemma 1 is a "local" theorem, we can assume that K is an arc, xy, of C, and not all of C.

By Hypothesis (2), in $xy - x - y$ there are two points, a and b, such that the radial limit of f exists at each, but is not the same at a as it is at b. Let R_a, R_b be the radii to a and b, respectively. The set $R_a \cup R_b$ divides D into two sectors, D' and D'', where D' is the one whose boundary contains the arc ab of xy. Since x and y are in $\overline{D}''$ but not in $\overline{D}'$, it follows that for n large, K_n intersects both R_a and R_b. *However, there is a radius R in $\overline{D}''$ such that, for n large, R does not meet K_n.* Otherwise, every point of $C \cap \overline{D}''$ would be in K, implying falsely that $K = C$.

If n is so large that $K_n \cap R$ is empty and $K_n \cap R_a \neq 0 \neq K_n \cap R_b$, then some component K_n' of $K_n \cap D'$ has a closure that intersects both R_a and R_b. Perhaps the most accessible proof of this is to note that K_n is locally compact, locally connected and connected, and so is arcwise connected. It follows that in K_n there is an arc uv irreducible from $R_a \cap K_n$ to $R_b \cap K_n$. The open arc uv is in D' or in D''. Since it cannot lie in D'' without meeting R, it lies in D'. Let K_n' be the component of $K_n \cap D'$ containing the open arc uv.

The sequence $\{K_n'\}$ thus obtained is defined for all values of n sufficiently large, and will have all of ab as limit set. There may be other components of $f^{-1}(P) \cap D'$ that have closures that meet both R_a and R_b, and there may be components of $f^{-1}(P) \cap D'$ whose closures meet only one of R_a or R_b. In any case, by Lemma 3, no component of $f^{-1}(P) \cap D'$ contains a limit point of the union of the other components, though it could happen that the closure of one might contain such a limit point. Also, $f^{-1}(P) \cap D'$ can have only a countable number of components, since each component is relatively open in $f^{-1}(P) \cap D'$.

If A is any component of $f^{-1}(P) \cap D'$ that has a closure that meets both R_a and R_b but that does not meet the origin, then $\overline{A}$ and $R_a \cup R_b$ are two continua whose intersection is not connected, so that $A \cup R_a \cup R_b$ separates the plane. It follows that A separates $D' \cup ab$ into at least two sets, one containing points near the origin (since a neighborhood of the origin contains points in D' and D'' that are separated by $\overline{A} \cup R_a \cup R_b$), the other containing ab (for the same sort of reason). General theorems about separation of two points by disjoint sets ([9], p. 42 et seq.) show that given two components of $f^{-1}(P) \cap D'$, both spanning $R_a \cup R_b$, one separates the other from ab in $D' \cap ab$. Further, it follows from compactness and Lemma 3 that the order type of all these components in the order induced by separation is that of the integers. We order them accordingly into a sequence $A_1, A_2, A_3, \ldots$, where for each n A_{n+1}, separates A_n from ab in $D' \cup ab$.

For each n, there is just one component, D_n, of $D' - f^{-1}(P)$ that has boundary points in A_n and in A_{n+1}; D_n has no boundary points in any other set A_k.

One argument for this is to note first that no component of $f^{-1}(P) \cap D'$ separates A_n from A_{n+1} in D'. For (i) no component that has limit points both in R_a and R_b can separate them, since such a component is an A_k, and this would violate the meaning of our ordering; (ii) no component that has limit points in R_a (or R_b) alone can separate them, because one can avoid such a component and go from A_n to A_{n+1} by an arc that stays in D' very close to R_b; and (iii) no component whose closure is entirely in D' separates them, for the same reason. Now let x_n, x_{n+1} be points of A_n, A_{n+1}, respectively. Compactify D' by adjoining an ideal point ω; then $D' \cup \omega$ is a 2-sphere. Add ω to all the components K_a of $f^{-1}(P) \cap D' - A_n - A_{n+1}$ that have limit points in $R_a \cup R_b \cup ab$. Then the collection $\{K_a \cup \omega\}$ is a countable collection of continua no one of which separates x_n from x_{n+1} in S, each two of which intersect exactly in ω, and whose union is closed. It follows from the Rutt-Roberts Theorem that $\bigcup (K_a \cup \omega)$ also fails to separate x_n from x_{n+1}. This gives us a connected open set D^* in $D' - \bigcup K_a$ that contains x_n and x_{n+1}. The compact components of $f^{-1}(P) \cap D^*$ are countable in number, and no one of them separates x_n from x_{n+1} in D^* (for they would also separate them in D'). Therefore their union fails to separate these points in D^*. We now have a connected open set D^{**} in D' that contains x_n

