Let us suppose that η-ldim $G < \omega$, i.e. η-ldim $G = n$ where $n < \omega$. Then $G \cong G' \subseteq {}^nP$ according to Theorem 4 and this is impossible because card $G = 2^{\aleph_0}$, card ${}^nP = \aleph_0$. Therefore η-ldim $G = \omega$.

References

[1] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914.

[2] W. Sierpiński, *Sur une propriété des ensembles ordonnés*, Fund. Math. 36 (1949), pp. 56-67.

[3] M. Novotný, *Sur la répresentation des ensembles ordonnés*, Fund. Math. 39 (1952), pp. 97-102.

[4] — *O podobnosti uspořádaných kontinuí typů τ a τ**, Časopis pro pestování matematiky 78 (1953), pp. 59-60

[5] I. Fleischer, *Embedding linearly ordered sets in real lexicographic product*, Fund. Math. 49 (1961), pp. 147-150.

[6] H. Komm, *On the dimension of partially ordered sets*, Amer. Journ. of Math. (1948), p. 507.

[7] V. Novák, *On the pseudodimension of ordered sets*, Czechoslovak Mathematical Journal, 1963.

[8] J. Novák, *On partition of an ordered continuum*, Fund. Math. 39 (1952), pp. 53-64.

[9] D. Kurepa, *Ensembles ordonnés et ramifiés*, Thése, Paris 1935, Publ. Math. Belgrade 4 (1935), pp. 1-138.

[10] F. Hausdorff, *Untersuchungen über Ordnungstypen*, Berichte über die Verhandlungen d. königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-physische Klasse 58 (1906).

[11] G. Birkhoff, *Lattice theory*, New York 1948.

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Semigroups and clusters of indecomposability *

by

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In [4] and [9] we have generalized indecomposable continua in various ways; here we wish to consider these types of continua as topological semigroups. The examples in [4] and [9] are based upon Wilder's constructions for his Theorems 1 and 8 of [15], pp. 275-278, 290-292; these constructions and examples are complicated. However, we also give below simpler examples for which our definitions and theorems hold.

Below, S is a topological semigroup, which we call a semigroup, such that there is a continuous mapping $m\colon S\times S\to S$, called multiplication, where S is a Hausdorff space and m is associative. For $x, y \in S$, we write $xy = m(x, y)$; and $AB = \{xy\colon x \in A,\ y \in B\}$. We let u be the unit of S and 0 be the zero, if these exist, where, for all $x \in S$, $xu = x = ux$ and $x0 = 0 = 0x$. We use E to denote the set of idempotents of S, where for $e \in E$, $ee = e$. We recall that a non-null subset A of S is a *left ideal* if and only if $SA \subset A$ and it is a *right ideal* if $AS \subset A$; it is an *ideal* if and only if it is both a left and a right ideal. We denote the *minimal ideal* by K and the null set by $\emptyset$.

Basic definitions and results concerning semigroups are in [14]; for topology they are in [6] and [16]. By a continuum, or a subcontinuum of S, we mean a connected subset of S which is closed in S. We think of S imbedded in another space, so that the connected semigroup S need not be the same as its closure $\bar{S}$; but then the multiplication operation m is extendable to $\bar{S}$; this is true for the examples of connected semigroups in [5] and [7].

Definitions. We say, for $A \subset B$, that A is *region-containing* in B if the interior of A with respect to B is non-null; that is if there exists a region (neighbourhood) R such that $A \supset R \cap B$: if $x \in R \cap B$, we say that A is region-containing at x. The connected set S has an *n-fold set* $\bigcup Z_j$ $(j = 1, 2, \ldots, n)$ *of indecomposability* if and only if every region-containing connected subset W of S is such that $\bar{W}$, the closure of W in S, contains some Z_j, and we take each Z_j non-null: if $n = 1$, we let $Z = Z_1$ and say S has a set Z of indecomposability, and if S is a con-

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tinuum, then S is a *continuum with a set Z of indecomposability* [9]. Let $\{Z_i\}$ and $\{Z_i'\}$ $(i=1,2,\ldots)$ be disjoint classes of non-null disjoint subsets of a connected set S; let $Z=\bigcup Z_i$ and $Z'=\bigcup Z_i'$. Then we say that S is a *connected set with cluster pair (Z,Z') of indecomposability* if and only if every region-containing connected subset W of S is such that $\overline{W}$ contains either Z_i or Z_i' for each i $(i=1,2,\ldots)$.

EXAMPLE 1.0. Let $I=\{x\colon 0\leqslant x\leqslant 1\}$, C be a Cantor ternary set and let S' be the cartesian product $C\times I$, where however we take as the same point all $(c,0)$ for $c\in C$; see the Rees Quotient of Theorem 8 below; this is what we call a *Cantor* ternary *triangle* in Example 1 of [11], p. 267 or of Example 4 [12], p. 125. Then S' is a topological semigroup with multiplication $(c,x)(c',x')=\big(\min(c,c'),xx'\big)$, where xx' is the multiplication of real numbers and C retains the ternary number system from its construction on a unit interval, so that $\min(c,c')$ has meaning. Let $(c,0)=(0,0)$. Then S' is a semigroup continuum with set $Z=(0,0)$ of indecomposability, and Z is its minimal ideal K. In Theorem 2 of [7], p. 247, we showed that there exists a biconnected semigroup B, dense in S', with dispersion point $(0,0)$; thus B is a connected set with set Z of indecomposability; as a semigroup it has unit $u=(1,1)$ and zero $0=(0,0)$, as does S'. Let now S_i $(i=1,2,3,4)$ be homeomorphic to a semigroup S' or B, $S=\bigcup S_i$, $S_j\cap S_{j+1}$ $(j=1,2,3)$ be Z_{j+1} where Z_{j+1} is the unit of S_j and the zero of S_{j+1}; if $x\in S_i$, $y\in S_j$ and $j<i$, let $xy=yx=y$; if $i=j$, $xy=\big(\min(c,c'),xx'\big)$ as above. Therefore S is a topological semigroup connected set with 4-fold set $\bigcup Z_i$ of indecomposability, with minimal ideal $K=Z_1=(0,0)$ of S_1; the unit of S is the unit of S_4; $\overline{S}$ inherits the multiplication of S. For similar multiplication, see Hunter in [2], pp. 242-243. The set B above can also be taken as the biconnected semigroup of Theorem 1 or 2 of [5], pp. 234, 237.