and x_{n+1}, but that does not meet $f^{-1}(P)-A_n-A_{n+1}$. At least one component of $D^{**}-A_n-A_{n+1}$ has boundary points both in A_n and A_{n+1}, for an arc in D^{**} from x_n to x_{n+1} will contain a subarc lying in D^{**} except for one end point in A_n and one in A_{n+1}. To show that there is exactly one such component, let x_a be a point on R_a that lies between $R_a \cap A_n$ and $R_a \cap A_{n+1}$ on R_a, and x_b be chosen similarly on R_b. There is an arc $x_a x_b$ in $(D'-A_n-A_{n-1}) \cup x_a \cup x_b$. There is a point y_a in $x_a x_b$ that is the last point of $x_a x_b$ to meet R_a, or to meet a component of $f^{-1}(P) \cap D'$ whose closure does not meet R_b, but that does meet R_a. There is also a first point y_b of $x_a b_b$ following y_a that meets R_b or a component of $f^{-1}(P) \cap D'$ whose closure does not meet R_a but that does meet R_b. There is no loss in assuming that $y_a y_b$ meets no component of $f^{-1}(P)$ lying entirely in D^*; then $y_a y_b$ separates A_n from A_{n+1} in D^{**}, and must meet every component of $D^{**}-A_n-A_{n+1}$ that has boundary points both in A_n and A_{n+1}. But $y_a y_b \cap D^{**}$ is connected, so that there can be just one such component. Let this component be denoted by D_n.

We remark next that, for each n, the complement of D_n has only a finite number of components, since $\mathrm{Bdry} D_n$ lies in the locally connected set $R_a \cup R_b \cup f^{-1}(P)$, since $\bar{D}_n$ is compact. The boundary of each such component is, then, a Peano continuum. It follows that $\bar{D}_n$ is also a Peano continuum ([9], VI, 2.3). Let J be the simple closed curve in $\mathrm{Bdry} D_n$ that separates D_n from the origin. The set J contains two arcs each irreducible from $A_{n+1} \cap J$ to $A_n \cap J$. If one of these arcs failed to meet either R_a or R_b it would follow that the union of that arc, with A_n and with A_{n+1} would be a connected subset of $f^{-1}(P) \cap D'$, contradicting the definition of $\{A_k\}$. It is not hard to verify, but I omit details, that one of these arcs meets R_a and the other meets R_b. It follows that D_n *meets both* R_a *and* R_b.

We show next that *there is a component* F *of* S^2-P *such that infinitely many of the sets* D_n *are mapped by* f *into* F. Since P is a Peano continuum, the components $F_1, F_2, F_3, \ldots$ of S^2-P form a contracting sequence; that is, either there are only a finite number of them, or the diameter of F_n approaches zero as n increases ([9], VI, 4.4). Hence any infinite sequence of components of S^2-F contains a subsequence converging to a point of P. Each set D_n is mapped by f into one of the sets F_k. If my remark were false, there would exist a sequence $n_1, n_2, n_3, \ldots$ of distinct values of n and a sequence $k_1, k_2, k_3, \ldots$ of distinct values of k such that $f(D_{n_j})$ is contained in F_{k_j} and such that $\{F_{k_j}\}$ converges to a point y, necessarily in P. If R is any radius leading to an interior point of ab, R intersects all the sets D_{n_j} by the last paragraph. Hence if the radial limit on R exists, it must be y. But this means that there is only one radial limit in ab. This contradicts (2), and proves the assertion.

The same argument shows also that in ab any radial limit must be in the part of P that bounds F.

Each of the sets $\bar{F}$, S^2-F, and $\bar{F}-F$ are Peano continua ([9], VI, 4.4), though the last two may just be P. With this remark, we have now proved:

LEMMA 4. *Under Hypotheses* (1) *and* (2) *if there is a Peano continuum* P *such that some component of* $f^{-1}(P)$ *is not locally connected, then there is a Peano continuum that does not separate* S^2 *with the same property.*

We now change the meaning of our notation. Let P now denote S^2-F; let $D_1, D_2, D_3, \ldots$ be those former sets D_{n_j} that map into F, the subscripts still indicating order by separation; and let $A_1, A_2, A_3, \ldots$ be the components of the complement in D' of the union of sets D_i that contains the old set A_j, the numbering being chosen so that D_n has boundary points in A_n and A_{n+1} as before.

We can give a topological characterization of P: since P does not separate S^2, it has 2-cells for its true cyclic elements, and a "boundary curve" for boundary ([9], VI, 2.3). It follows that *any point* x *of the boundary of* P *has arbitrarily small neighborhoods* V *such that* $F-\bar{V}$ *is connected.*

Recall that at a and at b the radial limits of f exist and are unequal; let these be α and β, α corresponding to a. There exist neighborhoods V_α and V_β of α and β, respectively, such that $\bar{V}_\alpha$ and $\bar{V}_\beta$ are disjoint, and such that $F-\bar{V}_\alpha-\bar{V}_\beta$ is connected. We have only to choose V_α so that $F-\bar{V}_\alpha$ is connected, and so that $\bar{V}_\alpha$ does not contain β, and apply our last remark to $F-\bar{V}_\alpha$ to get $\bar{V}_\beta$. Since $f(z)$ approaches α on R_a, there is an interval on R_a terminating at a and lying in $f^{-1}(\bar{V}_\alpha)$; and analogously on R_b. It follows that for all n sufficiently large D_n meets both $f^{-1}(\bar{V}_\alpha)$ and $f^{-1}(\bar{V}_\beta)$. Since these are disjoint sets, $D_n-f^{-1}(\bar{V}_\alpha)-f^{-1}(\bar{V}_\beta)$ is not empty; let E be one of its components. The boundary of E is a compact subset of $f^{-1}(P) \cup f^{-1}(\bar{V}_\alpha \cup \bar{V}_\beta)$. It follows from a theorem of Whyburn's, a variant of the maximum modulus principle ([9], VIII, 7.3), that the boundary of $f(E)$ is contained in $\bar{V}_\alpha \cup \bar{V}_\beta \cup P$. Then E must be mapped onto all of $F-\bar{V}_\alpha-\bar{V}_\beta$. Since V_α and V_β can be taken as small as we wish, this proves that *if* Y *is a subset of* F *whose closure does not contain* α *or* β, *then there is an integer* N *such that for all* $n > N$, Y *is contained in* $f(D_n)$. Further, see at once from this *that given a point* q *in the boundary of* F, *but distinct from* α *or* β, *then there is an integer* N *such that for all* $n > N$, q *is contained in* $f(\mathrm{Bdry} D_n)$.

If by some chance F were unbounded, but f were bounded, as it is in Theorem A, we would now have a contradiction since every point of F is in the image of some set D_n. If the original continuum P had neither separated S^2 nor had ∞ as an interior point, we would, then, have completed a proof of Theorem A. For these conditions would have assured that F is unbounded.

The remainder of the argument is a proof that points in the interior of P, as revised, are also taken on in D' arbitrarily closely to ab.

Suppose that G is a component of the interior of P. Then G is the interior of one of the 2-cell cyclic elements of P, and $\overline{G} \cap \overline{F}$ is a simple closed curve. Let q be a point of that curve, other than α or β. We have just proved that, for n large, q is contained in $f(\mathrm{Bdry}\, D_n)$. Now let r be a point of G. There is an arc qr lying in $G \cup q$, and there exist neighborhoods V_α, V_β of α, β whose closures do not meet qr, and such that $F-\overline{V}_\alpha-\overline{V}_\beta$ is connected. For each n for which such a point exists, let q_n be a point of $\mathrm{Bdry}\, D_n$ that maps into q. If n is large, any arc from q_n to a point outside D' must meet either $D_{n-1}, D_n, f^{-1}(V_\alpha)$, or $f^{-1}(V_\beta)$. It will simplify the argument slightly if we assume that qr does not contain the image of any branch point of f, except possibly r.

This is a safe assumption: Since the branch points of f form a countable set, we can take q not to be in the image of that set, and can also select qr to avoid the image of the branch points. With this simplification, we have a neighborhood of q_n on which f is a homeomorphism; and so there is an arc beginning at q_n that is mapped homeomorphically by f onto a subarc of qr. Now consider all arcs $\{q_n x\}$ with that property. It is easily seen that there is a maximal arc with that property. The construction is rather familiar to the analyst. In outline it is this. First if $q_n x_1, q_n x_2, q_n x_3, \ldots$ is a nested sequence of such arcs, all mapped homeomorphically onto a subarc of qr, and $\lim x_j = x$, then qx is an arc and x is in D'. It follows that a maximal nested sequence of such arcs has a greatest element, $q_n y$, lying in D'. If $f(y)$ is not r, we could extend $q_n y$, since f is a local homeomorphism at y, and contradict maximality. Thus we have proved that *the point r is taken on infinitely often in D'*.

Combining the last three italicized statements, we see that we have established that *if U is an open set containing ab, then $f(U \cap D)$ contains every point of S^2 with the possible exceptions of α and β*. This implies that f is unbounded, and so establishes Theorem A.