DEFINITION. Let $A\subset V\subset S$. Let Q be open in V, W be closed in V and W be connected. We say $T(A,V)=V-\{x\colon$ there exist Q and W such that $x\in Q\subset W\subset V-A\}$; we take $T(A,S)=T(A)$ ([11], [12]).

LEMMA 1. *Let S be a connected set with n-fold set $\bigcup Z_i$ of indecomposability and let $p_i\in Z_i$. Then $S=T(\bigcup p_i)$.*

Proof. Suppose $x\notin T(\bigcup p_i)$. Then there exist open Q and a connected and closed in S subset W such that $x\in Q\subset W\subset S-\bigcup p_i$. Thus W is a region-containing subcontinuum of S, and so, by definition of S as a connected set with n-fold set $\bigcup Z_i$ of indecomposability, W must contain some Z_i; hence $p_i\in W\subset S-\bigcup p_i$ is a contradiction. Therefore $x\in T(\bigcup p_i)$, and so $S\subset T(\bigcup p_i)\subset S$. See [12], p. 127.

LEMMA 2. *Let p_i $(i=1,2,\ldots,n)$ be n distinct points of the cartesian product S of two nondegenerate connected Hausdorff spaces V and W. Then in any region R about any $p\in S$ there exists $x\in S$ and $x\notin T(\bigcup p_i)$.*

Proof. We follow Jones' proof of his Theorem 7 in [3], p. 406. Let $S=\{(v,w)\colon v\in V,\ w\in W\}$ and let $p_i=(v_i,w_i)$; it is known that S is a Hausdorff space ([17], p. 92). Since a finite subset in a Hausdorff space is not connected, let $(v'',w'')\in S$, where v'' is not any v_i; in R about any $p\in S$ let $(v',w')=x$, where w' is not any w_i. Let $V(w_i)=\{(v,w)\colon v\in V,\ w=w_i\}$. Since $V(w_i)$ is a closed subset of S, $\bigcup V(w_i)$ is also a closed subset of S, and so S contains an open subset Q such that $x\in Q$ and $\overline{Q}\cap\big(\bigcup V(w_i)\big)=\emptyset$. Now let $W(v'')=\{(v,w)\colon v=v'',\ w\in W\}$; then $W(v'')$ is a continuum and $p_i\notin W(v'')$, for any i.

Let $H_0=\bigcup(V\times w_0)$ for all $w_0\in W$ such that $(V\times w_0)\cap\overline{Q}\neq\emptyset$. The set H_0 is closed, contains Q, but does not contain any p_i, since $\overline{Q}\cap V(w_i)=\emptyset$. Let $H=H_0\cup W(v'')$. Then H is a continuum, since each $(V\times w_0)=V(w_0)$ is connected and intersects $W(v'')$, which is connected; and $H\supset Q$ and $H\cap(\bigcup p_i)=\emptyset$. Hence $x\in Q\subset H\subset S-\bigcup p_i$, and so $x\notin T(\bigcup p_i)$. Thus the lemma is true.

THEOREM 1. *Let $S=ES\cup SE$ and let S be a compact semigroup continuum with n-fold set $\bigcup Z_i$ of indecomposability. Then the minimal ideal K contains some Z_i; if S has a zero 0, then some Z_i is 0. If $n=1$, then $K\supset Z_1=Z$.*

Proof. If S has a zero 0, then $K=0$; if S is compact, then K is known to exist. Suppose that K does not contain any Z_i. Let $p_i\in Z_i-K$. By Lemma 1, $S=T(\bigcup p_i)$. Let $x\in K$; then $x\in T(\bigcup p_i)$. But then, by Corollary 1.1 of [11], p. 266, $K\cap(\bigcup p_i)=\emptyset$, contrary to the way p_i was taken above. Therefore K contains some Z_i, and so the theorem is true.

THEOREM 2. *Let S be a compact semigroup continuum with n-fold set $\bigcup Z_i$ of indecomposability. Then neither S, nor K, — if $K\supset Z_1$ and $K\supset R$ a region of S, — is the cartesian product of two nondegenerate continua; and hence K then is either a group or, for all $x,y\in K$, either all $xy=x$ or all $xy=y$.*

Proof. The case for S is true at once from Lemmas 1 and 2. Suppose that K is the cartesian product of two non-degenerate continua. By hypothesis, K contains some Z_i and let $K\supset\bigcup Z_i$ $(i=1,2,\ldots,f)$, but $K\not\supset Z_j$ $(j=f+1,f+2,\ldots,n)$; let $p_t\in Z_t$ $(t=1,2,\ldots,n)$, but $p_j\notin K$. By hypothesis, $K\supset R\subset S$ and, by Lemma 2, there exists $x\notin T(\bigcup p_i,K)$, $x\in R$; by definition, there exist region-containing continuum W in K and open set Q such that $x\in Q\subset W\subset K-\bigcup p_i\subset S-\bigcup p_i$. Since $K\supset R\subset S$, Q may be taken open in S; hence $x\notin T(\bigcup p_i)$, and so $x\notin S$ by Lemma 1. Hence K cannot be the cartesian product of two nondegenerate continua, and so by Corollary 1 of [13], p. 278, either K is a group or, for $x,y\in K$, all $xy=x$ or all $xy=y$.

COROLLARY 2.1. *Let S be a semigroup and compact continuum with n-fold set $\bigcup Z_i$ of indecomposability, each Z_i be region-containing, $S = ES \cup SE$, and E be countable. Then K is a group.*

Proof. Since S is a compact continuum and $S = ES \cup SE$, K is known to be a continuum. We note if K is a point, $K = 0$ and so K is a group. Suppose that K is not a group. By Theorem 1 and hypothesis, $K \supset$ some $Z_i \supset R$ a region of S; then, by Theorem 2, for all $x, y \in K$, all $xy = x$ or $xy = y$, and so all $xx = x \in E$, which is countable. Thus K is a continuum containing only countably many points, which is false in a compact Hausdorff space.

EXAMPLE 2.0. Let D be the complex number unit disc, C be the Cantor ternary set, let $S'' = D \times C$, with $(0, c) = (0, 0) = Z$ for all $c \in C$ (i.e., S'' is a pile of discs with $(0, 0)$ in common); and $(d, c)(d', c') = (dd', \min(c, c'))$; in $S = \bigcup S_i$ $(i = 1, 2, 3, 4)$ of Example 1.0, let now S_j $(j = 2, 3, 4)$ be homeomorphic to S''; let S_1 be a solenoid with multiplication such that S_1 is a group. Let otherwise the multiplication be as in Example 1.0. Let $Z_1 = S_1 \cup Z_2 = S_1$. Then S is a continuum with 3-fold set $\bigcup Z_i$ of indecomposability and S is a semigroup, as it was in Example 1.0. Then $K = S_1$ is region-containing and is a group as in Theorem 2.

EXAMPLE 2.01. Let C be the Cantor ternary set, $I' = \{x\colon 0 < x \leqslant 1\}$ and let $S' = C \times I'$. Let $(1, 1)$ of this be at $(1, 1, 1)$ of the xyz-space and let S' spiral down upon the square Z in the xy-plane with diagonal from $(0, 0, 0)$ to $(1, 1, 0)$ so that Z is the limiting set of S' and S' projects onto Z; let S then be $S' \cup Z$. For $(x, y, z), (x', y', z') \in S$, let $(x, y, z)(x', y', z') = (x, y', 0)$. Then S is a semigroup and a continuum with set Z of indecomposability; and $K = Z$, which is the cartesian product of two nondegenerate continua, but the conclusion of Theorem 2 is not true here; nor is its hypothesis for K.

EXAMPLE 2.02. Let S be as in Example 2.01, but let $(x, y, z)(x', y', z') = (xx', yy', 0)$; then $K = (0, 0, 0) \not\supset Z$, contrary to the conclusion of Theorem 1; but $S \neq ES \cup SE$.

LEMMA 3. *Let S be a semigroup, cancellative in $Y \subset S$ and let $C \subset S$. If $y \in Y$, $c \in C$ and $\bar{C}$ is not region-containing at c, then $y\bar{C}$ is not region-containing at yc. If $K = S$ is a group and $k \in K$, then $k\bar{C}$ is region-containing at kc if and only if $\bar{C}$ is region-containing at c.*

Proof. Suppose that $y\bar{C}$ is region-containing at yc. Since multiplication m is continuous, for a region $R'' \subset y\bar{C} \subset \bar{S}$ and $yc \in R''$, there exist regions R, R' of $\bar{S}$ such that $y \in R$, $c \in R'$ and $yc \in m(R \times R') = RR' \subset R''$. Since $\bar{C}$ is not region-containing at c, there exists $s \in R' - \bar{C}$ such that $ys \in RR' \subset R'' \subset y\bar{C}$. Therefore there exists $c' \in \bar{C}$ such that $ys = yc'$.

Since S is cancellative in Y, $s = c' \notin \bar{C}$, which is a contradiction. Hence if $\bar{C}$ is not region-containing at c, $y\bar{C}$ is not region-containing at yc.

The second conclusion, with $S = K$ a group, is well known, but follows quickly from the above.

THEOREM 3. *Let S be a semigroup connected set with n-fold set $\bigcup Z_i$ of indecomposability, let the minimal ideal K exist and be closed and let every nondegenerate region of K contain a region of S; let $K \supset Z_1$. Then $K = T(\bigcup z_i, K)$, for all i where $z_i \in Z_i$ and $K \supset Z_i$. If there are m of these z_i, then K is a connected set with m-fold set $\bigcup (K \cap Z_i)$ of indecomposability. (Also true for all $K \cap Z_i \neq \emptyset$.)*

Proof. Since $K = SxS$ for $x \in K$, K is connected. By definition $K \supset T(\bigcup z_i, K)$. Suppose $T(\bigcup z_i, K) \not\supset K$; by Lemma 1 of [12], p. 114, $T(\bigcup z_i, K)$ is closed. Hence there exist in K a region R of S and $k \in K - T(\bigcup z_i, K)$ such that $k \in R$ and $\bar{R} \cap T(\bigcup z_i, K) = \emptyset$. Then there exist an open Q and a closed connected subset W such that $k \in Q \subset \overline{W} \subset K - \bigcup z_i \subset S - \bigcup z_j$ for all $z_j \in Z_j - K$ $(j = 1, 2, \ldots, n;\ i \neq j)$ and all $z_j = z_i$. Because $K \supset R$ of S then $k \notin T(\bigcup z_j, S)$, although by Lemma 1 $T(\bigcup z_j) = S$. Hence $T(\bigcup z_i, K) = K$.

Suppose now that there exists a connected subset W of K which is region-containing, but does not contain any $K \cap Z_i$; hence there exists $x \notin T(\bigcup z_i, K)$, $x \in K$. Since this is false, the theorem is true.