The rest follows easily. In selecting a and b, we had no restriction as to where on $yx-x-y$ they lay. It follows almost immediately that *if c is any point of $xy-x-y$, the range of f at c, $R(f,c)$ omits at most two points*. For suppose that $R(f,c)$ omits three points. Then there is an open set U containing c such that $f(U \cap D)$ omits these three points. Let a', b' be points of $xy-x-y$ so that the arc $a'b'$ of xy lies in U and contains c, and so that the radial limits of f exist and are unequal at a' and b'. But we have just proved that $f(U \cap D)$ contains all of S^2 except these two radial limits, a contradiction. Hence $R(f,c)$ is all S^2 except for two points.

If there is a point c in $xy-x-y$ such that $R(f,c)$ omits two points, let U be an open set containing c such that $f(U \cap D)$ omits these two points. Then at any point c' in $U \cap C$, $R(f,c')$ must omit precisely these two points. This readily gives the arc of C required in the conclusion of Theorem B if f is not Peano-reversing, but under the supposition that some $R(f,c)$ omits two points. If no set $R(f,c)$ omits two points, but one, $R(f,c_0)$, omits one point, w, select an open set U containing c_0 such that $f(U) = S^2-w$. For any point z in $U \cap xy-x-y$, $R(f,z)$ is S^2-w. Let w be one of the two points of Theorem B, and select the other at random. If no set $R(f,c)$ omits any point, the result is trivial. This completes the proof of Theorem B.

Now suppose Hypothesis (3); suppose that c is a point in $xy-x-y$ such that $R(f,c)$ omits a point w; and let U be an open set containing c such that $f(U \cap D)$ omits w. If $a'b'$ is an arc in $U \cap C$ such that the radial limits exist at a' and at b', we have proved that the only values omitted by f in $U \cap C$ are these two radial limits, α and β. If $\alpha \neq w \neq \beta$, we have a contradiction. But since there are three distinct radial limits in U, it is possible to choose α and β to get this contradiction. Thus each set $R(f,c)$ is all of S^2. This establishes Theorem C.

Our arguments are all essentially local. Thus we could state them locally. For example, Theorem A becomes:

If f satisfies (1), *c is a point of C near which f is bounded, and there is a neighborhood U of c such that given any arc of C in U, there are two points of the arc at which the radial limits exist and are unequal, then given a component of the inverse of a Peano-continuum, its closure is locally connected at c.*

I will not give the analogous other statements.

4. Some comments and examples. Professor C. Pommarenke has kindly pointed out to me that if X is a finite set of more than two points on the Riemann sphere, then the set S^2-X has a covering surface of hyperbolic type; and that the composition of conformal maps of the open disk onto this covering surface with the projection is an analytic or meromorphic function f such that the elements of X are exactly the radial limits of f, each being a radial limit at a dense set of points. This implies that even for analytic functions Hypotheses (3) and (4) are independent. It seems reasonable that Hypothesis (2) is also independent of the others, but I have no meromorphic example. Actually, the restriction to radial limits in the hypotheses was made only for verbal convenience, and at some loss of brevity the theorems could all be proved if "radial limit" were replaced throughout by "asymptotic value". For the axioms modified by this change, I can prove independence. Indeed, the following theorem, whose proof has appeared elsewhere [12], is true.

THEOREM. *Let W_1, W_2, W_3 be three finite sets of points on the Riemann sphere. Then there exist three disjoint countable subsets A_1, A_2, A_3, of the circle C, each dense in C, and a function f meromorphic in the open disk D, such that* (i) *f has asymptotic values only at points of $A_1 \cup A_2 \cup A_3$; and* (ii) *for each $j = 1, 2, 3$, each of the points of W_j is an asymptotic value at each point of A_j, there being no other asymptotic value in A_j.*

It would be interesting to get similar examples involving radial limits.

It is easy to give an example of a function continuous on the closed disk and light and interior on the open disk but which violates Hypothesis (2) and which is not Peano-reversing. Such a function can be obtained by taking an upper-semi-continuous collection filling up $D \cup C$ whose only non-degenerate element is an arc A of C. The decomposition space is again a closed disk, and we get a map $f\colon D \cup C \to D \cup C$. If K is a curve in D approaching A in the oscillatory style of $y = \sin(1/x)$, then the image of $K \cup A$ under f is an arc whose inverse does not have a locally connected closure.

There are various special cases of the theorem that I will not list. For example, if R is a rational curve, and f is Peano-reversing and interior and light, each component of $f^{-1}(R)$ has a rational curve for closure. Or if R is the closure of an open set each boundary curve of which is a Jordan curve, so is each component of $f^{-1}(R)$.

Perhaps here I should point out explicitly that the results of the paper depend on the fact that our functions are defined in the disk, rather than in any simply connected domain. If a domain does not have a locally connected closure, then even functions bounded and analytic on that domain need not be Peano-continuum reversing.

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