In Example 2.01, we see that K is not a connected set with set Z of indecomposability, although the hypothesis of Theorem 3 is almost satisfied.

NOTATION AND EXAMPLE. Below in Theorem 4 we say that S is a connected set with n-fold set $\bigcup Z_i$ of indecomposability, and with n minimal, each Z_i maximal and $\bigcup Z_i$ is unique. By this we mean that of the possible ways to choose the Z_i, we take one in which n has smallest possible value, then we take each Z_i maximal in size, and finally we consider only those cases for S where the class of Z_i thus can be taken in only one way. Consider Example 3 of [12], p. 124, where $S = \bigcup I_j$ $(j = 1, 2, 3)$ and S is a simple chain of the indecomposable continua I_j. As noted there S has 2-fold set $\bigcup Z_i$ of indecomposability (it also has 3-fold set), and the Z_i $(i = 1, 2)$ can be taken maximal in three different ways. Thus the class $\{Z_1, Z_2\}$ is not unique. If instead I_3 were a Cantor triangle with its vertex q not in I_2, then $S = \bigcup I_j$ would have n-fold set of indecomposability for $n = 2$, where n is minimal, the Z_i can be taken maximal and the class $\{Z_1, Z_2\}$ is unique (that is $Z_1 = I_1 \cap I_2$ and $Z_2 = q$).

THEOREM 4. *Let S be a connected set with n-fold set $\bigcup Z_i$ of indecomposability, the minimal ideal $K \supset Z_j$ $(j = 1, 2, \ldots, n';\ n' \geqslant 1)$ and K be a group and have n'-fold set $\bigcup Z_j$ of indecomposability; n is minimal,* *each Z_i is maximal and $\{Z_i\}$ is unique. Then $K = Z_1$; if also Z_1 is region-containing in S, then K is either a point or an indecomposable connected set.*

Proof. Let $\{\bar{C}\}$ be the class of region-containing subcontinua of K. By Lemma 3, $\{k\bar{C}\}$ is the same class, for $k \in K$. We see that $\bar{C} \supset Z_j$ implies $k\bar{C} \supset kZ_j$ implies $k^{-1}k\bar{C} = \bar{C} \supset k^{-1}kZ_j$, because $K \supset C \cup Z_j$; thus $\bar{C} \supset Z_j$ is equivalent to $k\bar{C} \supset kZ_j$. Hence $\{Z_j\}$ and $\{kZ_j\}$ are the same class, since $\{Z_i\}$ $(i = 1, 2, \ldots, n)$ is unique in order that S have n-fold set $\bigcup Z_i$ of indecomposability.

If $Z_j \subset K$, then $kZ_j \subset K$ by definition of an ideal; also if $kZ_j \subset K$, then $k^{-1}kZ_j = Z_j \subset K$. Hence $Z_j \subset K$ is equivalent to $kZ_j \subset K$; and by hypothesis $K \supset Z_1$. We wish to prove that $K = \bigcup Z_j$. Since $K \supset \bigcup Z_j$, let $k \in \bigcup Z_j$ and suppose $k' \in K - \bigcup Z_j$. Then $k' = (k'k^{-1})k$. Thus $k \in Z_j$ implies $k' \in (k'k^{-1})kZ_j$; but $(k'k^{-1})kZ_j$ is itself some Z_j contained in K. Therefore $k' \in \bigcup Z_j$, which is a contradiction. Hence $\bigcup Z_j \supset K = \bigcup Z_j$.

Let $\{C'\}$ be the class of region-containing connected subset of S. We note that $Z_j = \bigcap \{\bar{C}_a\colon C_a \in \{C'\} \text{ and } \bar{C}_a \supset Z_j\}$, and since the $\bar{C}_a$ are closed in S, Z_j is closed in S. Hence the connected set K is the union of n' closed subsets Z_j. Thus, if we suppose $n' > 1$, there exist two of these which intersect; say $Z_1 \cap Z_2 \neq \emptyset$. Each $C \in \{C'\}$, where either $\bar{C} \supset Z_1$ or $\bar{C} \supset Z_2$, is such that $\bar{C} \supset Z_1 \cap Z_2$. Hence $\{Z_1 \cap Z_2, Z_3, Z_4, \ldots, Z_n\}$ is a class of $n-1$ elements such that every $\bar{C}$ contains at least one of them: thus S is a connected set with $(n-1)$-fold set of indecomposability, and so n is not minimal contrary to hypothesis. Therefore $n' = 1$, and so $K = Z_1$ and K is a connected set with set Z_1 of indecomposability. Thus, by the definition of an indecomposable connected set, K is indecomposable, which however includes the case when K is a point.

Example 4.0. Let G be the complex number group on the unit circle and let S' be the clan with kernel G, irreducible from G to the unit of the example, following Wallace and Koch's Corollary 1, in [13], p. 286. Let C be the Cantor ternary set, form $S' \times C$, and shrink each $g \times C$ to a point for $g \in G$. We are following here Hunter's Example 2.2 in [1], p. 286. Thus we can get a semigroup S, which may be described as a band of spirals winding down upon G, where there is a spiral for each $c \in C$; and S is a continuum with set G of indecomposability, where $G = K$. Thus the first conclusion of Theorem 4 is true; K is not an indecomposable continuum as in the second conclusion, but the hypothesis of Theorem 4 is not satisfied.

Example 4.01. Let S be the union of an indecomposable continuum and a Cantor ternary triangle of Example 1.0, where these have intersection the point z, which is the vertex of the triangle; let the multiplication on the indecomposable continuum I' be that of a group and on the triangle be as in Example 1.0, with z at the unit of I'. Then S si a continuum semigroup with set I' of indecomposability and $K = I'$; this illustrates Theorem 4.

Example 4.1. This illustrates Theorem 3 and the second conclusion of Theorem 4. Let N be a topological group G and an indecomposable continuum. Let $I = \{x\colon 0 \leqslant x \leqslant 1\}$. Let C be a Cantor ternary subset of N, where C is taken such that $S = (C \times I) \cup N$ is a continuum with set N of indecomposability. Let the multiplication in N be that of G, and so for $c, c' \in C$, $cc' \in G$. For $x, x' \in I$, $g \in G$, let $(c, x)(c', x') = cc' \in G$, $(c, x)g = cg$ and $g(c, x) = gc$. Thus S is the desired semigroup above.

Corollary 4.2. *Let S be a nondegenerate compact semigroup continuum with n-fold set $\bigcup Z_i$ of indecomposability, where n is minimal, each Z_i is maximal and the class $\{Z_i\}$ is unique. If there exists $x \in S$ such that xS and Sx are both region-containing and if $S = K$, then S is a topological group and an indecomposable continuum.*

Proof. Neither, for all $x, y \in S$, is all $xy = x$ or all $xy = y$, since xS and Sx are region-containing and S is a nondegenerate compact continuum. Hence, by Theorem 2 above and by Corollary 1 of [13], p. 278, S is a group, and so, by Theorem 4, is an indecomposable continuum. For a related result, see Koch and Wallace's Corollary 1 in [13], p. 286.

Lemma 4. *Let $C, Y \subset S$, where $\bar{C}$ is compact, let $Z' \subset S$ and let $X = \{x\colon x \in Y, x\bar{C} \supset Z'\}$. Then X is closed in Y.*

Proof. Suppose that X is not closed in Y, and so let $x' \in Y$ be a limit point of X such that $x' \notin X$. Then $x'\bar{C} \not\supset Z'$, and since $x'\bar{C}$ is compact, there exist an open set $U \not\supset x'\bar{C}$ and a point $z' \in Z'$ such that $z' \notin U$. By continuity, there exist open sets U' and U'' such that $x' \in U'$ and $\bar{C} \subset U''$ and for which $U'U'' \subset U$. But $U' \supset x \in X$; hence $U \supset U'U'' \supset U'\bar{C} \supset x\bar{C} \supset Z' \supset z'$, which is a contradiction. Thus the lemma is true.

Definition. For $C, Y \subset S$, we say that the *right C-deflating subset in Y* is the set of all $y \in Y$ such that $y\bar{C}$ is not region-containing in S. (Or put left for right and $\bar{C}y$ for $y\bar{C}$.)

Example. Let S be the closed plane unit square with diagonal from $(0, 0)$ to $(1, 1)$ and the multiplication be coordinate-wise. Let Y be a subarc of S with a sequence y_i $(i = 1, 2, \ldots)$ on the x-axis: then each y_iS is not region-containing. Here the right S-deflating subset in Y is closed and is contained on the x and y axes.

Example. It is of interest to note that $y\bar{C}$ can be region-containing when $\bar{C}$ is not. On the x-axis, let S consists of the isolated point q and the sequence y_i $(i = 1, 2, \ldots)$ with sequential limit y_0. Let $C = y_0$, $y = y_1$ and $Y = \bigcup y_t$ $(t = 0, 1, 2, \ldots)$. Let $y_iy_j = q = q^2 = qy_t = y_tq$. Then yC is region-containing, although C is not and the C-deflating subset in Y is $\emptyset$.

THEOREM 5. *There exists a compact connected commutative semigroup S, with $Y \subset S$, such that the right S-deflating subset in Y is not closed.*

Proof. We first show the existence of a nonconnected semigroup S' on the the x-axis with this property. We let $[x', x''] = \{x: x' \leqslant x \leqslant x''\}$. Let C be the Cantor ternary set on $[0, 1]$, $y_i = 2+1/i$, $Y_i' = [y_{i+1}+\frac{1}{2}(y_i - y_{i+1}), y_i]$, Y_i be the Cantor ternary set on Y_i', $Y = \bigcup Y_i$ $(i = 1, 2, \ldots)$ and let $S' = C \cup Y \cup 2$.

Define $U_{ji} = [(j-1)/3^i, j/3^i] \cap C$, for i any natural number and j any odd natural number: thus the U_{ji} come from the nonomitted intervals of the Cantor ternary set construction, and, for any i, $\bigcup U_{ji} = C$ and only one U_{ji} then contains any $c \in C$. Note that the sets U_{ji} and Y_i are open subsets in S'.

We now define for S' a multiplication m: $S' \times S' \to S'$ as follows:

(a) $m(c, c') = 1$, for $c, c' \in C$;

(b) $m(y, y') = \max(y, y')$, for y, y' in the same Y_i;

(c) $m(y, y') = y_k$, for $y \in Y_i$, $y' \in Y_j$ and $k = \min(i, j)$, $i \neq j$;

(d) $m(2, y) = y_i = m(y, 2)$, for $y \in Y_i$;

(e) $m(c, y) = j/3^i = m(y, c)$, for $y \in Y_i$ and $c \in U_{ji}$;

(f) $m(2, c) = c = m(c, 2)$, for $c \in C$; and

(g) $m(2, 2) = 2$.

We remark: the above multiplication gives $S' = ES' = S'E$, $K = 1 \in C$ is the zero of S', 2 is a unit for C, and every element of Y is idempotent. It is easily seen that the operation m is commutative. (Here we may use Y_i' in place of Y_i, but then S below would not have a set z of indecomposability.)

We now show that S' is associative. Consider first $y(y'c)$ and $(yy')c$, for $y \in Y_i$, $y' \in Y_k$ and $c \in C$. Take first the case $i > k$. Then $(yy')c = y_k c$; since c is in some U_{tk}, say $c \in U_{jk}$; then $(yy')c = y_k c = j/3^k$. Then also $y(y'c) = y(j/3^k)$; then $j/3^k \in C$ and so in some U_{ti}, say $j/3^k \in U_{ni}$. Thus $y(j/3^k) = n/3^i$; but note $U_{ni} \cap U_{jk} \supset j/3^k$, and so their right end points are equal, that is $j/3^k = n/3^i$. Therefore $(yy')c = y(y'c)$. A somewhat similar proof shows this for the cases $i = k$ and $i < k$

We see that $(yc)c' = 1 = y(cc')$, for either $y = 2$ or $y \in Y$. Consider $yy'c$, for $y = 2$, $y' \in Y_k$, $c \in C$. Then $(2y')c = y_k c$ and $2(y'c) = 2(y_k c) = y_k c$.

Since m is commutative, any other permutations of the elements still is associative from the above arguments. Thus we conclude that S' is associative.

Consider now whether m is a continuous multiplication. We see first that m is continuous on $C \times C$ and on $Y_i \times Y_i$, since then the multiplication is trivial. Let now $y \in Y_i$, $y' \in Y_k$, and suppose $i \geqslant k$; then $yy' = y_k$. Let U be an open subset of S' containing y_k; now Y_i and Y_k are also open subsets of S' which contain y and y', respectively, and $m(Y_i \times Y_k) = y_k \in U$; therefore m is continuous on $Y_i \times Y_k$ for $i \geqslant k$. A similar proof shows this, when $i < k$.

Finally, let $y \in Y_k$ and consider $2y = y_k$. Let U be an open set containing y_k. There exists an open set V containing 2 such that $V \cap (\bigcup Y_i$ for $i = 1, 2, \ldots, k) = \emptyset = V \cap C$. Then $m(V \times Y_k) = y_k \in U$, and so m is continuous on $2 \times Y$. Thus m is continuous on $(Y \cup 2) \times (Y \cup 2)$. Consider now $Y \times C$.

Consider yc for $c \in C$, $y \in Y_i$; let $c \in U_{ji}$. Let U be any open subset of S' containing $yc = j/3^i$. Now U_{ji} and Y_i are open subsets of S' with $c \in U_{ji}$ and $y \in Y_i$, and $m(Y_i \times U_{ji}) = j/3^i \in U$. Hence m is continuous at (y, c).

Consider $2c$ and let $\varepsilon > 0$ and let $(c-\varepsilon, c+\varepsilon)$ be the open interval of C with these end points. There exist j and i such that $c \in U_{ji} \subset (c-\varepsilon, c+\varepsilon)$. The set $A = 2 \cup (\bigcup Y_k)$ $(k = i, i+1, i+2, \ldots)$ is open in S' about 2. It follows that $m(A \times U_{ji}) \subset U_{ji} \subset (c-\varepsilon, c+\varepsilon)$, and so m is continuous at $(2, c)$. Thus m is a continuous multiplication over $S' \times S'$.

For all i, the set $y_i S'$ consists of a finite subset of C together with the finite set $\bigcup y_k$ $(k = 1, 2, \ldots, i)$, and so it has a null interior. However $2S'$ contains C, and so has a non-null interior. Thus the right (and so the left) S'-deflating subset in Y is not closed; hence the right C-deflating subset in Y is also not closed.

Let now $S'' = S' \times I$, where I is the unit interval. Then define m': $S'' \times S'' \to S''$ by $m'\big((s, x), (s', x')\big) = \big(m(s, s'), xx'\big)$. We now form a connected space S by identifying the points $(s, 0) \in S''$ as one point and call this point z. Hence *S is a connected set with set z of indecomposability*. Let $z_i = (y_i, 1) \in Y$ and $z' = (2, 1) \in Y$. Then $z_i S$ is not region-containing, although $z'S$ is, and so the right S-deflating subset in Y is not closed.

Remark 5.0. In the definition of m in the proof of Theorem 5, change now (b) and (c) to (bc) and (d) to (d′) where

(bc) $m(y', y) = y'' = m(y, y')$, where $y'' = \max(y, y')$, for $y, y' \in Y$;

(d′) $m(y, 2) = m(2, y) = y$, for $y \in Y$.

Then a similar proof to that of Theorem 5 gives that *S is a connected set with set z of indecomposability, S is a commutative semigroup with unit at $(2, 1) \in S$, $K = z$, the right S-deflating subset in Y is closed, but the right C-deflating subset in Y is not closed.*

THEOREM 6. *Let S be a connected semigroup with n-fold set $\bigcup Z_i$ of indecomposability, where each Z_i is region-containing, C and Y are connected subset of S, $\overline{C}$ is compact and $\overline{C} \subset S$ and the right C-deflating subset* *in Y is closed in Y. Then, for $y \in Y$, either every $y\bar{C}$ is region-containing or every $y\bar{C}$ is not region-containing.*

Proof. Since C is connected, yC is connected. Let $X_i = \{x: x \in Y$ and $x\bar{C} \supset Z_i\}$. By the definition of S a connected set with set $\bigcup Z_i$ of indecomposability, if $y\bar{C}$ is region-containing, then $y\bar{C}$ contains some Z_i; thus y is an element of some X_i. By Lemma 4, each X_i is closed in Y. Let Y' be the set of elements in Y such that $y\bar{C}$ is not region-containing. By hypothesis, the right C-deflating subset in Y is closed in Y, and so Y' is closed in Y. Also, if $y\bar{C}$ is not region-containing, $y\bar{C} \not\supset Z_i$ for any i, since each Z_i is region-containing. Thus $Y' \cap (\bigcup X_i) = \emptyset$. Hence the connected set Y is the union of the disjoint, closed in Y, subsets Y' and $\bigcup X_i$. This is a contradiction unless one of these subsets is null, and so the theorem is true.

COROLLARY 6.1. *Let S be a compact continuum and semigroup with n-fold set $\bigcup Z_i$ of indecomposability, where each Z_i is region-containing. If the right S-deflating subset in S is closed and $s \in S$, then either sS is region-containing for all s or every sS is not; then, if zero $0 \in S$, no sS is region-containing, but if unit $u \in S$, every sS is region-containing; and S does not have both a zero and a unit.*

The proof follows from Theorem 6.

COROLLARY 6.2. *Let S be as in Corollary 6.1, and both the left and right S-deflating subset in S be closed; let unit $u \in \bigcap Z_i$. Then S is a topological group and an indecomposable continuum.*

Proof. By Corollary 6.1, for $s \in S$, sS is region-containing and so $sS \supset Z_i \supset u$ for some i. Hence there exists $s^{-1} \in S$ such that $ss^{-1} = u$. Similarly $Ss \supset Z_i \supset u$ for some i, and so there exists $s' \in S$ such that $s's = u$. Hence it follows that S is a topological group.

Since $u \in \bigcup Z_i$, each region-containing continuum contains u, and so $T(u) = S$. Suppose that there exist $x, y \in S$ such that $y \notin T(x)$. Then there exist open Q and continuum W such that $y \in Q \subset W \subset S - x$; hence $y'' = yx^{-1} \in Qx^{-1} \subset Wx^{-1} \subset Sx^{-1} - xx^{-1} = S - u$. Since S is a topological group, Qx^{-1} is open and Wx^{-1} is a continuum, and so $y'' \notin T(u) = S$. Hence $T(x) = S$ for all $x \in S$, and by Corollary 14.3 of [12], p. 128, S is an indecomposable continuum.

THEOREM 7. *If $S = AS$, S is closed and compact, and $S \supset A$, where A is countable, then there exists $a \in A$ such that aS is region-containing in S. If further S is a continuum with n-fold set $\bigcup U_i$ of indecomposability, each Z_i is region-containing, $A = E$ and the right S-deflating subset in S is closed, then every sS is region-containing and, for $n = 1$, each $e \in E$ is a left unit for $Z = Z_1$.*

Proof. Since $S = AS = S$, $S = \bigcup aS$ $(a \in A)$. Since a and S are compact, aS is closed. Suppose that no aS is region-containing in S.

Then the closure of $S - aS$ is S. Since A is countable, this is contrary to Theorem 15 of [6], p. 11. Hence aS is region-containing in S. Hence, by use of Corollary 6.1, the theorem follows without difficulty.

Notation. If S is a semigroup and I is an ideal of S, then the Rees Quotient $S/I = S - I \cup \{I\}$, with $x \cdot \{I\} = \{I\} = \{I\} \cdot x$ and xy the ordinary multiplication of S if $x, y \notin I$. Let θ: $S \to S/I$ be the natural mapping which is the identity on $S - I$ and sends any element of I into the point $\{I\}$. If S is a topological space, then the topology on S/I is defined as follows: a set $U \subset S/I$ is open if and only if $\theta^{-1}(U)$ is open in S. If I is closed, then S/I is compact Hausdorff, if S is.

THEOREM 8. *Let S be a connected semigroup with n-fold set $\bigcup Z_i$ of indecomposability; if I is an ideal in S, then S/I is a connected set with n'-fold set of indecomposability $\bigcup \theta(Z_i) \cup \theta(I)$.*

Proof. Suppose that M is a region-containing subcontinuum of S/I which does not contain $\theta(I)$; then $\theta^{-1}(M)$ is a region-containing subcontinuum of S, and thus contains a Z_i. Thus M contains a $\theta(Z_i)$.

COROLLARY 8.1. *Let $S = ES \cup SE$ be a compact continuum semigroup with n-fold set $\bigcup Z_i$ of indecomposability. Then S/K is a compact continuum with n'-fold set $\bigcup \theta(Z_i)$ of indecomposability ($n' \leqslant n+1$).*

9.0. MODIFIED WADA CONNECTED SET EXAMPLES. We now give a brief description of the construction of the continua, either with n-fold set or with cluster pair of indecomposability, of our "Clusters of indecomposability" [4]: we assume knowledge of the (C. 4) Network Construction of [9], pp. 84-88, which is based upon Wilder's construction for his Theorem 8 in [15], pp. 290-291. (We hope a more complete description of [4] will be published later.)

9.0.1. SETS OF INDECOMPOSABILITY CASE. In [9], pp. 84-85, filling up a connected domain D, we have a class of disjoint nettings $\{N_i\}$ $(i = 1, 2, \ldots)$ with various properties there described: with each N_i is associated an opening Q_i and a connection T_i. Here disregard the T_i, but associate with each N_i two disjoint openings Q_i and Q_i'; let also Z and Z' be Z' of [9], p. 87; let $\{H_i\}$ be the class of regions, as in [9], p. 86, that close down on the points of Z and let $\{H_i'\}$ be the similar class from Z'. The class of nettings $\{N_i\}$ is taken so that an H_j contains Q_i and H_k' contains Q_i' $(i = 1, 2, \ldots)$ as in [9], p. 86. With this modification in construction, a similar argument to that of Sections 6 and 9 of [9], pp. 84-85, 87-88, gives that $F(D_1) = S$ of Theorem 1 [9], p. 87 is a continuum with 2-fold set $\bigcup Z_i$ of indecomposability, where $Z \supset Z_1$ and $Z' \supset Z_2$; an obvious modification gives n-fold sets $(n = 1, 2, \ldots)$.

9.0.2. CLUSTER PAIR OF INDECOMPOSABILITY CASE. Let now $Z = \bigcup Z_i$ and $Z' = \bigcup Z_i'$ $(i = 1, 2, \ldots)$, where $\{Z_i\}$ and $\{Z_i'\}$ are disjoint classes of disjoint closed subsets. Let $\{H_{ij}\}$ be a class of regions closing down on Z_i in the perfectly separable manner and $\{H'_{ij}\}$ be a similar class closing down on Z'_i $(i=1,2,\ldots)$. Let N_1 with its two openings Q_i and Q'_i be as above in 9.0.1. Take now N_i with its opening Q_1 in H_{11}, N_2 with $Q_2\subset H_{21}$, N_3 with $Q_3\subset H_{12}$, N_4 with $Q_4\subset H_{13}$, N_5 with $Q_5\subset H_{22}$, N_6 with $Q_6\subset H_{31}$ and continue by the diagonal process (but modify as in line 12, p. 86 of [9]); and each N_i is also taken so that Q'_i is contained in corresponding H'_{ij} by the diagonal process. A similar proof to that of Theorem 1 of [9], p. 87, now will show that $F(D_1)=S$ there is now a continuum with cluster pair (Z,Z') of indecomposability, where each region-containing subcontinuum W of S must now contain either Z_i or Z'_i for each i.

A trivial example of a continuum with cluster pair (Z,Z') of indecomposability would be; let S be an indecomposable continuum in the xy-plane, which is bisected by the y-axis; let Z'_i be the points of S on $x=-1/i$ and Z_i those on $x=1/i$.

Semigroup example 9.0.3. We follow the method of Example 4.0. Let N be the continuum with cluster pair (Z,Z') of 9.0.2, let I be the unit interval, C be a Cantor ternary set on N, where C is taken such that $S=(C\times I)\cup N$ is still a continuum with cluster pair (Z,Z') of indecomposability, and let G be the semigroup N where $xy=x$, for $x,y\in N$. Let the multiplication on S be as in Example 4, that is $xy=x_py_p$, where x_p is the projection of $x\in S$ onto G and similarly y_p for y. Here $K=N=G$. In a similar manner we can take $N=K$ of a semigroup continuum with n-fold set of indecomposability.

Theorem 10. *Let S be a connected set with cluster pair (Z,Z') of indecomposability, where $Z=\bigcup Z_i$ and $Z'=\bigcup Z'_i$ $(i=1,2,\ldots)$. Then, for $i=f$, S is a connected set with 2-fold set $(Z_f\cup Z'_f)$ of indecomposability; if $p\in Z_f$ and $p'\in Z'_f$, then $S=T(p\cup p')$; if also S is a closed compact semigroup and $S=ES\cup SE$, then the minimal ideal K contains either Z_i or Z'_i for every i.*

Proof. Let W be a region-containing connected subset of S. By definition of S with cluster pair of indecomposability, we see that $\overline{W}$ contains either Z_f or Z'_f; hence, by definition, S is a connected set with 2-fold set $(Z_f\cup Z'_f)$ of indecomposability. That $S=T(p\cup p')$ then follows by Lemma 1, and the rest of the theorem by Theorem 1.

Remark 10.0. If S is a compact continuum semigroup with cluster pair (Z,Z') of indecomposability and $S=ES\cup SE$, and Z can be taken so that $\bigcup Z_i$ is dense in S, then $S=K$; for then K exists and is closed. See the trivial example at the end of 9.0.2.

Corollary 10.1. *Let $S=ES\cup SE$ and S be a compact semigroup continuum with cluster pair (Z,Z') of indecomposability. Then S does not have a zero.*

Proof. By definition the Z_i are disjoint and by Theorem 10 K contains Z_i or Z'_i for every i; hence $K=0$ cannot have this property.

Remark 10.2. Obviously one can restate Theorem 6 and Corollary 6.1 for S a connected set with cluster pair (Z,Z') of indecomeosability.

References

[1] R. P. Hunter, *On the semigroup structure of continua*, Trans. Amer. Math. Soc. 93 (1959), pp. 356-368.

[2] — *Note on arcs in semigroups*, Fund. Math. 49, 3 (1961), pp. 233-245.

[3] F. B. Jones, *Concerning nonaposyndetic continua*, Amer. Jour. Math. 70 (1948), pp. 403-413.

[4] R. L. Kelley and P. M. Swingle, *Clusters of indecomposability*. Amer. Math. Soc. Notices 6 (1959), pp. 643, 829.

[5] — *Concerning minimal generating subsets of semigroups*, Portugal. Math. 20 (1961), pp. 231-250.

[6] R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publications 13 (1932).

[7] P. M. Swingle, *Existence of widely connected and biconnected semigroups*, Proc. Amer. Math. Soc. 11 (1960), pp. 243-248.

[8] — *Widely connected and biconnected semigroups*, Proc. Amer. Math. Soc. 11 (1960), pp. 249-254.

[9] — *Connected sets of Wada*, Michigan Math. Jour. 8 (1961), pp. 77-95.

[10] — *Sums of connected sets with indecomposable properties*, Port. Math. 16 (1957), pp. 130-144.

[11] H. S. Davis, D. P. Stadtlander and P. M. Swingle, *Semigroups continua and the set functions T^m*, Duke Math. Jour. 29 (1962), pp. 265-280.

[12] — *Properties of the set functions T^n*, Portugal Math. 21 (1962), pp. 113-133.

[13] A. D. Wallace and R. J. Koch, *Admissibility of semigroup structure on continua*, Trans. Amer. Math. Soc. 88 (1958), pp. 277-287.

[14] A. D. Wallace *The structure of topological semigroups*, Bull. Amer. Math. 61 (1955) Soc. pp. 95-112.

[15] R. L. Wilder, *On the properties of domains and their boundaries in E_n*, Math. Annalen 109 (1933), pp. 273-306.

[16] — *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications 32 (1948).

[17] John L. Kelley, *General topology*, Princeton 1955.

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